

1-1-2014

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### Recommended Citation

ZHONG, XIANGGUI (2014) "On  $NR^*$ -subgroups of finite groups," *Turkish Journal of Mathematics*: Vol. 38: No. 2, Article 6. <https://doi.org/10.3906/mat-1305-12>  
Available at: <https://journals.tubitak.gov.tr/math/vol38/iss2/6>

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## On $NR^*$ -subgroups of finite groups

Xianggui ZHONG\*

Department of Mathematics, Guangxi Normal University, Guilin, Guangxi, P.R. China

Received: 08.05.2013 • Accepted: 23.08.2013 • Published Online: 27.01.2014 • Printed: 24.02.2014

**Abstract:** Let  $G$  be a finite group and let  $H$  be a subgroup of  $G$ .  $H$  is said to be an  $NR^*$ -subgroup of  $G$  if there exists a normal subgroup  $T$  of  $G$  such that  $G = HT$  and if whenever  $K \triangleleft H$  and  $g \in G$ , then  $K^g \cap H \cap T \leq K$ . A number of new characterizations of a group  $G$  are given, under the assumption that all Sylow subgroups of certain subgroups of  $G$  are  $NR^*$ -subgroups.

**Key words:** Finite group,  $NR^*$ -subgroup, the generalized Fitting subgroup, saturated formation

### 1. Introduction

All groups considered in this paper are finite. Throughout the following,  $G$  always stands for a finite group and if  $H$  and  $K$  are subgroups of  $G$  and  $K$  normalizes  $H$  then we shall use the notation  $[H]K$  to indicate that  $H \cap K = 1$ , i.e the product  $HK$  is a split extension of the normal subgroup  $H$  by the complement  $K$ . Other unexplained notations and terminology are standard, as in [6, 7]. A topic of some interest is to investigate the structure of  $G$  under certain restrictions on its certain subgroups (see [1-3, 8-10]). Following Berkovich [2], a subgroup  $H$  of  $G$  is called an  $NR$ -subgroup in  $G$  if whenever  $K \triangleleft H$ , then  $K^G \cap H = K$ .  $NR$ -subgroups play an important role in the following result of Berkovich (see [2, Proposition 11]).

**Theorem 1.1** *If all Sylow subgroups of a group  $G$  are  $NR$ -subgroups, then  $G$  is supersoluble.*

The following example indicates that it is not necessary that all Sylow subgroups of a supersolvable group  $G$  are  $NR$ -subgroups of  $G$ .

**Example 1.1.** Let  $H$  be an elementary abelian 3-group of order  $3^2$  and  $L$  be a cyclic group of order 2. Denote  $G = [H]L$  to be the corresponding semidirect product, where  $H = \langle a, b \mid a^3 = b^3 = 1 = [a, b] \rangle$ ,  $L = \langle x \rangle$ , and  $b^x = b, a^x = a^{-1}$ . Observe that a chief series  $1 \triangleleft \langle a \rangle \triangleleft \langle a, x \rangle \triangleleft G$  implies that  $G$  is supersoluble. However,  $H$  is not an  $NR$ -subgroup of  $G$ . As an illustration, let  $K = \langle ab \rangle$  be a maximal subgroup of  $H$ , and it is easy to see that  $K \neq H = K^G \cap H$ .

We hope to weaken the conditions on Sylow subgroups of  $G$  to generalize Theorem 1.1. In this article we work in this direction. We first analyze the counterexample  $G$  above. Note that  $\langle b, x \rangle$  is an  $NR$ -subgroup of prime index in  $G$ . This is the case that Tong-Viet studied (see [8, 9]). In fact, Tong-Viet in [9] proved that if

\*Correspondence: [xgzhang@mailbox.gxnu.edu.cn](mailto:xgzhang@mailbox.gxnu.edu.cn)

The author is supported by the National Natural Science Foundation of China (11261007) and the Natural Science Foundation of Guangxi Autonomous Region (2013GXNSFBA019003).

2010 *AMS Mathematics Subject Classification*: 20D20.

