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YAN DONG

XUEWEI CUI

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## Hölder regularity for weak solutions of diagonal divergence quasilinear degenerate elliptic systems

Yan DONG, Xuwei CUI\*

Department of Applied Mathematics; Key Laboratory of Space Applied Physics  
and Chemistry, Ministry of Education, Northwestern Polytechnical University, Xi'an, Shaanxi

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**Abstract:** In this paper, we establish Hölder regularity for weak solutions of a class of diagonal divergence quasilinear degenerate elliptic systems of Hörmander's vector fields when the coefficients belong to the class of  $VMO_X$  functions with respect to  $x$  and uniformly with respect to  $u$ , and the lower order terms satisfy a natural growth condition.

**Key words:** Quasilinear degenerate elliptic system, Hölder regularity, Hörmander's vector fields,  $VMO_X$  function, natural growth

### 1. Introduction

Let  $X = (X_1, \dots, X_q)$  be a family of real smooth vector fields in a neighborhood  $\tilde{\Omega}$  of some bounded domain  $\Omega \subset \mathbb{R}^n$  ( $q \leq n$ ) with the form

$$X_j = \sum_{k=1}^n b_{jk} \frac{\partial}{\partial x_k}, \quad j = 1, 2, \dots, q,$$

where  $b_{jk} \in C^\infty$ , and satisfying Hörmander's condition (see Section 2) free up to the order  $s$ . The paper is devoted to the study of the partial regularity of weak solutions to the following diagonal divergence quasilinear degenerate elliptic system constructed by Hörmander's vector fields  $\{X_j\} (j = 1, \dots, q)$

$$-X_\alpha^*(a_i^{\alpha\beta}(x, u)X_\beta u^i) = g_i(x, u, Xu) - X_\alpha^* f_i^\alpha(x), \quad (1.1)$$

where  $i = 1, 2, \dots, N$ ;  $\alpha, \beta = 1, 2, \dots, q$ ;  $X_j^* = -X_j + c_j$  ( $c_j = -\sum_{k=1}^n \frac{\partial b_{jk}}{\partial x_k} \in C^\infty(\Omega)$ ) is the transposed vector field of  $X_j$ , and  $\Omega$  is the bounded domain in  $\mathbb{R}^n$ .

In recent years, the regularity of weak solutions for divergence linear and nonlinear elliptic equations or systems in the classic Euclidean spaces ( $q = n, X_i = \frac{\partial}{\partial x_i}$ ) has been widely considered. De Giorgi [3] and Nash [18] investigated Hölder regularity for weak solutions of divergence linear elliptic equations with bounded and measurable coefficients by applying the technique of De Giorgi–Nash–Moser's iteration. Then many authors

\*Correspondence: c88xw@163.com

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obtained analogous results [12,14,16] for divergence nonlinear elliptic equations. In order to treat partial Hölder regularity and get a sharp Hölder exponent of weak solutions of nonlinear systems, authors in [10,11,19] always derived a higher integrability for weak solutions by using the reverse Hölder inequality. In the discussions of partial Hölder regularity of quasilinear equations and systems, one should consider what growth conditions the lower order terms satisfy. The authors of [22] exhibited a sharp Hölder exponent for weak solutions of the elliptic systems with discontinuous coefficients and the natural condition.

Recently, the regularity of divergence linear degenerate elliptic equations or systems of Hörmander’s vector fields has received extensive attention. Di Fazio and Fanciullo in [4] showed gradient estimates in Morrey spaces when  $p = 2$  and Hölder regularity by using an isomorphic relationship for a divergence linear degenerate elliptic system in Lipschitz vector fields. The interior Hölder regularity for weak solutions of linear degenerate elliptic systems was concluded by Dong and Niu in [5] with energy estimates and reverse Hölder inequality on the homogeneous spaces.

For nonlinear degenerate elliptic systems of Hörmander’s vector fields, Xu and Zuily [20] attained interior regularity for weak solutions. Partial regularity for weak solutions was examined by Gao and Niu in [6].

Our purpose in this paper is to establish Hölder regularity for weak solutions of (1.1), where the coefficients of (1.1) satisfy  $VMO_X$  conditions in  $x$  and are continuous with respect to  $u$ , and the lower order terms satisfy a natural growth condition. To do so, we first prove a Caccioppoli inequality (see Section 3) and then consider the homogeneous system with constant coefficients and the inhomogeneous system related to (1.1), respectively. Finally, partial Hölder regularity with a sharp Hölder exponent for weak solutions of (1.1) is deduced by using Morrey’s lemma in our setting.

To state the main result of the paper, let us first specify the definitions of Sobolev spaces, Hölder spaces, and  $BMO_X$  and  $VMO_X$  function spaces induced by Hörmander’s vector fields. Then the assumptions on  $a_i^{\alpha\beta}(z, u)$ ,  $g_i(z, u, Xu)$  and  $f_i^\alpha(z)$  in (1.1) are listed.

**Definition 1.1 (Sobolev spaces)** Let  $1 \leq p \leq +\infty$ ,  $k$  be a positive integer. If  $u \in L^p(\Omega, \mathbb{R}^N)$  satisfies

$$\|u\|_{S_X^{k,p}(\Omega, \mathbb{R}^N)} \equiv \|u\|_{L^p(\Omega, \mathbb{R}^N)} + \sum_{h=1}^k \sum_{j_h=1}^q \|X_{j_1} X_{j_2} \cdots X_{j_h} u\|_{L^p(\Omega, \mathbb{R}^N)} < +\infty,$$

then we say that  $u$  belongs to the Sobolev space  $S_X^{k,p}(\Omega, \mathbb{R}^N)$ .

The space  $S_X^{k,p}(\Omega, \mathbb{R}^N)$  is the closure of  $C_0^\infty(\Omega, \mathbb{R}^N)$  in  $S_X^{k,p}(\Omega, \mathbb{R}^N)$  with respect to the above norm. If  $p = 2$  and  $k = 1$ , then we simply denote  $S_X^{1,2}(\Omega, \mathbb{R}^N)$  by  $S_X^1(\Omega, \mathbb{R}^N)$ .

