

1-1-2014

## Pullback diagram of $H^*$ -algebras

MAHNAZ KHANEHGIR

MARYAM AMYARI

MARZIEH MORADIAN KHIBARY

Follow this and additional works at: <https://journals.tubitak.gov.tr/math>



Part of the [Mathematics Commons](#)

---

### Recommended Citation

KHANEHGIR, MAHNAZ; AMYARI, MARYAM; and KHIBARY, MARZIEH MORADIAN (2014) "Pullback diagram of  $H^*$ -algebras," *Turkish Journal of Mathematics*: Vol. 38: No. 2, Article 13. <https://doi.org/10.3906/mat-1303-16>

Available at: <https://journals.tubitak.gov.tr/math/vol38/iss2/13>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact [academic.publications@tubitak.gov.tr](mailto:academic.publications@tubitak.gov.tr).

## Pullback diagram of $H^*$ -algebras

Mahnaz KHANEHGIR, Maryam AMYARI\*, Marzieh MORADIAN Khibary  
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran

Received: 08.03.2013 • Accepted: 14.06.2013 • Published Online: 27.01.2014 • Printed: 24.02.2014

**Abstract:** In this paper we obtain some properties for the pullback diagram of  $H^*$ -algebras. More precisely, we prove that the commutative diagram of  $H^*$ -algebras and morphisms

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array}$$

is pullback and  $\psi_1$  is an injection if and only if  $\psi_1$  is a surjection,  $\psi_2$  is an injection, and  $\ker \varphi_1 \cap \ker \psi_1 = \{0\}$ .

**Key words:**  $H^*$ -algebra, morphisms, pullback diagram, trace-class

### 1. Introduction and preliminaries

Pedersen [5] introduced the notion of a pullback diagram in the category of  $C^*$ -algebras and investigated some properties of these diagrams. The authors of [2] generalized the construction of a pullback diagram in the category of Hilbert  $C^*$ -modules. Some properties of pullback diagrams are stable under  $H^*$ -algebras [2, 5]. In this paper we use these properties to discover new ones for pullback diagrams of  $H^*$ -algebras.

An  $H^*$ -algebra, introduced by Ambrose [1] in the associative case, is a Banach algebra  $A$ , satisfying the following conditions:

- (i)  $A$  is itself a Hilbert space under an inner product  $\langle \cdot, \cdot \rangle$ ;
- (ii) For each  $a$  in  $A$  there is an element  $a^*$  in  $A$ , the so-called adjoint of  $a$ , such that we have both  $\langle ab, c \rangle = \langle b, a^*c \rangle$  and  $\langle ab, c \rangle = \langle a, cb^* \rangle$  for all  $b, c \in A$ .

**Example 1.1** The Hilbert space  $l^2 = \{(a_n) : a_n \in \mathbb{C}, \sum_{n=1}^{\infty} |a_n|^2 < \infty\}$  is an  $H^*$ -algebra, where for each  $(a_n)$  and  $(b_n)$  in  $l^2$ ,  $(a_n)(b_n) = (a_nb_n)$  and  $(a_n)^* = (\overline{a_n})$ .

**Example 1.2** Any Hilbert space is an  $H^*$ -algebra, where the product of each pair of elements is to be zero. Of course, in this case, the adjoint  $a^*$  of a need not be unique; in fact, every element is an adjoint of every element.

\*Correspondence: amyari@mshdiau.ac.ir

2010 AMS Mathematics Subject Classification: Primary 46H05, Secondary 46C05.

Using the axiom of choice we can assign to each element  $a$  of an  $H^*$ -algebra  $A$  a unique element  $a^* \in A$ . From now on any  $H^*$ -algebra is assumed to be equipped with a choice function, and so we have a well-defined function  $a \mapsto a^*$  on any  $H^*$ -algebra. If  $A$  and  $B$  are  $H^*$ -algebras, then a continuous  $*$ -homomorphism  $\varphi : A \rightarrow B$  is called a morphism.

Recall that  $A_0 = \{a \in A : aA = \{0\}\} = \{a \in A : Aa = \{0\}\}$  is called the annihilator ideal of  $A$ . A proper  $H^*$ -algebra is an  $H^*$ -algebra with zero annihilator ideal. Ambrose [1] proved that an  $H^*$ -algebra is proper if and only if every element has a unique adjoint and showed that every  $H^*$ -algebra  $A$  can be presented as a direct sum of the form  $A_0 \oplus A_0^\perp$ , where  $A_0^\perp = \{a \in A : \langle a, b \rangle = 0, \text{ for all } b \in A_0\}$ .