$G$  has an  $NR$ -subgroup  $H$  of prime index in  $G$  and  $H$  is supersolvable, then  $G$  is supersolvable. On the other hand, notice that  $\{\langle a \rangle, \langle b \rangle, \langle ab \rangle, \langle a^{-1}b \rangle\}$  is the set of nontrivial normal subgroups of  $H$ . Moreover, if  $L = \langle ab \rangle$  or  $\langle a^{-1}b \rangle$ , then there exists a normal subgroup  $T = \langle b, x \rangle$  of  $G$  such that  $G = LT$  and  $L^g \cap H \cap T = 1$  for all  $g \in G$ , and if  $L = \langle a \rangle$  or  $\langle b \rangle$ , then there exists a normal subgroup  $T = G$  of  $G$  such that  $G = LT$  and  $L^g \cap H \cap T = L$  for all  $g \in G$ .

We start with the following new concept.

**Definition 1.1.** Let  $G$  be a group and let  $K \triangleleft H \leq G$ . A triple  $(G, H, K)$  is said to be *quasispecial* in  $G$  if there exists a normal subgroup  $T$  of  $G$  such that  $G = KT$  and  $K^g \cap H \cap T \leq K$  for all  $g \in G$ . A subgroup  $H$  is said to be an  $NR^*$ -subgroup of  $G$  if, whenever  $K$  is normal in  $H$ , the triple  $(G, H, K)$  is quasispecial in  $G$ .

It is clear that every  $NR$ -subgroup of  $G$  is an  $NR^*$ -subgroup. The converse is not true in general. For instance, let  $G = [H]L$ , where  $H = \langle a, b \mid a^3 = b^3 = 1 = [a, b] \rangle$ ,  $L = \langle x \rangle$ , and  $b^x = b, a^x = a^{-1}$ . Then  $H$  is an  $NR^*$ -subgroup in  $G$ . However, we know that  $H$  is not an  $NR$ -subgroup.

We extend the former result by replacing conditions on “ $NR$ -subgroups” by conditions referring to only some “ $NR^*$ -subgroups”. Furthermore, we work within the framework of formation theory and use  $NR^*$ -subgroup conditions on the Sylow subgroups of  $F^*(G)$  to characterize the structure of a group  $G$ . Our main results are Theorems 3.7 and 3.8 (see Section 3). These results generalize some classical and recent results as particular cases.

## 2. Preliminaries

The following 2 lemmas will be used frequently and without comment.

**Lemma 2.1.** *Let  $K$  be a subgroup and let  $H$  be an  $NR^*$ -subgroup of  $G$ . Then the following holds:*

- (1) *If  $H \leq K$ , then  $H$  is an  $NR^*$ -subgroup of  $K$ .*
- (2) *If  $N \triangleleft G$  and  $N \leq H$ , then  $H/N$  is an  $NR^*$ -subgroup of  $G/N$ .*

**Proof** (1) Let  $L$  be any normal subgroup of  $H$ . Since  $H$  is an  $NR^*$ -subgroup of  $G$ , the triple  $(G, H, K)$  is quasispecial in  $G$ , and hence there exists a normal subgroup  $T$  of  $G$  such that  $G = LT$  and  $L^g \cap H \cap T \leq L$  for all  $g \in G$ . Note that  $H \leq K$ ; we have  $K = K \cap LT = L(K \cap T)$  by Dedekind’s law and hence  $K \cap T \triangleleft K$ . This implies that  $H$  is an  $NR^*$ -subgroup of  $K$ .

(2) Let  $L$  be any normal subgroup of  $H$ . Since  $H$  is an  $NR^*$ -subgroup of  $G$ , there exists a normal subgroup  $T$  of  $G$  such that  $G = LT$  and  $L^g \cap H \cap T \leq L$  for all  $g \in G$ . It follows that  $G/N = (L/N)(TN/N)$  and whenever  $L/N \triangleleft H/N$  and  $g \in G$ , then  $(L/N)^{gN} \cap H/N \cap TN/N = (L^g \cap H \cap TN)/N = (L^g \cap H \cap T)N/N \leq L/N$ . Thus,  $H/N$  is an  $NR^*$ -subgroup of  $G/N$ . □

**Lemma 2.2.** *Let  $H$  be a  $p$ -subgroup of  $G$  and let  $N$  be a normal  $p'$ -subgroup. Then  $H$  is an  $NR^*$ -subgroup of  $G$  if and only if  $HN/N$  is an  $NR^*$ -subgroup of  $G/N$ .*