**Definition 1.2** Let  $\alpha \in (0, 1)$ . The space  $C_X^{0,\alpha}(\bar{\Omega}, \mathbb{R}^N)$  is the set of functions satisfying

$$\|u\|_{C_X^{0,\alpha}(\bar{\Omega}, \mathbb{R}^N)} = \sup_{\Omega} |u| + \sup_{\Omega} \frac{|u(x) - u(y)|}{[d(x, y)]^\alpha} < +\infty.$$

Clearly, it is a Banach space under the norm above.

**Definition 1.3 ( $BMO_X$  and  $VMO_X$  spaces)** Let  $u \in L^1_{loc}(\Omega, \mathbb{R}^N)$ . If

$$\|u\|_{BMO_X(\Omega, \mathbb{R}^N)} = \sup_{x_0 \in \Omega, 0 < R < d_0} \frac{1}{|\Omega \cap B(x_0, R)|} \int_{\Omega \cap B(x_0, R)} |u - u_R| dx < +\infty,$$

then we say that  $u \in BMO_X(\Omega, \mathbb{R}^N)$  (Bounded Mean Oscillation). Moreover, if  $u \in BMO_X(\Omega, \mathbb{R}^N)$  and

$$\eta_R(u) = \sup_{x_0 \in \Omega, 0 < \rho < R} \frac{1}{|\Omega \cap B(x_0, \rho)|} \int_{\Omega \cap B(x_0, \rho)} |u - u_\rho| dx \rightarrow 0, \quad R \rightarrow 0,$$

then we say that  $u \in VMO_X(\Omega, \mathbb{R}^N)$  (Vanishing Mean Oscillation).

The following are the relevant assumptions.

(H1) (ellipticity condition) there exists a positive constant  $\delta$ , such that for a.e.  $x \in \Omega$  and for any  $\xi \in \mathbb{R}^{qN}$ ,

$$a_i^{\alpha\beta}(x, u) \xi_\alpha^i \xi_\beta^i \geq \delta |\xi|^2;$$

(H2) ( $VMO_X \cap L^\infty$ ) assume that  $a_i^{\alpha\beta}(x, u_0) \in VMO_X \cap L^\infty(\Omega)$  uniformly on  $x$  for some fixed  $u_0 \in \mathbb{R}^N$ , that is,

$$\lim_{R \rightarrow 0} \eta_R(a_i^{\alpha\beta}(\cdot, u_0)) = 0,$$

and there exists a constant  $L > 0$ , such that for any  $x \in \Omega$  and  $u \in \mathbb{R}^N$ , it holds  $|a_i^{\alpha\beta}(x, u)| \leq L$ ;

(H3) (continuity) there exist a positive constant  $c$  and a continuous concave function  $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $\omega(0) = 0$ ,  $0 \leq \omega \leq 1$ , such that for any  $x \in \Omega$ ,  $u, v \in \mathbb{R}^N$ ,

$$|a_i^{\alpha\beta}(x, u) - a_i^{\alpha\beta}(x, v)| \leq c\omega(|u - v|^2);$$

(H4) (natural growth condition) let  $u \in S_X^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ ,  $f_i^\alpha \in L^m(\Omega)$  with  $m > \frac{pQ}{2}$  ( $p \in [2, 2 + \frac{2Q}{2+Q}\varepsilon]$ ),  $Q$  is the homogeneous dimension relative to  $\Omega$ , see Section 2),  $g_i(x, u, Xu)$  satisfies

$$|g_i(x, u, Xu)| \leq \mu(M) (|Xu|^2 + g^i(x)),$$

where  $g^i(x) \in L^m(\Omega)$  ( $i = 1, 2, \dots, N$ ),  $M = \sup_{x \in \Omega} |u(x)|$ .

**Definition 1.4** If  $u \in S_X^1(\Omega, \mathbb{R}^N)$  and for any  $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$ ,

$$-\int_{\Omega} a_i^{\alpha\beta}(x, u) X_\beta u^i X_\alpha \varphi^i dx = \int_{\Omega} (g_i(x, u, Xu) \varphi^i - f_i^\alpha X_\alpha \varphi^i) dx,$$

then we say that  $u$  is a weak solution of (1.1).

Our main result in the paper is as follows.

**Theorem 1.5** Let  $u \in S_X^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$  be a weak solution of (1.1). Suppose that assumptions (H1–H4) are satisfied and  $2\mu(M)M < \delta$  for any  $|u| \leq M$ . Then there exists an open subset  $\Omega_0 \subset \Omega$  with  $\dim_H(\Omega \setminus \Omega_0) \leq Q - 2$  (here  $\dim_H(\Omega \setminus \Omega_0)$  is the Hausdorff dimension of the set  $\Omega \setminus \Omega_0$ ) such that

$$u \in C^{0,k}(\Omega_0, \mathbb{R}^N),$$

where  $k = 1 - \frac{pQ}{2m}$ .

This paper is organized as follows. In section 2, we introduce Hörmander’s vector fields and properties. Some definitions and known conclusions are described. Section 3 is devoted to establishing a Caccioppoli inequality and 2 useful results, Lemma 3.1 and 3.2, for the homogeneous system with constant coefficients and the inhomogeneous system related to (1.1), respectively. The proof of Theorem 1.1 is given in Section 4.

**2. Preliminaries and some known results**

For every multi-index  $\beta = (\beta_1, \beta_2, \dots, \beta_d) (1 \leq \beta_i \leq q, i = 1, \dots, d, |\beta| = d)$ , write the commutator of length  $d$  by  $X_\beta = [X_{\beta_d}, [X_{\beta_{d-1}}, \dots [X_{\beta_2}, X_{\beta_1}], \dots]]$ .