The trace-class  $\tau(A)$  of  $A$  is defined by the set  $\tau(A) = \{ab : a, b \in A\}$ . The trace functional  $tr$  on  $\tau(A)$  is defined by  $tr(ab) = \langle b, a^* \rangle = \langle a, b^* \rangle = tr(ba)$  for each  $a, b \in A$ , in particular  $tr(aa^*) = \langle a, a \rangle = \|a\|^2$ , for all  $a \in A$ . Since Ambrose [1] up to now, there are many mathematicians who have worked on  $H^*$ -algebras and developed it in several directions; see [4], [3], [6], and the references cited therein. In this paper, we generalize the concept of the pullback diagram in the framework of  $H^*$ -algebras and investigate some conditions under which a diagram of  $H^*$ -algebras is pullback.

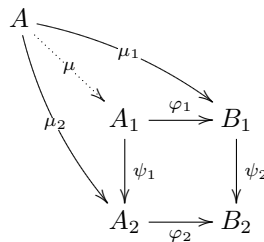
### 2. Pullback constructions in $H^*$ -algebras

In this section we introduce a pullback diagram of  $H^*$ -algebras and investigate some of its properties. For this, we need the following definition.

**Definition 2.1** *A commutative diagram of  $H^*$ -algebras and morphisms*

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array}$$

is pullback if  $\ker(\varphi_1) \cap \ker(\psi_1) = \{0\}$  and for every other pair of morphisms  $\mu_1 : A \rightarrow B_1$  and  $\mu_2 : A \rightarrow A_2$  from an  $H^*$ -algebra  $A$  that satisfies condition  $\psi_2\mu_1 = \varphi_2\mu_2$ , there is a unique morphism  $\mu : A \rightarrow A_1$  such that  $\mu_1 = \varphi_1\mu$  and  $\mu_2 = \psi_1\mu$ .



The following theorem is proven in the framework of  $C^*$ -algebras. It is easy to show this theorem in the category of  $H^*$ -algebras.

**Theorem 2.2** (see [5, proposition 3.1]) A commutative diagram of  $H^*$ -algebras and morphisms

$$\begin{array}{ccc}
 A_1 & \xrightarrow{\varphi_1} & B_1 \\
 \downarrow \psi_1 & & \downarrow \psi_2 \\
 A_2 & \xrightarrow{\varphi_2} & B_2
 \end{array} \tag{1}$$

is pullback if and only if the following conditions hold:

- (i)  $\ker(\varphi_1) \cap \ker(\psi_1) = \{0\}$ ,
- (ii)  $\psi_1(\ker(\varphi_1)) = \ker(\varphi_2)$ ,
- (iii)  $\psi_2^{-1}(\varphi_2(A_2)) = \varphi_1(A_1)$ .

Note that we can replace (i) by  $\ker \varphi_1 \subseteq (\ker \psi_1)^\perp$  in the category of  $H^*$ -algebras. Recall that  $\ker \varphi_1$  and  $\ker \psi_1$  are the ideals of  $A$ . For each  $a \in \ker \varphi_1$  and  $b \in \ker \psi_1$ , we have  $ab^* \in \ker \varphi_1 \ker \psi_1 = \ker \varphi_1 \cap \ker \psi_1 = \{0\}$ . So by [6],  $\langle a, b \rangle = \text{tr}(b^*a) = \text{tr}(ab^*) = 0$ , and then  $\ker \varphi_1$  is orthogonal to  $\ker \psi_1$ .

**Theorem 2.3** Let (1) be a pullback diagram of  $H^*$ -algebras. Then  $\psi_1$  is surjective and  $\psi_1^{-1}(\ker \varphi_2) = \ker \varphi_1 \oplus \ker \psi_1$ . Also,  $\psi_1$  is injective if and only if  $\psi_2$  is injective.