**Proof** Let  $L$  be any normal subgroup of  $H$ . Assume that  $H$  is an  $NR^*$ -subgroup of  $G$  and the triple  $(G, H, K)$  is quasispecial in  $G$ ; hence, there exists a normal subgroup  $T$  of  $G$  such that  $G = LT$  and  $L^g \cap H \cap T \leq L$  for all  $g \in G$ . By the assumption that  $H$  is a  $p$ -group and  $N$  is a  $p'$ -group, we have  $N \leq T$ , since  $(|N|, |G/T|) = 1$ , and hence  $G/N = (LN/N)(T/N)$ . Let  $M/N$  be a normal subgroup of  $HN/N$ . Then there exists a normal

subgroup  $L$  of  $H$  such that  $M = LN$ . Note that  $M^g \cap HN \cap T = L^g N \cap (H \cap T)N = (L^g N \cap H \cap T)N$ . Let  $P$  be a Sylow  $p$ -subgroup of  $L^g N$ . Since  $L^g N \cap H \cap T$  is a  $p$ -group of  $L^g N$  and  $L^g N$  contains a Sylow  $p$ -subgroup  $L^g$ , we have  $(L^g N \cap H \cap T)^{n_1} \leq P \leq L^{g n_2}$  for suitable  $n_1, n_2 \in N$ . Then it follows that  $L^g N \cap H \cap T \leq L^{g n_2 n_1^{-1}} \cap H \cap T$ , and hence  $(L^g N \cap H \cap T)N \leq (L^{g n_2 n_1^{-1}} \cap H \cap T)N \leq LN = M$ . Thus,  $(M/N)^{gN} \cap (HN/N) \cap T/N = (L^g \cap H \cap T)N/N \leq M/N$ . Therefore,  $HN/N$  is an  $NR^*$ -subgroup of  $G/N$ .

Conversely, assume that  $HN/N$  is an  $NR^*$ -subgroup of  $G/N$ . Let  $L$  be a normal subgroup of  $H$ ; then  $LN/N \triangleleft HN/N$  and hence there exists a normal subgroup  $T/N$  of  $G/N$  such that  $G/N = (LN/N)(T/N)$  and  $(LN/N)^{gN} \cap HN/N \cap T/N \leq LN/N$ . It follows that  $L^g \cap H \cap T \leq (L^g N \cap H \cap T)N \leq LN$ , so  $G = LT$  and  $L^g \cap H \cap T \leq LN \cap H = L(N \cap H) = L$ , since  $N \cap H = 1$ . This follows easily from the assumption that  $H$  is a  $p$ -group and  $N$  is a  $p'$ -group. Thus,  $H$  is an  $NR^*$ -subgroup of  $G$ .  $\square$

Recall that a subgroup  $H$  of a group  $G$  is an  $\mathcal{H}$ -subgroup in  $G$  if  $H^g \cap N_G(H) \leq H$  for all  $g$  in  $G$ . A subgroup  $H$  of  $G$  is called weakly  $\mathcal{H}$ -subgroup in  $G$  [1] if there exists a normal subgroup  $T$  of  $G$  such that  $G = HT$  and  $H \cap T$  is an  $\mathcal{H}$ -subgroup in  $G$ . For groups  $H \leq T \leq G$  we say that  $H$  is strongly closed in  $T$  with respect to  $G$  if  $H^g \cap T \leq H$  for all  $g \in G$ . Noting if  $H$  is a  $p$ -subgroup of  $G$ , then  $H$  is an  $\mathcal{H}$ -subgroup of  $G$  if and only if  $H$  is strongly closed in  $P$  with respect to  $G$  for some Sylow  $p$ -subgroup  $P$  of  $G$  containing  $H$ . We obtain the next result.