**Definition 2.1** *If for every  $x \in \Omega \subset \mathbb{R}^n$ ,  $\{X_\beta(x)\}_{|\beta| \leq s}$  spans  $\mathbb{R}^n$ , then we say that the system  $X = (X_1, \dots, X_q)$  satisfies the Hörmander’s condition of step  $s$ .*

*Following [20], we can assume that Hörmander’s vector fields  $X_1, \dots, X_q$  are free up to the order  $s$ . For every multi-index  $I = (i_1, i_2, \dots, i_k)$ , the length of  $I$  is denoted by  $|I| = k$ . If  $i_k \leq q$ , then we set*

$$X_I = X_{i_1} X_{i_2} \dots X_{i_k}.$$

**Definition 2.2 (Carnot–Carathéodory distance)** *Let  $\Omega$  be a bounded set in  $\mathbb{R}^n$ . An absolutely continuous curve  $\gamma : [0, T] \rightarrow \Omega$  is called a sub-unit curve with respect to the system  $X = (X_1, \dots, X_q)$ , if  $\gamma'(t)$  exists for a.e.  $t$  and satisfies*

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^q \langle X_j(\gamma(t)), \xi \rangle^2, \quad \text{a.e. } t \in [0, T].$$

*We denote the length of this curve by  $l_S(\gamma) = T$ . Given any  $x, y \in \Omega$ , let  $\Phi(x, y)$  be the collection of all sub-unit curves connecting  $x$  and  $y$ . The Carnot–Carathéodory distance induced by the system  $X$  is defined by*

$$d_X(x, y) = \inf\{l_S(\gamma) : \gamma \in \Phi(x, y)\}.$$

*Let us denote by*

$$B(x_0, R) = \{x \in \Omega : d(x_0, x) < R\}$$

*the metric ball centered at  $x_0$  of radius  $R$ . When one does not consider the center of a ball, we simply write  $B_R$  instead of  $B(x, R)$ .*

*It is well known that the doubling property (see [17]) for metric balls holds true, i.e., there exist positive constants  $C_D$  and  $R_D$ , such that for any  $x_0 \in \Omega$ ,  $0 < 2R < R_D$ ,  $B_{2R} \subset \Omega$ , one has*

$$|B(x_0, 2R)| \leq C_D |B(x_0, R)|.$$

*Furthermore, it follows that for any  $R \leq R_D$  and  $t \in (0, 1)$ ,*

$$|B_{tR}| \geq C_D^{-1} t^Q |B_R|.$$

*The number  $Q = \log_2 C_D$  is called a locally homogeneous dimension relative to  $\Omega$ . As in [20], we assume that there exist 2 positive constants  $C_1$  and  $C_2$ , such that*

$$C_1 R^Q \leq |B_R| \leq C_2 R^Q. \tag{2.1}$$

**Theorem 2.3 (Sobolev–Poincaré inequality, see [7], [15])** For the open set  $\Omega$ , there exist positive constants  $R_0$  and  $c$ , such that for any  $0 < R \leq R_0$ ,  $B_R \subset \Omega$  and  $u \in C^\infty(\overline{B_R})$ , it holds

$$\left( \frac{1}{|B_R|} \int_{B_R} |u - u_R|^{p'} dx \right)^{\frac{1}{p'}} \leq cR \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{1}{p}},$$

where  $1 < p < Q$ ,  $1 \leq p' \leq \frac{pQ}{Q-p}$ ,  $u_R = \frac{1}{|B_R|} \int_{B_R} u(x)dx$ ,  $R_0$  and  $c$  depend on  $\Omega$ .

In particular, if  $p = p'$ , then

$$\int_{B_R} |u - u_R|^p dx \leq cR^p \int_{B_R} |Xu|^p dx \leq c \int_{B_R} |Xu|^p dx. \tag{2.2}$$

When  $u \in C_0^\infty(\overline{B_R})$ , one has

$$\int_{B_R} |u|^p dx \leq cR^p \int_{B_R} |Xu|^p dx \leq c \int_{B_R} |Xu|^p dx. \tag{2.3}$$

**Theorem 2.4 (reverse Hölder inequality, see [8], [21])** Let  $\tilde{g}, \tilde{f} \geq 0$  satisfy

$$\tilde{g} \in L^{\tilde{q}}(\Omega) (\tilde{q} > 1), \tilde{f} \in L^{q'}(\Omega) (q' > \tilde{q}).$$

Fix a ball  $B_{R_0} = B(x_0, R_0)$  and assume that for any  $x \in B_{R_0}$  and  $R < \frac{1}{2} \text{dist}(x, \partial B_{R_0})$ , there exist  $b > 0$  and  $\theta \in [0, 1)$ , such that

$$\frac{1}{|B_R|} \int_{B_R} \tilde{g}^{\tilde{q}} dx \leq b \left[ \left( \frac{1}{|B_{4R/3}|} \int_{B_{4R/3}} \tilde{g} dx \right)^{\tilde{q}} + \frac{1}{|B_{4R/3}|} \int_{B_{4R/3}} \tilde{f}^{\tilde{q}} dx \right] + \theta \frac{1}{|B_{4R/3}|} \int_{B_{4R/3}} \tilde{g}^{\tilde{q}} dx.$$

Then there exist  $\varepsilon > 0$  and  $c > 0$  such that for any  $r \in [\tilde{q}, \tilde{q} + \varepsilon)$ , it yields  $\tilde{g} \in L^r_{loc}(B_{R_0})$ . Moreover, for any  $B_{2R} \subset \subset \Omega$ , we have

$$\left( \frac{1}{|B_R|} \int_{B_R} \tilde{g}^r dx \right)^{\frac{1}{r}} \leq c \left[ \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} \tilde{g}^{\tilde{q}} dx \right)^{\frac{1}{\tilde{q}}} + \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} \tilde{f}^{\tilde{q}} dx \right)^{\frac{1}{\tilde{q}}} \right],$$

where  $c$  and  $\varepsilon$  are positive constants depending only on  $b, \theta, \tilde{q}$ , and  $q'$ .

**Lemma 2.5 (see [1, 9, 13, 16])** Let  $H(\rho)$  be a nonnegative almost increasing function in  $[0, R_0]$ ,  $R_0 = \text{dist}(x_0, \partial\Omega)$ , satisfying that for any  $0 < \rho < R \leq R_0$ ,

$$H(\rho) \leq A \left( \left( \frac{\rho}{R} \right)^\alpha + \varepsilon \right) H(R) + BR^\beta,$$

where  $A, B, \alpha, \beta$  are positive constants and  $\alpha > \beta$ . Then there exist positive constants  $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta)$  and  $c = c(A, \alpha, \beta)$ , such that if  $\varepsilon < \varepsilon_0$ , then

$$H(\rho) \leq c \left( \left( \frac{\rho}{R} \right)^\beta H(R) + B\rho^\beta \right), \text{ for any } 0 < \rho < R < R_0.$$

The following Morrey-type lemma is from [23].