**Proof** Since the diagram is pullback, the conditions of Theorem 2.2 hold.

First we show that  $\psi_1$  is surjective. On the contrary, suppose that there exists  $a_2 \in A_2$  such that  $a_2 \notin \psi_1(A_1)$  for each  $a_1 \in A_1$ . So, by (ii),  $a_2 \notin \ker \varphi_2$ . By (iii), there exists  $a'_1 \in A_1$  such that  $\psi_2\varphi_1(a'_1) = \varphi_2(a_2)$ . By commutativity of the diagram  $\varphi_2(a_2) = \varphi_2(\psi_1(a'_1))$  and  $a_2 - \psi_1(a'_1) \in \ker \varphi_2$ . By (ii), there exists  $a''_1 \in \ker \varphi_1$  such that  $a_2 - \psi_1(a'_1) = \psi_1(a''_1)$ . Thus,  $a_2 = \psi_1(a'_1 + a''_1)$ . This contradicts the initial assumption. Hence,  $\psi_1$  is surjective.

For the second part, let  $a_1 + b_1 \in \ker \varphi_1 \oplus \ker \psi_1$ . By (ii), we have  $\psi_1(a_1 + b_1) = \psi_1(a_1) \in \ker \varphi_2$ . Then  $\ker \varphi_1 \oplus \ker \psi_1 \subseteq \psi_1^{-1}(\ker \varphi_2)$ .

Conversely, let  $a_1 \in \psi_1^{-1}(\ker \varphi_2)$  be arbitrary. Then there exists  $a_2 \in \ker \varphi_2$  such that  $\psi_1(a_1) = a_2$ . We also have  $A_1 = (\ker \psi_1)^\perp \oplus \ker \psi_1$ , since  $\ker \psi_1$  is a closed ideal of  $H^*$ -algebra  $A_1$ . We can write  $a_1 = b_1 + b'_1$ , for some  $b_1 \in (\ker \psi_1)^\perp$  and  $b'_1 \in \ker \psi_1$ . Hence,  $\psi_1(a_1) = \psi_1(b'_1) + \psi_1(b_1) = \psi_1(b_1)$  and by (ii), there exists  $b''_1 \in \ker \varphi_1 \subseteq (\ker \psi_1)^\perp$  such that  $\psi_1(b''_1) = a_2$ . By the above discussion it is enough to show that  $b_1 \in \ker \varphi_1$ , but  $b''_1 - b_1 \in \ker \psi_1$ , since  $\psi_1(b''_1 - b_1) = a_2 - a_2 = 0$ . On the other hand,  $b''_1 - b_1 \in (\ker \psi_1)^\perp$ . Thus,  $b_1 = b''_1 \in \ker \varphi_1$ .

Now suppose that  $\psi_1$  is injective. By (ii),  $\ker \varphi_1 = \psi_1^{-1}(\ker \varphi_2)$ . By (iii), if  $b_1 \in \ker \psi_2 = \psi_2^{-1}(\{0\}) \subseteq \psi_2^{-1}(\varphi_2(A_2)) = \varphi_1(A_1)$ , there exists  $a_1 \in A_1$  such that  $b_1 = \varphi_1(a_1)$ . Thus,  $\varphi_2\psi_1(a_1) = \psi_2\varphi_1(a_1) = \psi_2(b_1) = 0$ . This forces that  $\psi_1(a_1) \in \ker \varphi_2$  or  $a_1 \in \psi_1^{-1}(\ker \varphi_2) = \ker \varphi_1$ . Then  $b_1 = \varphi_1(a_1) = 0$  and  $\psi_2$  is injective.

Suppose that  $\psi_2$  is injective. To prove injectivity of  $\psi_1$ , we show that  $\ker \psi_1 = 0$  or  $\psi_1^{-1}(\ker \varphi_2) = \ker \varphi_1 \oplus 0$ . By (ii),  $\ker \varphi_1 \subseteq \psi_1^{-1}(\ker \varphi_2)$ . It is enough to show that  $\psi_1^{-1}(\ker \varphi_2) \subseteq \ker \varphi_1$ . Let  $a_1 \in \psi_1^{-1}(\ker \varphi_2)$ . Then there exists  $a_2 \in \ker \varphi_2$  such that  $\psi_1(a_1) = a_2$ , but  $0 = \varphi_2(a_2) = \varphi_2\psi_1(a_1) = \psi_2\varphi_1(a_1)$ . Since  $\psi_2$  is injective,  $\varphi_1(a_1) = 0$ . Hence,  $a_1 \in \ker \varphi_1$ . □

In the next theorem we introduce some properties under which a diagram of  $H^*$ -algebras is pullback.

