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## Some results on $\mathcal{T}$ -nonsingular modules

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**Abstract:** The notion of  $\mathcal{T}$ -nonsingularity of a module has been introduced and studied recently. In this article, a number of new results of this property are provided. It is shown that over a commutative semilocal ring  $R$  such that  $Jac(R)$  is a nil ideal, every  $\mathcal{T}$ -nonsingular module is semisimple. We prove that for a perfect ring  $R$ , the class of  $\mathcal{T}$ -nonsingular modules is closed under direct sums if and only if  $R$  is a primary decomposable ring. Finitely generated  $\mathcal{T}$ -nonsingular modules over commutative rings are shown to be precisely those having zero Jacobson radical. We also show that for a simple module  $S$ ,  $E(S) \oplus S$  is  $\mathcal{T}$ -nonsingular if and only if  $S$  is injective. Connections of  $\mathcal{T}$ -nonsingular modules to their endomorphism rings are investigated.

**Key words:** Small submodules,  $\mathcal{T}$ -nonsingular modules, endomorphism rings

### 1. Introduction

The concept of  $\mathcal{T}$ -nonsingularity of a module was introduced and studied recently by Tütüncü and Tribak in 2009 [20] as a dual notion of the  $\mathcal{K}$ -nonsingularity that was introduced and studied by Rizvi-Roman [14, 15]. It was shown in [21] that every dual Baer module is  $\mathcal{T}$ -nonsingular and that every  $\mathcal{T}$ -nonsingular lifting module is dual Baer. We note also that dual Rickart modules were introduced and studied by Lee et al. in 2011 [12] and it is easy to see that every dual Rickart module is  $\mathcal{T}$ -nonsingular. These links of the  $\mathcal{T}$ -nonsingularity with the dual Rickart and dual Baer properties are the motivations for the investigations in this paper. We obtain some new useful properties of this kind of module.

Throughout,  $R$  will denote an associative ring with unity,  $Jac(R)$  will denote the Jacobson radical of  $R$ , and  $Z(R)$  will stand for the right singular ideal of  $R$ . For an  $R$ -module  $M$ , we write  $E(M)$  and  $Rad(M)$  for the injective hull and the Jacobson radical of  $M$ , respectively. If  $N$  is a submodule of an  $R$ -module  $M$ , then the notation  $N \ll M$  means that  $N$  is small in  $M$ .

In Section 2 we investigate general properties of  $\mathcal{T}$ -nonsingular modules. We provide conditions for a  $\mathcal{T}$ -nonsingular module to have zero Jacobson radical. Among other results, we show that every finitely generated  $\mathcal{T}$ -nonsingular module over a commutative ring has zero Jacobson radical. The class of commutative rings  $R$  for which every cyclic  $R$ -module is  $\mathcal{T}$ -nonsingular is characterized as that of von Neumann regular rings, while the class of commutative rings  $R$  for which every finitely generated  $\mathcal{T}$ -nonsingular  $R$ -module is semisimple is shown to be precisely that of semilocal rings. It is also shown that over a commutative semilocal ring  $R$  such that  $Jac(R)$  is a nil ideal, every  $\mathcal{T}$ -nonsingular  $R$ -module is semisimple.

Section 3 is devoted to some results on direct sums of  $\mathcal{T}$ -nonsingular modules. We show that for a

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simple module  $S$ ,  $E(S) \oplus S$  is  $\mathcal{T}$ -noncosingular if and only if  $S$  is injective. We prove that for a perfect ring  $R$ , the class of  $\mathcal{T}$ -noncosingular modules is closed under direct sums if and only if  $R$  is a primary decomposable ring.

The focus of our investigations in Section 4 is on connections of a  $\mathcal{T}$ -noncosingular module to its endomorphism ring.

## 2. Some properties of $\mathcal{T}$ -noncosingular modules

**Definition 2.1** Let  $M$  and  $N$  be 2 modules.

(1) We say that  $M$  is  $\mathcal{T}$ -noncosingular relative to  $N$  if  $\forall \varphi \in \text{Hom}_R(M, N)$ ,  $\text{Im } \varphi \ll N$  implies  $\varphi = 0$ .

(2) The module  $M$  is called  $\mathcal{T}$ -noncosingular if  $M$  is  $\mathcal{T}$ -noncosingular relative to  $M$  (or, equivalently,  $\forall \varphi \in \text{End}_R(M)$ ,  $\text{Im } \varphi \ll M \Rightarrow \varphi = 0$ ).

Many examples of  $\mathcal{T}$ -noncosingular modules are exhibited in [21] and [20]. Before presenting another example, we recall that a module  $M$  is called *radical* if  $M$  has no maximal submodules, i.e.  $\text{Rad}(M) = M$ .

**Example 2.2** Let  $M$  be a simple radical module. That is,  $M$  is a nonzero radical module that has no nonzero radical submodules (e.g., we can consider the  $\mathbb{Z}$ -modules  $\mathbb{Z}(p^\infty)$  and  $\mathbb{Q}$ , where  $p$  is a prime number). Let  $\varphi$  be a nonzero endomorphism of  $M$ . Then  $\text{Im } \varphi \cong M/\text{Ker } \varphi$  and so  $\text{Rad}(\text{Im } \varphi) = \text{Im } \varphi$ . Therefore,  $\text{Im } \varphi = M$ . Hence,  $M$  is  $\mathcal{T}$ -noncosingular.