**Lemma 2.6** *Suppose that  $u \in S_X^1(B_R, \mathbb{R}^N)$  satisfies the following inequality*

$$\int_{B_R} |Xu|^2 dx \leq M_0^2 R^{Q-2+2k},$$

for any  $B_R \subset \Omega$ , where  $M_0 > 0$  and  $k \in (0, 1)$ . Then we have  $u \in C_{loc}^k(\Omega, \mathbb{R}^N)$ , and for any  $\Omega' \subset\subset \Omega$ , there holds

$$\sup_{\Omega'} |u| + \sup_{x_1, x_2 \in \Omega', x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{d_X(x_1, x_2)^k} \leq c \left( M_0 + \|u\|_{L^2(\Omega)} \right),$$

where  $c = c(Q, k, \Omega', \Omega) > 0$ .

**Definition 2.7** *Let  $E \subseteq \mathbb{R}^n$ ,  $0 \leq \alpha < +\infty$ ,  $0 < \delta \leq +\infty$ ,  $\{F_j\}$  be a system of open sets of  $\mathbb{R}^n$ . We say that*

$$H_\alpha(E) = \lim_{\delta \rightarrow 0} H_\alpha^\delta(E) = \sup_{\delta > 0} H_\alpha^\delta(E)$$

is the  $\alpha$ -Hausdorff dimension of  $E$ , where  $H_\alpha^\delta(E) = \omega_\alpha 2^{-\alpha} \inf \left\{ \sum_j (\text{diam } F_j)^\alpha \mid \bigcup_j F_j \supseteq E, \text{diam } F_j < \delta \right\}$ ,  $\omega_\alpha = \Gamma^\alpha(\frac{1}{2}) / \Gamma(\frac{\alpha}{2} + 1)$ .

**Lemma 2.8** *Let  $u \in L_{loc}^1(\Omega, \mathbb{R}^N)$ . If for  $0 \leq \alpha < Q$ , it has*

$$E_\alpha = \left\{ x \in \Omega \mid \limsup_{\rho \rightarrow 0} \rho^{-\alpha} \int_{B_\rho(x)} |u(x)| dx > 0 \right\},$$

then  $H_\alpha(E_\alpha) = 0$ .

We omit its proof, since it is similar to [2]. The following result is classic.

**Lemma 2.9 (Jensen inequality)** *Suppose that  $f : U \rightarrow \mathbb{R}$  is an integrable function for any bounded open subset  $U \subset \mathbb{R}^n$ . If  $\omega : \mathbb{R} \rightarrow \mathbb{R}$  is a concave function, then*

$$\frac{1}{|U|} \int_U \omega(f) dx \leq \omega \left( \frac{1}{|U|} \int_U f dx \right).$$

**Remark 2.10** *The constant  $c$  takes different values in different places. To simplify the notations, we denote*

$$\sqrt{\sum_{i=1}^q |X_i u|^2} \text{ and } B(x, R) \text{ by } |Xu| \text{ and } B_R, \text{ respectively. We also write } u_R = \frac{1}{|B_R|} \int_{B_R} u(x) dx, \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R = \frac{1}{|B_R|} \int_{B_R} a_i^{\alpha\beta}(x, u_R) dx.$$

**3. Auxiliary theorems**

**Theorem 3.1 (Caccioppoli inequality)** *Let  $u \in S_X^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$  be a weak solution of (1.1). Then for any  $\rho, R: 0 < \rho < R, B_R \subset\subset \Omega$ , we have*

$$\int_{B_\rho} |Xu|^2 dx \leq \frac{c}{(R-\rho)^2} \int_{B_R} |u - u_R|^2 dx + c \int_{B_R} \left( \sum_i |g^i|^2 + |f|^2 \right) dx,$$

where the positive constant  $c$  depends on  $\delta, Q, L, M, \mu(M)$ .

**Proof** Let  $\eta \in C_0^\infty(B_R)$  be a cutoff function satisfying

$$0 \leq \eta \leq 1 \text{ in } B_R, \eta = 1 \text{ on } B_\rho, |X\eta| \leq \frac{c}{R-\rho}.$$

Multiplying both sides of (1.1) by  $(u^i - u_R)\eta^2$  and integrating on  $B_R$  leads to

$$- \int_{B_R} a_i^{\alpha\beta}(x, u) X_\beta u^i X_\alpha ((u^i - u_R)\eta^2) dx = \int_{B_R} [g_i(u^i - u_R)\eta^2 - f_i^\alpha X_\alpha ((u^i - u_R)\eta^2)] dx,$$

that is

$$\begin{aligned} & - \int_{B_R} \left( 2a_i^{\alpha\beta}(x, u)\eta(u^i - u_R)X_\beta u^i X_\alpha \eta + a_i^{\alpha\beta}(x, u)\eta^2 X_\beta u^i X_\alpha u^i \right) dx \\ & = \int_{B_R} (g_i(u^i - u_R)\eta^2 - 2\eta(u^i - u_R)f_i^\alpha X_\alpha \eta - \eta^2 f_i^\alpha X_\alpha u^i) dx. \end{aligned}$$

Using (H1), (H4), and Young's inequality, one has

$$\begin{aligned} & \delta \int_{B_R} |Xu|^2 \eta^2 dx \leq \int_{B_R} a_i^{\alpha\beta}(x, u)\eta^2 X_\beta u^i X_\alpha u^i dx \\ & \leq 2L \int_{B_R} |(u^i - u_R)X_\alpha \eta| |\eta X_\beta u^i| dx + \int_{B_R} |g_i| |(u^i - u_R)| \eta^2 dx \\ & \quad + 2 \int_{B_R} |\eta f_i^\alpha| |(u^i - u_R)X_\alpha \eta| dx + \int_{B_R} |\eta f_i^\alpha| |\eta X_\alpha u^i| dx \\ & \leq c_\varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + \varepsilon \int_{B_R} |Xu|^2 \eta^2 dx \\ & \quad + \int_{B_R} \mu(M) \left( |Xu|^2 + \sum_i |g^i| \right) |u - u_R| \eta^2 dx + c_\varepsilon \int_{B_R} |f|^2 \eta^2 dx \\ & \quad + \varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + c_\varepsilon \int_{B_R} |f|^2 \eta^2 dx + \varepsilon \int_{B_R} |Xu|^2 \eta^2 dx \\ & \leq c_\varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + 2\varepsilon \int_{B_R} |Xu|^2 \eta^2 dx + c_\varepsilon \int_{B_R} |f|^2 \eta^2 dx \\ & \quad + \mu(M) \int_{B_R} |Xu|^2 |u - u_R| \eta^2 dx + \mu(M) \int_{B_R} \sum_i |g^i| |u - u_R| \eta^2 dx \\ & \leq c_\varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + 2\varepsilon \int_{B_R} |Xu|^2 \eta^2 dx + c_\varepsilon \int_{B_R} |f|^2 \eta^2 dx \\ & \quad + 2\mu(M)M \int_{B_R} |Xu|^2 \eta^2 dx + \mu(M)c_\varepsilon \int_{B_R} \sum_i |g^i|^2 \eta^2 dx + \mu(M)\varepsilon \int_{B_R} |u - u_R|^2 \eta^2 dx, \end{aligned}$$