**Theorem 2.4** *Suppose*

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array}$$

*is a commutative diagram of  $H^*$ -algebras and morphisms. Then this diagram is pullback and  $\psi_1$  is an injection if and only if  $\psi_1$  is a surjection,  $\psi_2$  is an injection, and  $\ker \varphi_1 \cap \ker \psi_1 = \{0\}$ .*

**Proof** Suppose that the diagram is pullback and  $\psi_1$  is an injection. Then by Theorem 2.3,  $\psi_1$  is surjective,  $\psi_2$  is injective, and  $\ker \varphi_1 \cap \ker \psi_1 = \{0\}$ .

Conversely, let  $\psi_1$  be a surjection,  $\psi_2$  be an injection, and  $\ker \varphi_1 \cap \ker \psi_1 = \{0\}$ . We are going to show that  $\psi_1(\ker \varphi_1) = \ker \varphi_2$  and  $\psi_2^{-1}(\varphi_2(A_2)) = \varphi_1(A_1)$ .

Let  $a_2 \in \psi_1(\ker \varphi_1)$ . Then there exists  $a_1 \in \ker \varphi_1$  such that  $a_2 = \psi_1(a_1)$ . By commutativity of the diagram and injectivity of  $\psi_2$ , we have  $\varphi_2(a_2) = \varphi_2\psi_1(a_1) = \psi_2\varphi_1(a_1) = 0$ . Hence,  $a_2 \in \ker \varphi_2$  and  $\psi_1(\ker \varphi_1) \subseteq \ker \varphi_2$ .

Conversely, let  $a_2 \in \ker \varphi_2$ . Then  $\psi_1(a_1) = a_2$  for some  $a_1 \in A_1$ , since  $\psi_1$  is onto. But  $\varphi_2(a_2) = \varphi_2\psi_1(a_1) = \psi_2\varphi_1(a_1) = 0$ . Hence,  $\varphi_1(a_1) = 0$ , since  $\psi_2$  is an injection. It implies that  $a_1 \in \ker \varphi_1$  and  $a_2 = \psi_1(a_1) \in \psi_1(\ker \varphi_1)$ . Hence,  $\ker \varphi_2 \subseteq \psi_1(\ker \varphi_1)$ .

For the next part, let  $b_1 \in \varphi_1(A_1)$ . Then  $b_1 = \varphi_1(a_1)$  for some  $a_1 \in A_1$ . Put  $a_2 = \psi_1(a_1)$ . Hence,  $\psi_2^{-1}(\varphi_2(a_2)) = \psi_2^{-1}(\varphi_2(\psi_1(a_1))) = \psi_2^{-1}(\psi_2(\varphi_1(a_1))) = \varphi_1(a_1) = b_1$ . Therefore,  $b_1 \in \psi_2^{-1}(\varphi_2(A_2))$ . Conversely, suppose that  $b_1 \in \psi_2^{-1}(\varphi_2(A_2))$ . Then there exists  $a_2 \in A_2$  such that  $b_1 = \psi_2^{-1}\varphi_2(a_2)$ . By surjectivity of  $\psi_1$  there exists  $a_1 \in A_1$  such that  $\psi_1(a_1) = a_2$ . Thus,  $b_1 = \psi_2^{-1}(\varphi_2(\psi_1(a_1))) = \psi_2^{-1}(\psi_2(\varphi_1(a_1))) = \varphi_1(a_1)$ , since  $\psi_2$  is injective. So  $b_1 \in \varphi_1(A_1)$ . Since  $\psi_2$  is injective,  $\psi_1$  is too by Theorem 2.3.  $\square$

Now we want to show that the pullback diagram of  $H^*$ -algebras have some further properties beyond corresponding to the case of the pullback diagram of  $C^*$ -algebras.

**Theorem 2.5** *Suppose that the diagram (1) is pullback, and then the following statements are true.*

- (i)  $\psi_1((\ker \varphi_1)^\perp) = (\ker \varphi_2)^\perp$ ,
- (ii)  $\psi_1^*((\ker \varphi_2)^\perp) \subseteq (\ker \varphi_1)^\perp$ ,
- (iii)  $\psi_2\varphi_1((\ker \varphi_1)^\perp) = \varphi_2\psi_1((\ker \varphi_1)^\perp)$ ,
- (iv)  $\psi_2^{-1}(\varphi_2((\ker \varphi_2)^\perp)) = \varphi_1((\ker \varphi_1)^\perp)$ ,
- (v)  $\ker \varphi_1 \cap \ker \psi_1 = \{0\}$ .