A ring  $R$  is called a *right V-ring* if every simple right  $R$ -module is injective. This is equivalent to the condition that for any right  $R$ -module  $M$ , we have  $\text{Rad}(M) = 0$ . Recall that a module  $M$  is called  $\mathcal{K}$ -nonsingular if, for every  $0 \neq \varphi \in \text{End}_R(M)$ ,  $\text{Ker } \varphi$  is not essential in  $M$  (see [15]). The next example shows the existence of a  $\mathcal{T}$ -noncosingular module that is not  $\mathcal{K}$ -nonsingular and provides a  $\mathcal{K}$ -nonsingular module that is not  $\mathcal{T}$ -noncosingular.

**Example 2.3** (1) Let  $R$  be a right  $V$ -ring that is not semisimple (e.g., we can take  $R = \prod_{i=1}^{\infty} F_i$  with  $F_i = F$  is a field for all  $i \geq 1$ ). By [20, Proposition 2.13], every  $R$ -module is  $\mathcal{T}$ -noncosingular. On the other hand, from [15, Corollary 2.21] it follows that  $R$  has a module  $M$  that is not  $\mathcal{K}$ -nonsingular.

(2) Let  $F$  be a field and set  $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ . Then  $\text{Jac}(R) = \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$ , and hence  $R_R$  is not  $\mathcal{T}$ -noncosingular by [20, Corollary 2.7]. On the other hand, we have  $Z(R_R) = 0$  by [4, Corollary 4.3]. Applying [15, Corollary 2.4], we conclude that  $R_R$  is  $\mathcal{K}$ -nonsingular.

In [20, Proposition 2.3] it was showed that the  $\mathcal{T}$ -noncosingularity is inherited by direct summands. Next, we show that the  $\mathcal{T}$ -noncosingularity property does not always transfer from a module to each of its submodules and factor modules.

**Example 2.4** (1) Note that the  $\mathbb{Z}$ -module  $\mathbb{Z}/8\mathbb{Z}$  is not  $\mathcal{T}$ -noncosingular, while  $\mathbb{Z}$  is  $\mathcal{T}$ -noncosingular (see [20, Proposition 2.10]).

(2) Consider the  $\mathbb{Z}$ -module  $M = \mathbb{Z}(2^\infty) \oplus \mathbb{Z}(2^\infty)$ . Then  $M$  has a submodule  $N$ , which is isomorphic to  $\mathbb{Z}(2^\infty) \oplus \mathbb{Z}/2\mathbb{Z}$ . Then  $M$  is  $\mathcal{T}$ -noncosingular as every factor module of  $M$  is injective, while  $N$  is not  $\mathcal{T}$ -noncosingular by [20, Example 2.12].

Recall that a ring  $R$  is said to be a *right H-ring* if, whenever  $S_1$  and  $S_2$  are simple  $R$ -modules such that  $\text{Hom}_R(E(S_1), E(S_2)) \neq 0$ , then  $S_1 \cong S_2$ . It is well known that every commutative Noetherian ring is an  $H$ -ring (see, e.g., [16]). Next, we deal with the  $\mathcal{T}$ -nonsingularity of injective hulls of simple modules. First note that for any prime number  $p$ , the  $\mathbb{Z}$ -module  $E(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty)$  is  $\mathcal{T}$ -nonsingular.

**Proposition 2.5** *Assume that  $R$  is a ring that has a unique simple right  $R$ -module (up to isomorphism) or  $R$  is a right  $H$ -ring. If  $S$  is a simple  $R$ -module such that  $E(S)$  is  $\mathcal{T}$ -nonsingular, then  $S$  is injective or  $\text{Rad}(E(S)) = E(S)$ .*

**Proof** Suppose that  $S$  is not injective and  $\text{Rad}(E(S)) \neq E(S)$ . Then  $S \ll E(S)$  and  $E(S)$  has a maximal submodule  $N$ . Let  $S'$  denote the simple  $R$ -module  $E(S)/N$ . Taking the canonical projection  $\pi : E(S) \rightarrow S'$  and the inclusion map  $\alpha : S' \rightarrow E(S')$ , the homomorphism  $\alpha\pi : E(S) \rightarrow E(S')$  is nonzero and  $\text{Im}(\alpha\pi) = S'$ . By hypothesis, we get that  $S' \cong S$ . Hence, there exists a nonzero endomorphism  $\varphi$  of  $E(S)$  such that  $\text{Im} \varphi = S \ll E(S)$ . This contradicts the fact that  $M$  is  $\mathcal{T}$ -nonsingular.  $\square$

**Corollary 2.6** *Let  $m$  be a maximal ideal of a commutative Artinian ring  $R$ . Then  $E(R/m)$  is  $\mathcal{T}$ -nonsingular if and only if  $R/m$  is injective.*

**Proof** Note that  $\text{Rad}(E(R/m)) \neq E(R/m)$  and  $R$  is Noetherian by [2, Theorem 15.20 and Corollary 15.21]. Hence,  $R$  is an  $H$ -ring, and the result follows from Proposition 2.5.  $\square$

We recall that a ring  $R$  is called a *right max ring* if  $\text{Rad}(M) \neq M$  for all nonzero right  $R$ -modules  $M$ .