namely,

$$\begin{aligned} & (\delta - 2\mu(M)M - 2\varepsilon) \int_{B_R} |Xu|^2 \eta^2 dx \\ & \leq c_\varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + \mu(M)\varepsilon \int_{B_R} |u - u_R|^2 \eta^2 dx \\ & \quad + \mu(M)c_\varepsilon \int_{B_R} \sum_i |g^i|^2 \eta^2 dx + c_\varepsilon \int_{B_R} |f|^2 \eta^2 dx. \end{aligned}$$

Because of  $2\mu(M)M < \delta$ , we can choose  $\varepsilon$  small enough such that  $\delta - 2\mu(M)M - 2\varepsilon > 0$ , and obtain

$$\int_{B_R} |Xu|^2 \eta^2 dx \leq c \int_{B_R} |u - u_R|^2 (|X\eta|^2 + \eta^2) dx + c \int_{B_R} \left( \sum_i |g^i|^2 + |f|^2 \right) \eta^2 dx.$$



By properties of  $\eta$ , this conclusion is proved. □

By Theorem 3.1 and Theorem 2.2, we easily know that the following result is true by using a similar proof as in [5, Theorem 3.2].

**Theorem 3.2** *Let  $u \in S^1_X(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$  be a weak solution of (1.1). Then there exist constants  $\varepsilon > 0$  and  $c > 0$  such that for any  $p \in [2, 2 + \frac{2Q}{2+Q}\varepsilon)$ , it follows  $u \in S^{1,p}_{X,loc}(\Omega, \mathbb{R}^N)$  and*

$$\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \leq c \left[ \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |Xu|^2 dx \right)^{\frac{p}{2}} + \frac{1}{|B_{2R}|} \int_{B_{2R}} \left( \sum_i |g^i|^p + |f|^p \right) dx \right]$$

for any  $B_{2R} \subset\subset \Omega$ .

In order to prove Hölder regularity for weak solutions of (1.1), we need to consider the homogeneous system and the inhomogeneous system. Let  $v, w = u - v$  be the weak solution of the following systems

$$\begin{cases} -X_\alpha^* \left( \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R X_\beta v^i \right) = 0, & \text{in } B_R, \\ v = u, & \text{on } \partial_p B_R, \end{cases} \tag{3.1}$$

and

$$\begin{cases} -X_\alpha^* \left( \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R X_\beta w^i \right) = -X_\alpha^* \left( \left( \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u) \right) X_\beta u^i \right) + g_i - X_\alpha^* f_i^\alpha, & \text{in } B_R, \\ w = 0, & \text{on } \partial_p B_R, \end{cases} \tag{3.2}$$

respectively. The system (3.1) allows the following result; see [5] for a detailed proof.

**Lemma 3.3** *Let  $v \in S^1_X(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$  be a weak solution of (3.1). Then for any  $0 \leq \rho \leq R$ ,  $B_R(x_0) \subset \Omega$ , one has*

$$\int_{B_\rho} |Xv|^2 dx \leq c \left( \frac{\rho}{R} \right)^Q \int_{B_R} |Xv|^2 dx.$$

For the system (3.2), we have

**Lemma 3.4** *Let  $w \in S^1_{X,0}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$  be the weak solution of (3.2). Then for any  $0 \leq \rho \leq R$ ,  $B_R(x_0) \subset \Omega$ , it holds*

$$\int_{B_R} |Xw|^2 dx \leq c \int_{B_R} |Xu|^2 dx, \tag{3.3}$$

where the positive constant  $c$  depends on  $L$  and  $\delta$ .

**Proof** Multiplying both sides of (3.1) by  $u^i - w^i$  and integrating on  $B_R$ ,

$$\int_{B_R} \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R X_\beta v^i X_\alpha (u^i - w^i) dx = 0.$$

It yields by inserting  $v = u - w$  into the above equation that

$$\int_{B_R} \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R X_\beta (u^i - w^i) X_\alpha (u^i - w^i) dx = 0,$$

that is,

$$\begin{aligned} & \int_{B_R} \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R X_\alpha w^i X_\beta w^i dx \\ &= \int_{B_R} \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R X_\alpha w^i X_\beta u^i dx + \int_{B_R} \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R X_\alpha u^i X_\beta w^i dx \\ & \quad - \int_{B_R} \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R X_\alpha u^i X_\beta u^i dx. \end{aligned}$$

By (H2) and Young’s inequality,

$$\delta \int_{B_R} |Xw|^2 dx \leq c_\varepsilon \int_{B_R} |Xu|^2 dx + \varepsilon \int_{B_R} |Xw|^2 dx.$$

Taking  $\varepsilon$  small enough, the proof is ended. □

#### 4. Proof of the main result

**Proof of Theorem 1.1** Multiplying both sides of the system in (3.2) by  $w^i$  and integrating on  $B_R$ , it implies

$$\begin{aligned} & \int_{B_R} \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R X_\beta w^i X_\alpha w^i dx \\ &= \int_{B_R} \left( \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u) \right) X_\beta u^i X_\alpha w^i dx + \int_{B_R} f_i^\alpha X_\alpha w^i dx - \int_{B_R} g_i w^i dx \\ &= \int_{B_R} \left( \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u_R) \right) X_\beta u^i X_\alpha w^i dx \\ & \quad + \int_{B_R} \left( a_i^{\alpha\beta}(x, u_R) - a_i^{\alpha\beta}(x, u) \right) X_\beta u^i X_\alpha w^i dx + \int_{B_R} (f_i^\alpha X_\alpha w^i - g_i w^i) dx \\ & \triangleq I + II + III. \end{aligned} \tag{4.1}$$