*Conversely, if the diagram (1) is commutative, it satisfies in the conditions (iv) and (v), and  $\psi_1$  is surjective, then it is a pullback diagram.*

**Proof**

(i) Suppose that the diagram is pullback. Then the conditions of 2.2 hold and by Theorem 2.3  $\psi_1$  is surjective. First we show that  $\psi_1((\ker \varphi_1)^\perp) \subseteq (\psi_1(\ker \varphi_1))^\perp = (\ker \varphi_2)^\perp$ . Let  $\psi_1(a_1)$  and  $\psi_1(a'_1)$  be elements in  $\psi_1((\ker \varphi_1)^\perp)$  and  $\psi_1(\ker \varphi_1)$ , respectively. Then  $\langle \psi_1(a_1), \psi_1(a'_1) \rangle = tr(\psi_1(a_1^*a'_1)) = tr(0) = 0$  (since

$(\ker \varphi_1)^\perp$  is a closed ideal of  $A_1$  and  $a_1^* a'_1 \in (\ker \varphi_1)^\perp (\ker \varphi_1) = (\ker \varphi_1)^\perp \cap (\ker \varphi_1) = \{0\}$ .

Conversely, let  $a''_2 \in (\ker \varphi_2)^\perp$  and  $a'_2 \in (\ker \varphi_2)$  be arbitrary. Then  $a'_2 + a''_2 = a_2 \in A_2$ . By surjectivity of  $\psi_1$ , there exists  $a_1 \in A_1$  such that  $a_2 = \psi_1(a_1)$ . On the other hand,  $A_1 = \ker \varphi_1 \oplus (\ker \varphi_1)^\perp$ , so  $a_1 = a'_1 + a''_1$  for some  $a'_1 \in \ker \varphi_1$  and  $a''_1 \in (\ker \varphi_1)^\perp$ , and by linearity of  $\psi_1$ , we can write  $a_2 = \psi_1(a'_1) + \psi_1(a''_1)$ .

Therefore,  $a'_2 - \psi_1(a'_1) = \psi_1(a''_1) - a''_2$ . The left side of equality is in  $\ker \varphi_2$  and by the above inclusion the right side belongs to  $(\ker \varphi_2)^\perp$ . This implies that  $\psi_1(a''_1) = a''_2$ . Hence,  $(\ker \varphi_2)^\perp \subseteq \psi_1((\ker \varphi_1)^\perp)$ .

(ii) Let  $a_1 \in \psi_1^*((\ker \varphi_2)^\perp)$ . Then there exists  $a'_2 \in (\ker \varphi_2)^\perp$  such that  $a_1 = \psi_1^*(a'_2)$ . For each  $a'_1 \in \ker \varphi_1$ , we have  $\langle a_1, a'_1 \rangle = \langle \psi_1^*(a'_2), a'_1 \rangle = \langle a'_2, \psi_1(a'_1) \rangle = 0$ . Then  $a_1 \in (\ker \varphi_1)^\perp$ .

(Recall that by pullbackness of diagram  $\psi_1(\ker \varphi_1) = \ker \varphi_2$  and  $\psi_1^*$  is the adjoint of  $\psi_1$  in  $B(A_1, A_2)$ .)

Parts (iii) and (v) are obvious and for proving (iv),

$$\begin{aligned} \psi_2^{-1}(\varphi_2(\ker \varphi_2)^\perp) &= \psi_2^{-1}(\varphi_2(\ker \varphi_2 \oplus (\ker \varphi_2)^\perp)) \\ &= \psi_2^{-1}(\varphi_2(A_2)) \\ &= \varphi_1(A_1) \\ &= \varphi_1(\ker \varphi_1 \oplus (\ker \varphi_1)^\perp) \\ &= \varphi_1((\ker \varphi_1)^\perp). \end{aligned}$$