**Lemma 2.3.** *Let  $N$  be a normal subgroup of  $G$ . Assume that  $p$  is a prime dividing  $|G|$  and  $P$  is a Sylow  $p$ -subgroup of  $N$ . If  $H$  is normal in  $P$  and  $(G, P, H)$  is quasispecial in  $G$ , then  $H$  is a weakly  $\mathcal{H}$ -subgroup of  $G$ .*

**Proof** By the hypotheses, for any normal subgroup  $H$  of  $P$ , there exists a normal subgroup  $T$  of  $G$  such that  $G = HT$  and  $H^g \cap P \cap T \leq H$  for all  $g \in G$ . Let  $S$  be a Sylow  $p$ -subgroup of  $G$  containing  $P$ . Observe that  $N \cap S = P$  implies that  $(H \cap T)^g \cap S = H^g \cap T \cap N \cap S = H^g \cap T \cap P \leq H$  for every  $g \in G$ . Thus,  $H \cap T$  is an  $\mathcal{H}$ -subgroup of  $G$ . This ends the proof.  $\square$

**Lemma 2.4.** *Let  $N$  be a minimal normal subgroup of  $G$  and let  $H$  be a subgroup of  $N$ . If  $H$  is an  $NR^*$ -subgroup of  $G$ , then  $H$  is an  $NR$ -subgroup of  $G$ .*

**Proof** By our hypotheses, for any normal subgroup  $L$  of  $H$ , there exists a normal subgroup  $T$  of  $G$  such that  $G = LT$  and  $L^g \cap H \cap T \leq L$  for all  $g \in G$ . Since  $N = L(N \cap T)$  by Dedekind's law and  $N \cap T \triangleleft G$ , the minimality of  $N$  implies that  $N \leq T$  or  $N \cap T = 1$ . If  $N \leq T$ , then  $G = T$  and so  $H$  is an  $NR$ -subgroup in  $G$ . If  $N \cap T = 1$ , then  $N = L \leq H$  and hence  $H = N$  is normal in  $G$  and, of course, an  $NR$ -subgroup in  $G$ .  $\square$

We shall need the following lemma.

**Lemma 2.5** ([10, Lemma 3.1]). *Let  $P$  be a Sylow  $p$ -subgroup of a group  $G$  and let  $T$  be a normal subgroup of  $G$ . If  $N_G(P)$  is  $p$ -nilpotent, then  $\langle T \cap P \cap (P')^g | g \in G \rangle = T \cap P \cap \langle (P')^g | g \in G \rangle$ .*

### 3. Main results

Our main result in this section gives detailed information about the  $NR^*$ -subgroup conditions of certain subgroups of  $G$ .

**Theorem 3.1** *Let  $p$  be a prime dividing  $|G|$  and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Then  $G$  is  $p$ -nilpotent if and only if  $P$  is an  $NR^*$ -subgroup of  $G$  and  $N_G(P)$  is  $p$ -nilpotent.*

**Proof** Suppose, first, that  $G$  is  $p$ -nilpotent. Then there exists a normal subgroup  $T$  of  $G$  such that  $G = PT$  and  $P \cap T = 1$ . Let  $P_1$  be any normal subgroup of  $P$  and let  $g \in G$ . Then  $g = at$  with  $a \in P$  and  $t \in T$ . Clearly,  $P_1^g = P_1^t$  and  $P_1T$  is a normal subgroup of  $G$ . Observe that  $P_1 \triangleleft N_{P_1T}(P_1)$  and  $P_1 \in \text{Syl}_p(N_{P_1T}(P_1))$  implies that  $P_1^t \cap P \leq P_1$  and  $P_1$  is  $NR^*$ -subgroup of  $G$ . Clearly,  $N_G(P)$  is  $p$ -nilpotent.