By (H1), Young’s inequality, and Hölder’s inequality, we have

$$\begin{aligned} I &= \int_{B_R} \left( \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u_R) \right) X_\beta u^i X_\alpha w^i dx \\ &\leq c_\varepsilon \int_{B_R} \left| \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u_R) \right|^2 |Xu|^2 dx + \varepsilon \int_{B_R} |Xw|^2 dx \\ &\leq c_\varepsilon \left( \int_{B_R} \left| \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u_R) \right|^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}} \left( \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \\ & \quad + \varepsilon \int_{B_R} |Xw|^2 dx. \end{aligned} \tag{4.2}$$

Due to (H2),

$$\begin{aligned}
 & \left( \int_{B_R} \left| \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u_R) \right|^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}} \\
 &= \left( \int_{B_R} \left| \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u_R) \right|^{\frac{p+2}{p-2}} \left| \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u_R) \right| dx \right)^{\frac{p-2}{p}} \\
 &\leq (2L)^{\frac{p+2}{p}} \left( \int_{B_R} \left| \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u_R) \right|^{\frac{p-2}{p}} dx \right)^{\frac{p-2}{p}} \\
 &= (2L)^{\frac{p+2}{p}} |B_R|^{\frac{p-2}{p}} \left( \frac{1}{|B_R|} \int_{B_R} \left| \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R - a_i^{\alpha\beta}(x, u_R) \right| dx \right)^{\frac{p-2}{p}} \\
 &\leq c |B_R|^{\frac{p-2}{p}} \left( \eta_R(a_i^{\alpha\beta}(x, u_R)) \right)^{\frac{p-2}{p}}.
 \end{aligned}$$

Putting the above into (4.2), it follows that

$$\begin{aligned}
 I &\leq c_\varepsilon |B_R|^{\frac{p-2}{p}} \left( \eta_R(a_i^{\alpha\beta}(x, u_R)) \right)^{\frac{p-2}{p}} \left( \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} + \varepsilon \int_{B_R} |Xw|^2 dx \\
 &= c_\varepsilon |B_R| \left( \eta_R(a_i^{\alpha\beta}(x, u_R)) \right)^{\frac{p-2}{p}} \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} + \varepsilon \int_{B_R} |Xw|^2 dx. \tag{4.3}
 \end{aligned}$$

By (H3), Young’s inequality, and Hölder’s inequality,

$$\begin{aligned}
 II &= \int_{B_R} \left( a_i^{\alpha\beta}(x, u_R) - a_i^{\alpha\beta}(x, u) \right) X_\beta u^i X_\alpha w^i dx \\
 &\leq c_\varepsilon \int_{B_R} \left| a_i^{\alpha\beta}(x, u_R) - a_i^{\alpha\beta}(x, u) \right|^2 |Xu|^2 dx + \varepsilon \int_{B_R} |Xw|^2 dx \\
 &\leq c_\varepsilon \int_{B_R} \omega^2 \left( |u - u_R|^2 \right) |Xu|^2 dx + \varepsilon \int_{B_R} |Xw|^2 dx \\
 &\leq c_\varepsilon \left( \int_{B_R} \left( \omega \left( |u - u_R|^2 \right) \right)^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}} \left( \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} + \varepsilon \int_{B_R} |Xw|^2 dx \\
 &= c_\varepsilon \left( \int_{B_R} \left( \omega \left( |u - u_R|^2 \right) \right)^{\frac{p+2}{p-2}} \omega \left( |u - u_R|^2 \right) dx \right)^{\frac{p-2}{p}} \left( \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} + \varepsilon \int_{B_R} |Xw|^2 dx \\
 &\leq c |B_R| \left( \frac{1}{|B_R|} \int_{B_R} \omega \left( |u - u_R|^2 \right) dx \right)^{\frac{p-2}{p}} \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} + \varepsilon \int_{B_R} |Xw|^2 dx.
 \end{aligned}$$

Using Lemma 2.4 leads to

$$\frac{1}{|B_R|} \int_{B_R} \omega \left( |u - u_R|^2 \right) dx \leq \omega \left( \frac{1}{|B_R|} \int_{B_R} |u - u_R|^2 dx \right).$$

It follows that

$$II \leq c |B_R| \omega^{\frac{p-2}{p}} \left( \frac{1}{|B_R|} \int_{B_R} |u - u_R|^2 dx \right) \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} + \varepsilon \int_{B_R} |Xw|^2 dx. \tag{4.4}$$

By (H4) and (2.3),

$$\begin{aligned}
 III &= \int_{B_R} (f_i^\alpha X_\alpha w^i - g_i w^i) dx \\
 &\leq \int_{B_R} (|f_i^\alpha| |X_\alpha w^i| + |g_i| |w^i|) dx \\
 &\leq \int_{B_R} |f_i^\alpha| |X_\alpha w^i| dx + \mu(M) \int_{B_R} (|Xu|^2 + g^i) |w^i| dx \\
 &\leq c_\varepsilon \int_{B_R} |f|^2 dx + \varepsilon \int_{B_R} |Xw|^2 dx + \mu(M) \int_{B_R} |Xu|^2 |w| dx + \mu(M) \int_{B_R} \sum_i g^i |w| dx \\
 &\leq c_\varepsilon \int_{B_R} |f|^2 dx + \varepsilon \int_{B_R} |Xw|^2 dx + \mu(M) \int_{B_R} |Xu|^2 |w| dx \\
 &\quad + \mu(M)c_\varepsilon \int_{B_R} \sum_i |g^i|^2 dx + \mu(M)\varepsilon \int_{B_R} |w|^2 dx \\
 &\leq \varepsilon \int_{B_R} |Xw|^2 dx + \mu(M) \int_{B_R} |Xu|^2 |w| dx + c \int_{B_R} \left( \sum_i |g^i|^2 + |f|^2 \right) dx + \mu(M)\varepsilon R^2 \int_{B_R} |Xw|^2 dx \\
 &\leq 2\varepsilon \int_{B_R} |Xw|^2 dx + \mu(M) \int_{B_R} |Xu|^2 |w| dx + c \int_{B_R} \left( \sum_i |g^i|^2 + |f|^2 \right) dx. \tag{4.5}
 \end{aligned}$$