Conversely, by surjectivity of  $\psi_1$  and commutativity of the diagram, we have  $\psi_1(\ker \varphi_1) \subseteq \ker \varphi_2$ . If  $a'_2 \in \ker \varphi_2$  and  $a''_2 \in (\ker \varphi_1)^\perp$ , then  $a'_2 + a''_2 \in \ker \varphi_2 \oplus (\ker \varphi_2)^\perp = A_2 = \psi_1(A_1) = \psi_1(\ker \varphi_1 \oplus (\ker \varphi_1)^\perp)$ . Therefore, there exists  $a'_1 + a''_1 \in \ker \varphi_1 \oplus (\ker \varphi_1)^\perp$ , such that  $a'_2 + a''_2 = \psi_1(a'_1 + a''_1) = \psi_1(a'_1) + \psi_1(a''_1)$ . Then  $a'_2 - \psi_1(a'_1) = \psi_1(a''_1) - a''_2$ . The left side of equality is in  $\ker \varphi_2$  and the right side belongs to  $(\ker \varphi_2)^\perp$ . It implies that  $a'_2 = \psi_1(a'_1)$ . Then  $\ker \varphi_2 \subseteq \psi_1(\ker \varphi_1)$ . Now by (iv) we have,

$$\begin{aligned} \psi_2^{-1}(\varphi_2(A_2)) &= \psi_2^{-1}(\varphi_2(\ker \varphi_2 \oplus (\ker \varphi_2)^\perp)) \\ &= \psi_2^{-1}(\varphi_2(\ker \varphi_2)^\perp) \\ &= \varphi_1((\ker \varphi_1)^\perp) \\ &= \varphi_1(\ker \varphi_1 \oplus (\ker \varphi_1)^\perp) \\ &= \varphi_1(A_1). \end{aligned}$$

By (v) all conditions of 2.2 holds. Then the diagram is pullback. □

We can verify that Theorem 2.2 holds for diagrams of trace classes of  $H^*$ -algebras.

**Theorem 2.6** Consider the following left commutative diagram of  $H^*$ -algebras and morphisms, in which  $\psi_2$  is injective.

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array} \qquad \begin{array}{ccc} \tau(A_1) & \xrightarrow{\varphi'_1} & \tau(B_1) \\ \downarrow \psi'_1 & & \downarrow \psi'_2 \\ \tau(A_2) & \xrightarrow{\varphi'_2} & \tau(B_2) \end{array}$$

If the left diagram is pullback, then the right diagram is pullback. Conversely, if the right commutative diagram is pullback,  $\psi_1(A_1)$  is closed, and  $A_2$  is a proper  $H^*$ -algebra, then the left diagram is pullback.

**Proof** Suppose that the left diagram is pullback. It thus satisfies in the conditions of Theorem 2.2. We prove that the right diagram satisfies in the conditions of Theorem 2.2. Recall that for  $i = 1, 2$   $\varphi'_i = \varphi_i|_{\tau(A_i)}$  and  $\psi'_1 = \psi_1|_{\tau(A_1)}$ ,  $\psi'_2 = \psi_2|_{\tau(A_2)}$ .

(i)  $\ker \varphi'_1 \cap \ker \psi'_1 \subseteq \ker \varphi_1 \cap \ker \psi_1 = \{0\}$ .

(ii) We will show that  $\psi'_1(\ker \varphi'_1) = \ker \varphi'_2$ . Let  $a_2a'_2 \in \ker \varphi'_2 \subseteq \ker \varphi_2$ . By Theorem 2.3,  $\psi_1$  is a surjection, so  $\psi_1(a_1) = a_2$  and  $\psi_1(a'_1) = a'_2$  for some  $a_1$  and  $a'_1$  in  $A_1$ . By commutativity of diagram  $\psi_2\varphi_1(a_1a'_1) = \varphi_2(\psi_1(a_1)\psi_1(a'_1)) = \varphi_2(a_2a'_2) = 0$ . Since  $\psi_2$  is an injection,  $a_1a'_1 \in \ker \varphi'_1$ . Also,  $\psi'_1(a_1a'_1) = \psi_1(a_1a'_1) = a_2a'_2$ . Hence,  $a_2a'_2 \in \psi'_1(\ker \varphi'_1)$ .

Conversely, let  $a_1a'_1 \in \ker \varphi'_1 \subseteq \ker \varphi_1$ . Then  $\varphi_2\psi_1(a_1a'_1) = 0$ , since  $\psi_1(\ker \varphi_1) = \ker \varphi_2$ . This implies that  $\psi'_1(a_1)\psi'_1(a'_1) = \psi'_1(a_1a'_1) \in \ker \varphi'_2$ .