Conversely, suppose that  $N_G(P)$  is  $p$ -nilpotent and  $P$  is an  $NR^*$ -subgroup of  $G$ . Let  $P_1, P_2, \dots, P_s$  be maximal subgroups of  $P$  such that  $\bigcap_{i=1}^s P_i = \Phi(P)$ . Then there exists a normal subgroup  $T_i$  of  $G$  such that  $G = P_iT_i$  and  $P_i^g \cap T_i \cap P \leq P_i$  for all  $i \in \{1, 2, \dots, s\}$  and all  $g \in G$ , respectively, since  $P \cap (P')^g \leq (P')^g \leq (\Phi(P))^g \leq P_i^g$ . Let  $N = \bigcap_{i=1}^s T_i$ , a normal subgroup of  $G$ . Then we have  $N \cap P \cap (P')^g \leq \bigcap_{i=1}^s (T_i \cap P_i^g \cap P) \leq \bigcap_{i=1}^s P_i = \Phi(P)$  for all  $g \in G$ . By Grun's Theorem [7, IV, Theorem 3.7], we obtain  $N \cap P \cap G' = N \cap \langle P \cap (P')^g, P \cap (N_G(P))' | g \in G \rangle$ . Since  $N_G(P)$  is  $p$ -nilpotent, Lemma 2.5 implies that  $N \cap P \cap G' = N \cap \langle P \cap (P')^g | g \in G \rangle = N \cap P \cap \langle (P')^g | g \in G \rangle = \langle N \cap P \cap (P')^g | g \in G \rangle \leq \Phi(P)$ . By applying Tate's Theorem [7, IV, Theorem 4.7], we get that  $N \cap G'$  is  $p$ -nilpotent. Let  $B$  be a normal  $p$ -complement of  $(N \cap G')_p$  in  $N \cap G'$ . If  $B > 1$ , then  $B$  is normal in  $G$ . Consider the quotient group  $G/B$ . Since  $N_{G/B}(PB/B) = N_G(P)B/B$ , by Lemma 2.3(2),  $G/B$  is  $p$ -nilpotent and so is  $G$ . If  $B = 1$ , then  $N \cap G' \leq P$  and, therefore,  $N \cap G' = N \cap P \cap G' \leq \Phi(P)$ . By [7, III, Theorem 3.3(a)],  $N \cap G' \leq \Phi(G)$ . Observe that  $G/N$  being  $p$ -nilpotent implies that  $G/(N \cap G')$  is  $p$ -nilpotent. We get that  $G$  is  $p$ -nilpotent.  $\square$

**Corollary 3.2.** *Let  $p$  be a prime dividing  $|G|$  and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Then  $G$  is  $p$ -nilpotent if and only if  $N_G(P)$  is  $p$ -nilpotent and  $P$  is an  $NR$ -subgroup of  $G$ .*

**Remark 3.3.** In Theorem 3.1, the assumption that  $N_G(P)$  is  $p$ -nilpotent is essential. In order to illustrate the situation, we consider  $G = \langle a, b, c | a^9 = b^2 = c^2 = 1 = [b, c], ac = ba, ca = abc \rangle$ . Then the unique subgroup of order 3 is normal in  $G$ , but  $G$  is not 3-nilpotent. However, if  $p$  is the smallest prime dividing the order of a group, then the result holds. In fact, we obtain the following result.

**Theorem 3.4.** *Let  $p$  be the smallest prime dividing  $|G|$  and let  $P$  be a Sylow subgroup of  $G$ . Then  $G$  is  $p$ -nilpotent if and only if  $P$  is an  $NR^*$ -subgroup of  $G$ .*

**Proof** Since  $P$  is an  $NR^*$ -subgroup of  $G$ , by Lemma 2.1(1),  $P$  is an  $NR^*$ -subgroup of  $N_G(P)$ . If  $N_G(P) < G$ , then, by induction,  $N_G(P)$  is  $p$ -nilpotent. By Theorem 3.1,  $G$  is  $p$ -nilpotent. Thus,  $P$  is normal in  $G$ . Let  $H$  be a maximal subgroup of  $P$ ; then, by Lemma 2.3,  $H$  is a weakly  $\mathcal{H}$ -subgroup of  $G$ . By [1, Theorem 3.1],  $G$  is  $p$ -nilpotent.  $\square$

Recall that a class  $\mathcal{F}$  of groups is called a *formation* provided that (i)  $G \in \mathcal{F}$  and  $N$  is normal in  $G$  imply  $G/N \in \mathcal{F}$ , and (ii) if both  $G/N$  and  $G/M$  are in  $\mathcal{F}$ , then  $G/(N \cap M) \in \mathcal{F}$ . If, in addition,  $G/\Phi(G) \in \mathcal{F}$  implies  $G \in \mathcal{F}$ , then we say that  $\mathcal{F}$  is *saturated*. Note that, for a class  $\mathcal{F}$  of groups, a chief factor  $H/K$  of  $G$  is called  $\mathcal{F}$ -central if  $[H/K](G/C_G(H/K)) \in \mathcal{F}$ . The symbol  $Z_{\mathcal{F}}(G)$  denotes the  $\mathcal{F}$ -hypercenter of  $G$ , that is, the product of all such normal subgroups  $H$  of  $G$  whose  $G$ -chief factors are  $\mathcal{F}$ -central. We refer to [4, 5] for notation and terminology about the theory of formations. The following lemma plays a key role in the proof of Theorem 3.7.