In the light of (2.3) and (3.3),

$$\begin{aligned}
 \int_{B_R} |Xu|^2 |w| dx &\leq \left( \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \left( \int_{B_R} |w|^{\frac{p}{p-2}} dx \right)^{\frac{p-2}{p}} \\
 &= |B_R| \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \left( \frac{1}{|B_R|} \int_{B_R} |w| |w|^{\frac{2}{p-2}} dx \right)^{\frac{p-2}{p}} \\
 &\leq c |B_R| \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \left( \frac{1}{|B_R|} \int_{B_R} |w| dx \right)^{\frac{p-2}{p}} \\
 &\leq c |B_R| \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \left( \frac{1}{|B_R|} \int_{B_R} |w|^2 dx \right)^{\frac{p-2}{2p}} \\
 &\leq c |B_R| \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \left( \frac{R^2}{|B_R|} \int_{B_R} |Xw|^2 dx \right)^{\frac{p-2}{2p}} \\
 &\leq c |B_R| \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \left( \frac{R^2}{|B_R|} \int_{B_R} |Xu|^2 dx \right)^{\frac{p-2}{2p}}.
 \end{aligned}$$

Returning to (4.5), we have

$$\begin{aligned}
 III &\leq c |B_R| \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \left( \frac{R^2}{|B_R|} \int_{B_R} |Xu|^2 dx \right)^{\frac{p-2}{2p}} \\
 &\quad + 2\varepsilon \int_{B_R} |Xw|^2 dx + c \int_{B_R} \left( \sum_i |g^i|^2 + |f|^2 \right) dx.
 \end{aligned}$$

Summing up these estimates and using (2.1), we get

$$\begin{aligned}
 & \delta \int_{B_R} |Xw|^2 dx \leq \int_{B_R} \left( a_i^{\alpha\beta}(\cdot, u_R) \right)_R X_\beta w^i X_\alpha w^i dx \\
 & \leq c_\varepsilon |B_R| \left( \eta_R(a_i^{\alpha\beta}(x, u_R)) \right)^{\frac{p-2}{p}} \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} + 4\varepsilon \int_{B_R} |Xw|^2 dx \\
 & \quad + c |B_R| \omega^{\frac{p-2}{p}} \left( \frac{1}{|B_R|} \int_{B_R} |u - u_R|^2 dx \right) \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \\
 & \quad + c |B_R| \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \left( \frac{R^2}{|B_R|} \int_{B_R} |Xu|^2 dx \right)^{\frac{p-2}{2p}} + c \int_{B_R} \left( \sum_i |g^i|^2 + |f|^2 \right) dx \\
 & \leq c_\varepsilon |B_R| \left( \eta_R(a_i^{\alpha\beta}(x, u_R)) \right)^{\frac{p-2}{p}} \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} + 4\varepsilon \int_{B_R} |Xw|^2 dx \\
 & \quad + c |B_R| \omega^{\frac{p-2}{p}} \left( R^{2-Q} \int_{B_R} |Xu|^2 dx \right) \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \\
 & \quad + c |B_R| \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} \left( R^{2-Q} \int_{B_R} |Xu|^2 dx \right)^{\frac{p-2}{2p}} + c \int_{B_R} \left( \sum_i |g^i|^2 + |f|^2 \right) dx \\
 & \leq c |B_R| \vartheta(x, R) \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} + 4\varepsilon \int_{B_R} |Xw|^2 dx \\
 & \quad + c \int_{B_R} \left( \sum_i |g^i|^2 + |f|^2 \right) dx, \tag{4.6}
 \end{aligned}$$

where  $\vartheta(x, R) = \left( \eta_R(a_i^{\alpha\beta}(x, u_R)) \right)^{\frac{p-2}{p}} + \omega^{\frac{p-2}{p}} \left( R^{2-Q} \int_{B_R} |Xu|^2 dx \right) + \left( R^{2-Q} \int_{B_R} |Xu|^2 dx \right)^{\frac{p-2}{2p}}$ .

Denote  $E(x_0, R) = R^{2-Q} \int_{B_R} |Xu|^2 dx$  and set

$$\Omega_0 = \left\{ x_0 \in \Omega \mid \liminf_{\rho \rightarrow 0} \rho^{2-Q} \int_{B_\rho(x_0)} |Xu|^2 dx = 0 \right\}.$$

For any  $x_0 \in \Omega$  and  $\varepsilon_0 > 0$ , there exists  $R < R_0 \leq \text{dist}(x_0, \partial\Omega)$ , such that  $E(R) < \varepsilon_0$ . By properties of  $VMO_X$  functions, we know that for any  $\sigma > 0$ , if  $x_0 \in \Omega$ , then there exists a positive constant  $R < R_0$ , such that  $\vartheta(x, R) < \sigma$ . Taking small enough  $\varepsilon$  such that  $\delta - 4\varepsilon > 0$ , it infers by (4.6) that

$$\int_{B_R} |Xw|^2 dx \leq c |B_R| \sigma \left( \frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{2}{p}} + c \int_{B_R} \left( \sum_i |g^i|^2 + |f|^2 \right) dx. \tag{4.7}$$

From Lemma 3.1, we obtain that for any  $\rho \leq \frac{1}{2}R$ ,

$$\begin{aligned}
 & \int_{B_{2\rho}} |Xu|^2 dx \leq c \int_{B_{2\rho}} |Xv|^2 dx + c \int_{B_{2\rho}} |Xw|^2 dx \\
 & \leq c \left( \frac{\rho}{R} \right)^Q \int_{B_R} |Xv|^2 dx + c \int_{B_R} |Xw|^2 dx \leq c \left( \frac{\rho}{R} \right)^Q \int_{B_R} |Xu|^2 dx + c \int_{B_R} |Xw|^2 dx.
 \end{aligned}$$