**Corollary 2.7** *Let  $R$  be a right max local ring with maximal right ideal  $m$ . The following are equivalent:*

- (i)  $E(R/m)$  is  $\mathcal{T}$ -nonsingular;
- (ii)  $R/m$  is injective;
- (iii)  $R$  is a division ring.

**Proof** (i)  $\Rightarrow$  (ii) By Proposition 2.5 and the fact that  $R$  is a right max ring.

(ii)  $\Rightarrow$  (iii) By hypothesis, every simple  $R$ -module is injective. Thus,  $R$  is a right  $V$ -ring and  $m = \text{Rad}(R) = 0$ . Therefore,  $R$  is a division ring.

(iii)  $\Rightarrow$  (i) This is obvious.  $\square$

**Proposition 2.8** *Let  $M$  be a module with  $\text{Rad}(M) \neq 0$  and let  $N$  be a nonzero small submodule of  $M$ . If  $K$  is a module that is isomorphic to  $N$ , then the module  $M \oplus K$  is not  $\mathcal{T}$ -nonsingular.*

**Proof** By hypothesis, there exists an isomorphism  $\varphi : K \rightarrow N$ . Let  $\pi : M \oplus K \rightarrow K$  be the canonical projection and let  $\mu : N \rightarrow M$  and  $\rho : M \rightarrow M \oplus K$  be the inclusion maps. Then  $\rho\mu\varphi\pi$  is a nonzero endomorphism of  $M \oplus K$  such that  $\text{Im}(\rho\mu\varphi\pi) = N \oplus 0 \ll M \oplus K$ .  $\square$

It is easy to see that every module with zero Jacobson radical is  $\mathcal{T}$ -nonsingular and that the converse is not true, in general (e.g., for any prime integer  $p$ , the  $\mathbb{Z}$ -module  $\mathbb{Z}(p^\infty)$  is  $\mathcal{T}$ -nonsingular, but  $\text{Rad}(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p^\infty)$ ). In the next 3 results we present conditions under which the converse holds.

**Proposition 2.9** *Let  $M$  be a module such that every nonzero submodule contains a simple submodule. If  $M \oplus S$  is  $\mathcal{T}$ -nonsingular for every simple small submodule  $S \leq M$ , then  $\text{Rad}(M) = 0$ .*

**Proof** Assume that  $Rad(M) \neq 0$ . Then  $Rad(M)$  contains a simple submodule  $S$ . Thus,  $S \ll M$ . From Proposition 2.8 it follows that  $M \oplus S$  is not  $\mathcal{T}$ -nonsingular. This completes the proof.  $\square$

**Definition 2.10** A module  $M$  is said to be retractable if for every submodule  $N \leq M$ ,  $Hom(M, N) \neq 0$ .

Retractable modules have been studied extensively by different authors (see, e.g., [6, 7, 8, 9, 17]).

**Proposition 2.11** Let  $M$  be a retractable module. If  $M$  is  $\mathcal{T}$ -nonsingular, then  $Rad(M) = 0$ .

**Proof** Suppose that  $Rad(M) \neq 0$ . Then  $M$  contains a nonzero submodule  $N$  such that  $N \ll M$ . Since  $M$  is retractable, there exists a nonzero endomorphism  $f : M \rightarrow M$  with  $Im f \subseteq N$ . This contradicts the  $\mathcal{T}$ -nonsingularity of  $M$ .  $\square$

A ring  $R$  is said to be *right semi-Artinian* if every nonzero right  $R$ -module contains a simple submodule. Recall that if  $R$  is any ring, then a right  $R$ -module  $M$  is *nonsingular* if  $mE \neq 0$  for every nonzero element  $m$  of  $M$  and essential right ideal  $E$  of  $R$ .

**Corollary 2.12** If  $M$  is a nonzero module that satisfies one of the following conditions:

- (i)  $M$  is a module over a commutative semi-Artinian ring,
- (ii)  $M$  is a projective module over a commutative Noetherian ring,
- (iii)  $M$  is a finitely generated module over a commutative ring,
- (iv)  $M$  is a nonsingular module over a right self-injective ring,

then  $M$  is  $\mathcal{T}$ -nonsingular if and only if  $Rad(M) = 0$ .

**Proof** By Proposition 2.11 [5, Theorems 2.7 and 2.8] and [17, Proposition 1.17 and Corollary 2.12].  $\square$

**Corollary 2.13** The following are equivalent for a commutative ring  $R$ :

- (i) Every cyclic  $R$ -module is  $\mathcal{T}$ -nonsingular;
- (ii)  $R$  is a von Neumann regular ring.

**Proof** (i)  $\Rightarrow$  (ii) Let  $I$  be an ideal of  $R$ . By hypothesis, the  $R$ -module  $R/I$  is  $\mathcal{T}$ -nonsingular. Then  $Rad(R/I) = 0$  by Corollary 2.12. So  $R$  is a  $V$ -ring (see [19, Theorem 22.1]). Thus,  $R$  is von Neumann regular since  $R$  is commutative (see [19, Theorem 22.4]).