(iii) Finally, we prove that  $\psi_2^{-1}\varphi'_2(\tau(A_2)) = \varphi'_1(\tau(A_1))$ . Let  $a_2a'_2 \in \tau(A_2)$ . By surjectivity of  $\psi_1$  and commutativity of the diagram, we have

$$\begin{aligned} \psi_2^{-1}\varphi'_2(a_2a'_2) &= \psi_2^{-1}\varphi_2(a_2a'_2) \\ &= \psi_2^{-1}\varphi_2(\psi_1(a_1)\psi_1(a'_1)) && \text{(for some } a_1, a'_1 \in A_1) \\ &= \psi_2^{-1}\psi_2\varphi_1(a_1a'_1) && (\psi_2 \text{ is injective}) \\ &= \varphi_1(a_1a'_1) \\ &= \varphi'_1(a_1a'_1). \end{aligned}$$

Conversely, by Theorem 2.4, we will show that  $\psi_1$  is a surjection,  $\psi_2$  is an injection, and  $\ker \varphi_1 \cap \ker \psi_1 = \{0\}$ .

We know that  $\psi'_1$  is surjective. Now if on the contrary  $\psi_1(A_1) \neq A_2$ , then there exists a nonzero element  $e_2$  in  $A_2$  such that  $e_2 \perp \psi_1(A_1)$  (recall that  $A_2$  is a Hilbert space and  $\psi_1(A_1)$  is closed). Since  $A_2$  is proper, there exists  $a_0 \in A_2$  such that  $e_2a_0 \neq 0$ , and by [1, Lemma 2.2 ],  $e_2a_0a_0^*e_2^* \neq 0$ . Since  $\psi'_1$  is surjective, there is  $a_1a'_1 \in \tau(A_1)$  such that  $\psi'_1(a_1a'_1) = e_2a_0a_0^*e_2^*$ . On the other hand, we have,

$$\langle \psi'_1(a_1a'_1)e_2a_0a_0^* , e_2 \rangle = \langle \psi'_1(a_1a'_1) , e_2a_0a_0^*e_2^* \rangle = \| e_2a_0a_0^*e_2^* \|^2 \neq 0. (*)$$

Also,  $e_2a_0a_0^* = \psi'_1(b_1b_2)$  for some  $b_1b_2 \in \tau(A_1)$ , so  $\psi'_1(a_1a'_1)e_2a_0a_0^* = \psi'_1(a_1a'_1b_1b_2) \in \psi_1(A_1)$ . Hence,  $\langle \psi'_1(a_1a'_1)e_2a_0a_0^* , e_2 \rangle = 0$ . That is in contradiction with relation (\*). Then  $\psi_1$  is a surjection.

By assumption  $\psi_2$  is an injection. If  $a_1 \in \ker \varphi_1 \cap \ker \psi_1$ , then  $a_1a_1^* \in \ker \varphi'_1 \cap \ker \psi'_1 = \{0\}$ . So  $\langle a_1, a_1 \rangle = tr(a_1a_1^*) = 0$  forces  $a_1 = 0$ . Hence,  $\ker \varphi_1 \cap \ker \psi_1 = \{0\}$  □

### References

[1] Ambrose,W.: Structure theorems for a special class of Banach algebras. Trans. Amer. Math. Soc. 57, 364–386 (1945).  
 [2] Amyari, M., Chakoshi, M.: Pullback diagram of Hilbert  $C^*$ -modules. Math. Comm. 16, 569–575 (2011).

- [3] Bakic, D., Guljas, B.: Operators on Hilbert  $H^*$ -modules. *J. Operator Theory* 46, 123–137 (2001).
- [4] Balachandran, V.K., Swaminathan, N.: Real  $H^*$ -algebras. *J. Funct. Anal.* 65, 64–75 (1986).
- [5] Pedersen, G.K.: Pullback and pushout constructions in  $C^*$ -algebra theory. *J. Funct. Anal.* 167, 243–344 (1999).
- [6] Saworotnow, P.P., Friedell, J.C.: Trace-class for an arbitrary  $H^*$ -algebra. *Proc. Amer. Math. Soc.* 26, 95–100 (1970).