**Lemma 3.5.** *Let  $P$  be a normal  $p$ -subgroup of  $G$ . If  $P$  is an  $NR^*$ -subgroup of  $G$ , then  $P \leq Z_{\mathcal{U}}(G)$ .*

**Proof** Let  $H$  be an arbitrary maximal subgroup of  $P$ . Observe that  $P$  being an  $NR^*$ -subgroup of  $G$  implies that  $H$  is a weakly  $\mathcal{H}$ -subgroup of  $G$  by Lemma 2.3. We conclude by [1, Lemma 3.3] that  $P \leq Z_{\mathcal{U}}(G)$ .  $\square$

We now prove the following main results.

**Theorem 3.6.** *If all Sylow subgroups of  $G$  are  $NR^*$ -subgroups of  $G$ , then  $G$  is supersolvable.*

**Proof** Assume that the result is not true and let  $G$  be a counterexample of minimal order. By Theorem 3.4, we can conclude that  $G$  has a Sylow tower of supersolvable type. Let  $p$  be the largest prime dividing the order of  $G$  and  $P$  a Sylow  $p$ -subgroup of  $G$ . Then  $P \triangleleft G$ . If  $N$  is a nontrivial normal subgroup of  $G$  contained in  $P$ , then as  $G/N$  satisfies the hypothesis of the theorem by Lemmas 2.1(2) and 2.2, the minimality of  $G$  yields that  $G/N$  is supersolvable. Set  $L = P \cap \Phi(G)$ . Observe that  $L \neq 1$  implies that  $G/L$  is supersolvable, and hence  $G$  must be supersolvable, which is a contradiction. Hence,  $L$  must be 1. Then  $P$  is the direct product of minimal normal subgroups of  $G$ , which are contained in  $P$ . Now we wish to show that  $P$  is the unique minimal normal subgroup of  $G$ . Otherwise, let  $N_1$  and  $N_2$  be 2 distinct minimal normal  $p$ -subgroups of  $G$ ; arguing as above, we conclude that  $G/N_i$  are supersolvable and so  $G$  is supersolvable, which is a contradiction and our claim holds. Let  $P_1$  be any maximal subgroup of  $P$ . By Lemma 2.3,  $P_1$  is weakly  $\mathcal{H}$ -subgroup in  $G$ . Then there exists a normal subgroup  $K$  of  $G$  such that  $G = P_1K$  and  $P_1 \cap K$  is  $\mathcal{H}$ -subgroup in  $G$ . Clearly,  $P_1 \cap K$  is subnormal in  $G$ . By [3, Theorem 6(2)],  $P_1 \cap K \triangleleft G$ . It follows that  $P_1 \cap K = 1$  and so  $P = P \cap K$  is a cyclic group of order  $p$ , which implies that  $G$  is supersolvable, a final contradiction, and the proof of the theorem is now complete.  $\square$

**Theorem 3.7.** *Let  $\mathcal{F}$  be a saturated formation containing the class of supersolvable groups  $\mathcal{U}$ . Then  $G \in \mathcal{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathcal{F}$  and all Sylow subgroups of  $H$  are  $NR^*$ -subgroups of  $G$ .*

**Proof** We need only to prove the “if” part. We use induction on  $|G|$ . By our hypothesis and Lemma 2.1(1), every Sylow subgroup of  $H$  is an  $NR^*$ -subgroup in  $H$ . Then  $H$  is supersolvable by Theorem 3.6. Let  $p$  be the largest prime dividing  $|H|$  and  $P$  a Sylow  $p$ -subgroup of  $H$ . Then  $P$  is characteristic in  $H$  and so is normal in  $G$ . So, first by applying Lemma 2.1(2) and Lemma 2.2, all maximal subgroups of every Sylow subgroup of  $H/P$  are  $NR^*$ -subgroups in  $G/P$ . We get that  $(G/P, H/P)$  satisfies the hypothesis of Theorem 3.7 and so  $G/P \in \mathcal{F}$  by induction on  $|G|$ . Now, Lemma 3.5 applies, yielding that  $P \leq Z_{\mathcal{U}}(G)$ . Observe that  $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$  by [4, Proposition 3.11] implies that  $P \leq Z_{\mathcal{F}}(G)$  and so  $G \in \mathcal{F}$ .  $\square$