(ii)  $\Rightarrow$  (i) This follows from [20, Proposition 2.13] and [19, Theorem 22.4].  $\square$

Following [18], a module  $M$  is called *noncosingular* if  $\overline{Z}(M) = \cap\{N \mid M/N \text{ is small in its injective hull}\} = M$ . That is, for every nonzero module  $N$  and every nonzero homomorphism  $f : M \rightarrow N$ ,  $Im f$  is not a small submodule of  $N$ . This is obviously equivalent to the condition that  $M$  is  $\mathcal{T}$ -nonsingular relative to  $N$  for every module  $N$ . Clearly, every noncosingular module is  $\mathcal{T}$ -nonsingular. It is easy to check that if  $S$  is a simple module that is not injective, then  $S$  is  $\mathcal{T}$ -nonsingular but not noncosingular. In the next result, we give conditions under which the  $\mathcal{T}$ -nonsingularity of a module implies its noncosingularity. The following condition was studied in [1] as a dual notion of the retractability.

**Definition 2.14** A module  $M$  is called *coretractable* if, for any proper submodule  $K$  of  $M$ , there exists a nonzero homomorphism  $f : M \rightarrow M$  with  $f(K) = 0$ , that is,  $Hom_R(M/K, M) \neq 0$ .

**Proposition 2.15** *Let  $M$  be a coretractable injective module. If  $M$  is  $\mathcal{T}$ -noncosingular, then  $M$  is noncosingular.*

**Proof** Suppose that there exists a proper submodule  $X$  of  $M$  such that  $M/X \ll E(M/X)$ . Let  $\pi : M \rightarrow M/X$  be the canonical projection. Since  $M$  is coretractable, there exists a nonzero homomorphism  $\varphi : M/X \rightarrow M$ . Since  $M$  is injective,  $\varphi$  can be extended to a homomorphism  $\psi : E(M/X) \rightarrow M$ . Taking the inclusion map  $\alpha : M/X \rightarrow E(M/X)$ ,  $\psi\alpha\pi$  is a nonzero endomorphism of  $M$ . Since  $\alpha\pi(M) \ll E(M/X)$ ,  $\psi\alpha\pi(M) \ll M$  by [13, Lemma 4.2(3)]. This contradicts the  $\mathcal{T}$ -noncosingularity of  $M$ . Hence,  $M$  is noncosingular.  $\square$

The next proposition can be regarded as the dual of [15, Proposition 2.18]. First we prove the following elementary known result.

**Lemma 2.16** *Let  $N$  be a small submodule of a module  $M$ . If  $L$  is a submodule of  $M$  such that  $(L+N)/N \ll M/N$ , then  $L \ll M$ .*

**Proof** Let  $X$  be a submodule of  $M$  such that  $L+X=M$ . Then,  $[(L+N)/N] + [(X+N)/N] = M/N$ . By hypothesis, we have  $(X+N)/N = M/N$ . Therefore,  $X=M$  as  $N \ll M$ . So,  $L \ll M$ .  $\square$

**Proposition 2.17** *Let  $M$  be a module that has a projective cover  $f : P \rightarrow M$ . If  $P$  is  $\mathcal{T}$ -noncosingular, then so is  $M$ .*

**Proof** By hypothesis,  $f : P \rightarrow M$  is an epimorphism with  $Q = \text{Ker } f \ll P$ . Thus,  $P/Q \cong M$ . To prove the  $\mathcal{T}$ -noncosingularity of  $M$ , let  $\varphi \in \text{End}(P/Q)$  such that  $\text{Im } \varphi \ll P/Q$ . Consider the natural epimorphism  $\pi : P \rightarrow P/Q$ . Since  $P$  is projective, there exists a homomorphism  $\psi : P \rightarrow P$  such that  $\varphi\pi = \pi\psi$ . Therefore,  $\pi\psi(P) = \varphi(P/Q) \ll P/Q$ . So  $\psi(P) \ll P$  by Lemma 2.16. But  $P$  is  $\mathcal{T}$ -noncosingular. Then  $\psi = 0$ , and hence  $\varphi\pi = 0$ . This implies that  $\varphi = 0$ . Thus,  $M$  is  $\mathcal{T}$ -noncosingular.  $\square$

**Proposition 2.18** *The following are equivalent for a ring  $R$ :*

- (i)  $R_R$  is  $\mathcal{T}$ -noncosingular;
- (ii) Every projective  $R$ -module is  $\mathcal{T}$ -noncosingular;
- (iii) Every  $R$ -module having a projective cover is  $\mathcal{T}$ -noncosingular.

**Proof** (i)  $\Rightarrow$  (ii) By [20, Corollary 2.7],  $\text{Jac}(R) = 0$ . Let  $P$  be a projective  $R$ -module. Hence  $\text{Rad}(P) = P(\text{Jac}(R)) = 0$  by [2, Proposition 17.10]. So,  $P$  is  $\mathcal{T}$ -noncosingular.

(ii)  $\Rightarrow$  (iii) This follows from Proposition 2.17.