With the aid of (4.7), it follows that

$$\begin{aligned}
 & \int_{B_{2\rho}} |Xu|^2 dx \\
 & \leq c|B_R|^{\frac{p-2}{p}} \left(\frac{\rho}{R}\right)^Q \left(\int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} + c|B_R| \sigma \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} \\
 & \quad + c \int_{B_R} \left(\sum_i |g^i|^2 + |f|^2\right) dx \\
 & \leq c|B_R|^{\frac{p-2}{p}} \left[ \left(\left(\frac{\rho}{R}\right)^Q + \sigma\right) \left(\int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} + \left(\int_{B_R} \sum_i |g^i|^p dx\right)^{\frac{2}{p}} + \left(\int_{B_R} |f|^p dx\right)^{\frac{2}{p}} \right].
 \end{aligned}
 \tag{4.8}$$

Noting

$$\int_{B_\rho} |Xu|^p dx \leq c|B_\rho| \left(\frac{1}{|B_{2\rho}|} \int_{B_{2\rho}} |Xu|^2 dx\right)^{\frac{p}{2}} + c \int_{B_{2\rho}} \left(\sum_i |g^i|^p + |f|^p\right) dx,$$

we have by Theorem 3.2 that

$$\begin{aligned}
 & \int_{B_\rho} |Xu|^p dx \\
 & \leq c|B_\rho| \left(\frac{c|B_R|^{\frac{p-2}{p}}}{|B_{2\rho}|} \left[ \left(\left(\frac{\rho}{R}\right)^Q + \sigma\right) \left(\int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} + \left(\int_{B_R} \sum_i |g^i|^p dx\right)^{\frac{2}{p}} + \left(\int_{B_R} |f|^p dx\right)^{\frac{2}{p}} \right]\right)^{\frac{p}{2}} \\
 & \quad + c \int_{B_{2\rho}} \left(\sum_i |g^i|^p + |f|^p\right) dx \\
 & \leq c \left(\frac{|B_R|}{|B_\rho|}\right)^{\frac{p-2}{2}} \left[ \left(\left(\frac{\rho}{R}\right)^Q + \sigma\right)^{\frac{p}{2}} \int_{B_R} |Xu|^p dx + \int_{B_R} \left(\sum_i |g^i|^p + |f|^p\right) dx \right] \\
 & \leq c \left(\frac{|B_R|}{|B_\rho|}\right)^{\frac{p-2}{2}} \left[ \left(\left(\frac{\rho}{R}\right)^Q + \sigma\right)^{\frac{p}{2}} \int_{B_R} |Xu|^p dx + R^{Q-2+2k} \left(\sum_i |g^i|_{L^m}^p + \|f\|_{L^m}^p\right) \right],
 \end{aligned}$$

where  $k = 1 - \frac{pQ}{2m}$ . Therefore,

$$\begin{aligned}
 & \left(|B_\rho|^{\frac{p-2}{2}} \int_{B_\rho} |Xu|^p dx\right)^{\frac{2}{p}} \\
 & \leq c \left(\left(\frac{\rho}{R}\right)^Q + \sigma\right) \left(|B_R|^{\frac{p-2}{2}} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} + c \left(|B_R|^{\frac{p-2}{2}} R^{Q-2+2k} \left(\sum_i |g^i|_{L^m}^p + \|f\|_{L^m}^p\right)\right)^{\frac{2}{p}} \\
 & \leq c \left(\left(\frac{\rho}{R}\right)^Q + \sigma\right) \left(|B_R|^{\frac{p-2}{2}} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} + c \left(R^{\frac{pQ}{2}-2+2k} \left(\sum_i |g^i|_{L^m}^p + \|f\|_{L^m}^p\right)\right)^{\frac{2}{p}}.
 \end{aligned}$$

Now let us set  $H(\rho) = \left(|B_\rho|^{\frac{p-2}{2}} \int_{B_\rho} |Xu|^p dx\right)^{\frac{2}{p}}$ ,  $H(R) = \left(|B_R|^{\frac{p-2}{2}} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}}$ ,  $\alpha = Q$ ,

$\beta = \left(\frac{pQ}{2} - 2 + 2k\right) \frac{2}{p}$ ,  $\varepsilon = \sigma$ ,  $B = c \left(\sum_i |g^i|_{L^m}^p + \|f\|_{L^m}^p\right)^{\frac{2}{p}}$ . In terms of Lemma 2.1,

$$\begin{aligned}
 & \left(|B_\rho|^{\frac{p-2}{2}} \int_{B_\rho} |Xu|^p dx\right)^{\frac{2}{p}} \\
 & \leq c \left(\frac{\rho}{R}\right)^{\left(\frac{pQ}{2}-2+2k\right) \frac{2}{p}} \left(|B_R|^{\frac{p-2}{2}} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} + c \left(\rho^{\frac{pQ}{2}-2+2k} \left(\sum_i |g^i|_{L^m}^p + \|f\|_{L^m}^p\right)\right)^{\frac{2}{p}}.
 \end{aligned}$$

By (2.1),

$$\begin{aligned} & \int_{B_\rho} |Xu|^p dx \\ & \leq c \left(\frac{\rho}{R}\right)^{\frac{pQ}{2}-2+2k} \left(\frac{|B_R|}{|B_\rho|}\right)^{\frac{p-2}{2}} \int_{B_R} |Xu|^p dx + c |B_\rho|^{\frac{2-p}{2}} \rho^{\frac{pQ}{2}-2+2k} \left(\sum_i |g^i|_{L^m}^p + \|f\|_{L^m}^p\right) \\ & \leq c \left(\frac{\rho}{R}\right)^{Q-2+2k} \int_{B_R} |Xu|^p dx + c \rho^{Q-2+2k} \left(\sum_i |g^i|_{L^m}^p + \|f\|_{L^m}^p\right) \end{aligned}$$

and thus

$$\int_{B_\rho} |Xu|^p dx \leq c \rho^{Q-2+2k}.$$

It is observed by lemma 2.2 that if  $x_0 \in \Omega_0$ , then

$$u(x) \in C^k(\Omega_0, \mathbb{R}^N), \quad k = 1 - \frac{pQ}{2m},$$

and from lemma 2.3,

$$H_{Q-2}(\Omega \setminus \Omega_0) = 0,$$

The proof is completed. □

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