For a group  $G$ , the generalized Fitting subgroup  $F^*(G)$  of  $G$  is the set of all elements of  $G$  that induce an inner automorphism on every chief factor of  $G$ . If  $G \neq 1$ , then  $F^*(G) \neq 1$ . If  $N \triangleleft G$ , then  $F^*(N) \leq F^*(G)$ . In particular we have  $C_G(F^*(G)) \leq F(G)$ , and the solvability of  $F^*(G)$  implies that  $F^*(G) = F(G)$  (see [1, Lemma 2.7], see also [4, 6]). With these results, now we can prove the next theorem.

**Theorem 3.8.** *Let  $\mathcal{F}$  be a saturated formation containing the class of supersolvable groups  $\mathcal{U}$ . Then  $G \in \mathcal{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathcal{F}$  and all Sylow subgroups of  $F^*(H)$  are  $NR^*$ -subgroups of  $G$ .*

**Proof** We need only to prove the “if” part. By Lemma 2.2(1), we have that every Sylow subgroup of  $F^*(H)$  is an  $NR^*$ -subgroup in  $F^*(H)$ . By Theorem 3.6 with respect to  $F^*(H)$ , we have that  $F^*(H)$  is supersolvable and, hence,  $F^*(H) = F(H)$ . Since any subgroup that is characteristic in  $F(G)$  is normal in  $G$ , by Lemma 3.5,  $F(H) \leq Z_{\mathcal{U}}(G)$ . Observe that  $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$  implies that  $F(H) \leq Z_{\mathcal{F}}(G)$ . Hence,  $G/C_G(F(H)) \in \mathcal{F}$  by [4, Theorem 6.10]. Since  $G/H$  and  $G/C_G(F(H))$  are in  $\mathcal{F}$ , we have that  $G/C_H(F(H)) = G/(H \cap C_G(F(H)))$  is in  $\mathcal{F}$ . Finally, note that  $C_H(F^*(H)) \leq F(H)$  and the fact that  $F^*(H) = F(H)$ . Then  $G/F(H)$  is an

epimorphic image of  $G/C_H(F(H))$ , and thus  $G/F(H) \in \mathcal{F}$ . Now, by applying Theorem 3.7, we get  $G \in \mathcal{F}$ , which completes the proof.  $\square$

Theorem 3.8 immediately implies the following corollaries.

**Corollary 3.9.** *Let  $\mathcal{F}$  be a saturated formation containing the class of supersolvable groups  $\mathcal{U}$ . Then  $G \in \mathcal{F}$  if and only if  $G$  has a solvable normal subgroup  $H$  such that  $G/H \in \mathcal{F}$  and if all Sylow subgroups of  $F(H)$  are  $NR^*$ -subgroups of  $G$ .*

**Corollary 3.10.** *Let  $G$  be a group with a normal subgroup  $H$  such that  $G/H$  is supersolvable. If all Sylow subgroups of  $F^*(H)$  are  $NR^*$ -subgroups of  $G$ , then  $G$  is supersolvable.*

**Corollary 3.11.** *Let  $G$  be a group. If all Sylow subgroups of  $F^*(G)$  are  $NR^*$ -subgroups of  $G$ , then  $G$  is supersolvable.*

**Corollary 3.12.** *Let  $G$  be a group with a solvable normal subgroup  $H$  such that  $G/H$  is supersolvable. If all Sylow subgroups of  $F(H)$  are  $NR^*$ -subgroups of  $G$ , then  $G$  is supersolvable.*

**Corollary 3.13.** *Let  $G$  be a solvable group. If all Sylow subgroups of  $F(G)$  are  $NR^*$ -subgroups of  $G$ , then  $G$  is supersolvable.*

Corollary 3.13 is not true if the solvability of  $G$  is omitted, as the nonabelian simple groups show.

## Acknowledgments

The author would like to thank the referee for his/her valuable suggestions and useful comments on the original version, which make the present paper readable.

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