(iii)  $\Rightarrow$  (i) This is obvious.  $\square$

**Definition 2.19** (1) *A module  $M$  has  $D_1$  property (or is called lifting) if for every submodule  $N \leq M$ , there exists a direct summand  $K$  of  $M$  with  $K \subseteq N$  and  $N/K \ll M/K$ .  $M$  has  $D_3$  property if for any direct summands  $K, L$  of  $M$  with  $M = K + L$ ,  $K \cap L$  is a direct summand of  $M$ .*

(2) *A module  $M$  satisfying  $D_1$  and  $D_3$  is called quasi-discrete.*

**Definition 2.20** A module  $M$  is said to have the strong summand sum property, *SSSP*, if the sum of any family of direct summands is a direct summand of  $M$ .  $M$  is said to have the summand intersection property, *SIP*, if the intersection of any 2 direct summands is a direct summand of  $M$ .

In the next result, we provide an application of  $\mathcal{T}$ -noncosingularity to quasi-discrete modules. It can be regarded as the dual of [15, Proposition 4.1].

**Proposition 2.21** Let  $M$  be a quasi-discrete module. If  $M$  is  $\mathcal{T}$ -noncosingular, then  $M$  has *SSSP* and *SIP*.

**Proof** Since  $M$  is  $\mathcal{T}$ -noncosingular lifting,  $M$  has *SSSP* by [21, Theorems 2.1 and 2.14]. To prove *SIP*, let  $K_1$  and  $K_2$  be 2 direct summands of  $M$ . Then  $K = K_1 + K_2$  is a direct summand of  $M$ . Since  $M$  has  $(D_3)$ ,  $K$  has  $(D_3)$  by [13, Lemma 4.7]. Therefore,  $K_1 \cap K_2$  is a direct summand of  $K$ . Hence,  $K_1 \cap K_2$  is a direct summand of  $M$ .  $\square$

Recall that a ring  $R$  is said to be *semilocal* if the factor ring  $R/Jac(R)$  is semisimple.

We conclude this section by describing the structure of some classes of  $\mathcal{T}$ -noncosingular modules over commutative semilocal rings. First we prove the following lemma.

**Lemma 2.22** Let  $I$  be a nil ideal of a commutative ring  $R$ . If  $M$  is  $\mathcal{T}$ -noncosingular, then  $MI = 0$ .

**Proof** Let  $a \in I$  and consider the endomorphism  $\varphi_a$  of  $M$  defined by  $\varphi_a(x) = xa$  for all  $x \in M$ . Clearly, we have  $Im\varphi_a = Ma$ . Let  $X$  be a submodule of  $M$  such that  $M = Ma + X$ . By induction, we have  $M = Ma^n + X$  for every integer  $n \geq 1$ . Then  $X = M$  since the ideal  $I$  is nil. It follows that  $Ma \ll M$ . Thus,  $Ma = 0$  as  $M$  is  $\mathcal{T}$ -noncosingular.  $\square$

**Theorem 2.23** Let  $R$  be a commutative semilocal ring such that  $Jac(R)$  is a nil ideal of  $R$ . Then an  $R$ -module  $M$  is  $\mathcal{T}$ -noncosingular if and only if  $M$  is semisimple.

**Proof** Let  $M$  be a  $\mathcal{T}$ -noncosingular module. Since  $R$  is semilocal, we have  $Rad(M) = MJac(R)$  and  $M/Rad(M)$  is semisimple by [2, Corollary 15.18]. Therefore,  $Rad(M) = 0$  by Lemma 2.22. Thus,  $M$  is semisimple. The converse is immediate.  $\square$

**Corollary 2.24** Let  $R$  be a commutative perfect ring. Then an  $R$ -module  $M$  is  $\mathcal{T}$ -noncosingular if and only if  $M$  is semisimple.

**Proof** By Theorem 2.23 and [2, Theorem 28.4].  $\square$

**Theorem 2.25** The following statements are equivalent for a commutative ring  $R$ :

- (i) Every  $\mathcal{T}$ -noncosingular  $R$ -module  $M$  with  $Rad(M) \ll M$  is semisimple;
- (ii) Every finitely generated  $\mathcal{T}$ -noncosingular module is semisimple;
- (iii)  $R$  is semilocal.

**Proof** (i)  $\Rightarrow$  (ii) This is clear.

(ii)  $\Rightarrow$  (iii) Since  $Rad(R/Jac(R)) = 0$ , the  $R$ -module  $R/Jac(R)$  is  $\mathcal{T}$ -noncosingular. The result follows by (ii).

(iii)  $\Rightarrow$  (i) Let  $M$  be a  $\mathcal{T}$ -noncosingular  $R$ -module with  $Rad(M) \ll M$ . Since  $R$  is semilocal,  $M/Rad(M)$  is semisimple and  $Rad(M) = MJac(R)$  by [2, Corollary 15.18]. If  $a \in Jac(R)$  and  $\varphi_a$  is the endomorphism of  $M$  defined by  $\varphi_a(x) = xa$  for all  $x \in M$ , then we have  $Im \varphi_a = Ma \subseteq MJac(R) \ll M$ . But by  $\mathcal{T}$ -noncosingularity,  $Ma = 0$ . Thus,  $Rad(M) = 0$ . This implies that  $M$  is semisimple.  $\square$

### 3. Direct sums of $\mathcal{T}$ -noncosingular modules

It is shown in [20, Example 2.12] that, in general, a direct sum of 2  $\mathcal{T}$ -noncosingular modules is not  $\mathcal{T}$ -noncosingular. In this section we prove that for a simple module  $S$ ,  $E(S) \oplus S$  is  $\mathcal{T}$ -noncosingular if and only if  $S$  is injective (Proposition 3.4). The class of perfect rings for which arbitrary direct sums of  $\mathcal{T}$ -noncosingular modules are  $\mathcal{T}$ -noncosingular is shown to be exactly that of the primary decomposable rings (Theorem 3.7).

We begin with the next proposition, which is a direct consequence of [20, Corollary 2.7 and Proposition 2.11].

**Proposition 3.1** (i) *If  $M$  is a  $\mathcal{T}$ -noncosingular module, then every direct sum of copies of  $M$  is a  $\mathcal{T}$ -noncosingular module.*

(ii) *If  $R$  is a ring with  $Jac(R) = 0$ , then every free  $R$ -module is  $\mathcal{T}$ -noncosingular.*

Next, we provide a characterization for an arbitrary direct sum of  $\mathcal{T}$ -noncosingular modules to be  $\mathcal{T}$ -noncosingular when each module is fully invariant in the direct sum.

**Proposition 3.2** *Let  $M = \bigoplus_{i \in I} M_i$  be the direct sum of fully invariant submodules  $M_i$ . Then  $M$  is  $\mathcal{T}$ -noncosingular if and only if  $M_i$  is  $\mathcal{T}$ -noncosingular for all  $i \in I$ .*

**Proof** The necessity follows from [20, Proposition 2.3]. Conversely, we need only to show that  $M_i$  is a  $\mathcal{T}$ -noncosingular module relative to  $M_j$  for all  $i, j \in I$  with  $i \neq j$  (see [20, Proposition 2.11]). Let  $f : M_i \rightarrow M_j$  ( $i \neq j$ ) be a homomorphism. Let  $\pi_i : M \rightarrow M_i$  be the projection map and  $\alpha_j : M_j \rightarrow M$  be the inclusion map. Then  $g = \alpha_j f \pi_i \in End_R(M)$  and  $g(M) \subseteq M_j$ . Since  $M_i$  is fully invariant in  $M$ , we have  $g(M_i) \subseteq M_i$ . So,  $g(M_i) \subseteq M_i \cap M_j = 0$ . Hence,  $f = 0$ . Consequently,  $M$  is  $\mathcal{T}$ -noncosingular.  $\square$

**Proposition 3.3** *Let  $M = N \oplus (\bigoplus_{i \in I} S_i)$  such that  $S_i$  ( $i \in I$ ) are simple modules. The following are equivalent:*

- (i)  $M$  is  $\mathcal{T}$ -noncosingular;
- (ii) (a)  $N$  is  $\mathcal{T}$ -noncosingular, and
- (b) For every simple small submodule  $S$  of  $N$ ,  $S \not\cong S_i$  for all  $i \in I$ .

**Proof** (i)  $\Rightarrow$  (ii) By [20, Proposition 2.3],  $N$  and  $N \oplus S_i$  are  $\mathcal{T}$ -noncosingular modules for all  $i \in I$ . Proposition 2.8 now shows that condition (b) holds.

(ii)  $\Rightarrow$  (i) By (b), each  $S_i$  is  $\mathcal{T}$ -noncosingular relative to  $N$ . Applying [20, Proposition 2.11], we obtain that  $M$  is  $\mathcal{T}$ -noncosingular.  $\square$

Let  $R$  be a Dedekind domain that is not a field and  $P$  be a nonzero prime ideal of  $R$ . Let  $R(P^\infty)$  denote the  $P$ -primary component of the torsion  $R$ -module  $K/R$ , where  $K$  is the quotient field of  $R$ . In [20, Example 2.12] it is proven that the  $R$ -module  $R(P^\infty) \oplus R/P$  is not  $\mathcal{T}$ -noncosingular. That is,  $E(R/P) \oplus R/P$



is not  $\mathcal{T}$ -noncosingular. In the next result we provide a necessary and sufficient condition for  $E(S) \oplus S$  to be  $\mathcal{T}$ -noncosingular, where  $S$  is a simple module.

**Proposition 3.4** *Let  $S$  be a simple module. Then the module  $M = E(S) \oplus S$  is  $\mathcal{T}$ -noncosingular if and only if  $S$  is injective.*

**Proof** The sufficiency is obvious. Conversely, suppose that  $S$  is not injective. Then  $S \ll E(S)$ . Thus,  $M$  is not  $\mathcal{T}$ -noncosingular by Proposition 2.8.  $\square$

By combining [20, Proposition 2.13] and Proposition 3.4, we get the following result.

**Corollary 3.5** *The following are equivalent for a ring  $R$ :*

- (i) *Every  $R$ -module is  $\mathcal{T}$ -noncosingular;*
- (ii) *For every simple  $R$ -module  $S$ , the module  $E(S) \oplus S$  is  $\mathcal{T}$ -noncosingular;*
- (iii) *The ring  $R$  is a right  $V$ -ring.*

Next, we present other examples that show that the property of  $\mathcal{T}$ -noncosingularity does not go to direct sums of  $\mathcal{T}$ -noncosingular modules.

**Example 3.6** (1) *Let  $R$  be a right hereditary ring that is not a right  $V$ -ring. Therefore,  $R$  has a simple  $R$ -module  $S$  that is not injective (e.g., we can take a Dedekind domain  $R$  that is not a field and  $S$  any simple  $R$ -module). Then  $E(S)$  and  $S$  are both  $\mathcal{T}$ -noncosingular  $R$ -modules by [20, Example 2.1]. However, the  $R$ -module  $M = E(S) \oplus S$  is not  $\mathcal{T}$ -noncosingular by Proposition 3.4.*

(2) *Let  $R$  be an almost DVR with maximal ideal  $m$  and quotient field  $Q$  (i.e.  $R$  is a commutative local Noetherian domain of Krull dimension 1 and the integral closure  $R'$  of  $R$  in  $Q$  is a finitely generated  $R$ -module and is a discrete valuation ring). Note that  $E(R/m)$  is a simple radical  $R$ -module by [11, Proposition 4]. Therefore,  $E(R/m)$  is a  $\mathcal{T}$ -noncosingular  $R$ -module (see Example 2.2). Further, the  $R$ -module  $R/m$  is  $\mathcal{T}$ -noncosingular. On the other hand, the  $R$ -module  $E(R/m) \oplus R/m$  is not  $\mathcal{T}$ -noncosingular, since otherwise  $R$  will be a  $V$ -ring and  $m = 0$  by Corollary 3.5.*

Recall that a ring  $R$  is called *left (resp. right) perfect* if it is semilocal and every nonzero left  $R$ -module contains a maximal (resp. simple) submodule. A ring  $R$  is said to be *perfect* if it is right and left perfect. A perfect ring is said to be *primary* if the ring  $R/Jac(R)$  is simple Artinian. A perfect ring is called *primary decomposable* if it is isomorphic to a finite product of primary rings. A module  $M$  is called *supplemented* if, for every submodule  $N$  of  $M$ , there exists a submodule  $K \leq M$  such that  $M = N + K$  and  $N \cap K \ll K$  (see, e.g., [3], [13], and [22]). It is easy to check that if  $M$  is a module with zero Jacobson radical, then  $M$  is supplemented if and only if  $M$  is semisimple.

In the next result, we characterize the class of perfect rings  $R$  for which arbitrary direct sums of  $\mathcal{T}$ -noncosingular  $R$ -modules are  $\mathcal{T}$ -noncosingular.

**Theorem 3.7** *The following assertions are equivalent for a perfect ring  $R$ :*

- (i) *Every  $\mathcal{T}$ -noncosingular  $R$ -module is semisimple;*
- (ii) *Every direct sum of  $\mathcal{T}$ -noncosingular  $R$ -modules is  $\mathcal{T}$ -noncosingular;*
- (iii)  *$R$  is primary decomposable.*

**Proof** (i)  $\Rightarrow$  (ii) This is clear.

(ii)  $\Rightarrow$  (iii) Let  $M$  be a module such that  $S = \text{End}_R(M)$  is a division ring. Clearly,  $M$  is an indecomposable  $\mathcal{T}$ -noncosingular module. Since  $R$  is perfect, every  $R$ -module contains a simple submodule. Noting that  $M \oplus S$  is  $\mathcal{T}$ -noncosingular for every simple  $R$ -module  $S$ , we conclude from Proposition 2.9 that  $\text{Rad}(M) = 0$ . Since  $R$  is perfect,  $M$  is supplemented by [13, Theorem 4.41]. Thus,  $M$  is semisimple, but  $M$  is indecomposable. Then  $M$  is simple. So  $R$  is primary decomposable by [10, Theorem 1.2].

(iii)  $\Rightarrow$  (i) By hypothesis,  $R = R_1 \oplus \cdots \oplus R_n$  is a direct sum of perfect primary rings  $R_i$  ( $1 \leq i \leq n$ ). We can write  $1_R = e_1 + e_2 + \cdots + e_n$ , where  $1_R$  is the identity element of  $R$  and for each  $i$ ,  $e_i \in R_i$ . Then for each  $i$ ,  $e_i$  is the identity element of the ring  $R_i$ . Let  $M$  be an  $R$ -module. Then  $M = Me_1 \oplus Me_2 \oplus \cdots \oplus Me_n$ . Also,  $Me_i$  can be regarded as an  $R_i$ -module as well as an  $R$ -module, and its submodules are the same in both cases, because  $xe_i(r_1 + r_2 + \cdots + r_n) = xe_i r_i$ , where  $x \in M$  and  $r_j \in R_j$  for each  $j$ ,  $1 \leq j \leq n$ . Now assume that  $R$  has a  $\mathcal{T}$ -noncosingular module  $M$  that is not semisimple. Without loss of generality we can assume that  $M_1 = Me_1$  is not semisimple. Note that  $\text{End}_{R_1}(M_1) = \text{End}_R(M_1)$ . So  $(M_1)_{R_1}$  is  $\mathcal{T}$ -noncosingular by [20, Proposition 2.3]. Since  $R_1$  is a perfect ring,  $(M_1)_{R_1}$  is supplemented by [13, Theorem 4.41]. Therefore,  $\text{Rad}_{R_1}(M_1) \neq 0$ . Hence,  $\text{Rad}_{R_1}(M_1)$  contains a simple submodule  $S_1$ . Moreover,  $(M_1)_{R_1}$  contains a maximal submodule  $K_1$  since  $R_1$  is perfect. Consider the natural epimorphism  $\pi : M_1 \rightarrow M_1/K_1$  and the inclusion map  $\alpha : S_1 \rightarrow M_1$ . Since  $R_1$  is primary,  $R_1$  has a unique isomorphism class of simple modules. So, there exists an isomorphism  $\theta : M_1/K_1 \rightarrow S_1$ . It follows that  $\varphi = \alpha\theta\pi$  is a nonzero endomorphism of  $M_1$  such that  $\varphi(M_1) = S_1 \ll M_1$ . This shows that  $(M_1)_{R_1}$  is not  $\mathcal{T}$ -noncosingular, a contradiction.  $\square$

**Corollary 3.8** *If  $R$  is a finite product of local perfect rings (e.g.,  $R$  is commutative perfect), then every  $\mathcal{T}$ -noncosingular module is semisimple.*

**Proof** This is a direct consequence of Theorem 3.7.  $\square$

#### 4. The endomorphism ring of a $\mathcal{T}$ -noncosingular module

We conclude this paper by investigating the connection of the  $\mathcal{T}$ -noncosingularity of a module to its endomorphism ring. Recall that a ring  $R$  is called *reduced* if it has no nonzero nilpotent elements.

**Proposition 4.1** *Let  $M$  be a quasi-discrete module with  $S = \text{End}_R(M)$ . If  $M$  is  $\mathcal{T}$ -noncosingular, then  $S = S_1 \times S_2$  such that  $S_1$  is von Neumann regular and  $S_2$  is reduced.*

**Proof** Let  $\nabla(M) = \{\varphi \in S \mid \text{Im } \varphi \ll M\}$ . By [13, Proposition 5.7],  $S/\nabla(M) = S_1 \times S_2$  such that  $S_1$  is von Neumann regular and  $S_2$  is reduced. However, since  $M$  is  $\mathcal{T}$ -noncosingular,  $\nabla(M) = 0$ .  $\square$

**Proposition 4.2** *Let  $P$  be a quasi-projective module with  $S = \text{End}_R(P)$ . The following are equivalent:*

- (i)  $P$  is  $\mathcal{T}$ -noncosingular;
- (ii)  $\text{Jac}(S) = 0$ ;
- (iii)  ${}_S S$  is  $\mathcal{T}$ -noncosingular.

**Proof** This follows from [20, Corollary 2.7] and the fact that  $\varphi \in \text{Jac}(S)$  if and only if  $\text{Im } \varphi \ll P$  (see, e.g., [22, 22.2]).  $\square$

Proposition 4.2 is not true, in general, as the next example shows.

**Example 4.3** Consider the  $\mathbb{Z}$ -module  $M = \mathbb{Z}(p^\infty)$ , where  $p$  is a prime number. It is well known that  $S = \text{End}_{\mathbb{Z}}(M)$  is a local ring that is not a division ring. Then  $\text{Jac}(S) \neq 0$ , while  $M$  is  $\mathcal{T}$ -nonsingular.

**Definition 4.4** A module  $M$  has  $D_2$  property (or is called direct projective) if, for any direct summand  $K$  of  $M$  and submodule  $N$  of  $M$  with  $M/N \cong K$ ,  $N$  is a direct summand of  $M$ .

**Proposition 4.5** Let  $M$  be a direct projective module with  $S = \text{End}_R(M)$ . If  $\text{Jac}(S) = 0$ , then  $M$  is  $\mathcal{T}$ -nonsingular.

**Proof** By [22, 41.19(1)]. □

**Proposition 4.6** Let  $R$  be a commutative ring. If  $R$  is  $\mathcal{T}$ -nonsingular, then  $R$  is  $\mathcal{K}$ -nonsingular.

**Proof** By [15, Proposition 2.7], it suffices to show that  $R$  is nonsingular. Since  $R$  is  $\mathcal{T}$ -nonsingular, we have  $\text{Jac}(R) = 0$  by [20, Corollary 2.7]. So,  $R$  is a semiprime ring. Therefore,  $Z(R) = 0$  by [4, Proposition 1.27(b)]. □

The converse of Proposition 4.6 is not true, in general, as shown below.

**Example 4.7** Let  $R$  be a discrete valuation ring with maximal ideal  $m$ . It is clear that  $Z(R) = 0$ , while  $\text{Jac}(R) = m$ . So  $R$  is  $\mathcal{K}$ -nonsingular, but  $R$  is not  $\mathcal{T}$ -nonsingular by [20, Corollary 2.7] and [15, Proposition 2.7].

Following [22, p. 261], a module  $M$  is called *semi-injective* if for any monomorphism  $f : N \rightarrow M$ , where  $N$  is a factor module of  $M$ , and for any homomorphism  $g : N \rightarrow M$ , there exists  $h : M \rightarrow M$  such that  $hf = g$ . Note that every quasi-injective module is semi-injective.

In the next result, we provide a condition under which Proposition 4.6 holds true for modules.

**Proposition 4.8** Let  $M$  be a coretractable module and let  $S = \text{End}_R(M)$ . If  $M$  is  $\mathcal{T}$ -nonsingular, then  ${}_S S$  is  $\mathcal{K}$ -nonsingular. The converse holds when  $M$  is semi-injective.

**Proof** This follows from [1, Corollary 4.8] and [15, Proposition 2.7]. □

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