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Generalized derivations on Jordan ideals in prime rings

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Abstract: Let R be a 2-torsion free prime ring with center $Z(R)$, J be a nonzero Jordan ideal also a subring of R, and F be a generalized derivation with associated derivation d. In the present paper, we shall show that $J\subseteq Z(R)$ if any one of the following properties holds: (i) $[F(u),u]\in Z(R)$, (ii) $F(u)u=ud(u)$, (iii) $d(u^2)=2F(u)u$, (iv) $F(u^2)-2uF(u)=d(u^2)-2ud(u)$, (v) $F^2(u)+3d^2(u)=2Fd(u)+2dF(u)$, (vi) $F(u^2)=2uF(u)$ for all $u\in J$.

Key words: Prime rings, Jordan ideals, generalized derivations, derivations

1. Introduction

Let R denote an associative ring with center $Z(R)$. For any $x,y\in R$, we write the commutator $[x,y]=xy-yx$, and the Jordan product $x\circ y=xy+yx$. We recall that a ring R is called prime if for any $a,b\in R$, $aRb=(0)$ implies that either $a=0$ or $b=0$; it is called a semiprime if $aRa=(0)$ implies that $a=0$. A prime ring is clearly a semiprime ring. An additive mapping $d:R\to R$ is called a derivation if $d(xy)=d(x)y+xd(y)$ holds for all $x,y\in R$. An additive mapping $F:R\to R$ is called a generalized derivation if there exists a derivation $d:R\to R$ such that $F(xy)=F(x)y+xd(y)$ holds for all $x,y\in R$. A ring R is said to be n-torsion free, where $n\neq 0$ is a positive integer, if whenever $na=0$, with $a\in R$, then $a=0$. An additive subgroup J is said to be a Jordan ideal of R if $uor\in J$, for all $u\in J$ and $r\in R$. One may observe that every ideal of R is a Jordan ideal of R but the converse need not be true. An additive subgroup U of R is said to be a Lie ideal of R if $[u,r]\in U$, for all $u\in U$ and $r\in R$. It is clear that if $char R=2$, then the Jordan ideal and Lie ideal of R are the same. In [4] Huang proved: Let R be an associative prime ring with $\mathrm{char}\, R\neq 2$, U a Lie ideal of R such that $u^2\in U$ for all $u\in U$, and F a generalized derivation associated with $d\neq 0$. If any one of the following conditions holds: (1) $[d(x),F(y)]=0$, (2) $d(x)\circ F(y)=0$, (3) either $d(x)\circ F(y)=x\circ y$ or $d(x)\circ F(y)+x\circ y=0$, (4) either $[d(x),F(y)]=[x,y]$ or $[d(x),F(y)]+[x,y]=0$, (5) either $[d(x),F(y)]=(x\circ y)$ or $[d(x),F(y)]+(x\circ y)=0$, (6) either $d(x)\circ F(y)=[x,y]$ or $d(x)\circ F(y)+[x,y]$, (7) either $d(x)\circ F(y)+xy\in Z(R)$ or $d(x)\circ F(y)-xy\in Z(R)$for all $x,y\in U$, then either $d=0$ or $U\subseteq Z(R)$.

Motivated by the results of Huang, we continue this line of investigation. In this paper, we study generalized derivation F with derivation d if any one of the following conditions holds: (i) $[F(u),u]\in Z(R)$, (ii) $F(u)u=ud(u)$, (iii) $d(u^2)=2F(u)u$, (iv) $F(u^2)-2uF(u)=d(u^2)-2ud(u)$, (v) $F^2(u)+3d^2(u)=2Fd(u)+2dF(u)$, (vi) $F(u^2)=2uF(u)$ for all u in a Jordan ideal that is also a subring of R.

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2. Preliminaries

Throughout the present paper, we shall make use of the following 2 basic identities without any specific mention:

$$[xy, z] = x[y, z] + [x, z]y, \text{ for all } x, y, z \in R. \tag{2.1}$$

$$[x, yz] = y[x, z] + [x, y]z, \text{ for all } x, y, z \in R. \tag{2.2}$$

We begin with the following known results, which will be used to prove our theorems.

Lemma 2.1 [[5], Lemma 2.7]**.** *Let R be a prime ring with $\mathrm{char}R \neq 2$ and J a nonzero Jordan ideal of R. If J is a commutative Jordan ideal, then $J \subseteq Z(R)$.*

Lemma 2.2 [[5], Lemma 2.5]**.** *Let R be a prime ring and J a nonzero Jordan ideal of R. If $a \in R$ and $aJ = (0)$ (or $Ja = (0)$), then $a = 0$.*

Lemma 2.3 [[5], Lemma 2.6]**.** *Let R be a prime ring with $\mathrm{char}R \neq 2$ and J a nonzero Jordan ideal of R. If $a, b \in R$ and $aJb = (0)$, then either $a = 0$ or $b = 0$.*

Lemma 2.4 [[1], Lemma 2.5]**.** *Let R be a prime ring with $\mathrm{char}R \neq 2$ and J a nonzero Jordan ideal of R. Suppose that θ, ϕ are automorphisms of R. If R admits a (θ, ϕ)-derivation d such that $d(J) = (0)$, then either $d = 0$ or $J \subseteq Z(R)$.*

Lemma 2.5 [[5], Theorem 3.1]**.** *Let R be a prime ring with $\mathrm{char}R \neq 2$ and J be both a Jordan ideal and a subring of R. If θ is an automorphism of R and $G : R \to R$ is an additive mapping satisfying $G(u^2) = 2\theta(u)G(u)$ for all $u \in J$, then either $J \subseteq Z(R)$ or $G(J) = 0$.*

Lemma 2.6 [[4], Lemma 2.6]**.** *A group cannot be the union of 2 of its proper subgroups.*

Now, we will prove the following 2 lemmas, which will be used to prove our theorems.

Lemma 2.7 *Let R be a ring. If R admits a generalized derivation F associated with derivation $d \neq 0$, then the mapping $F - d$ is a left centralizer on R.*

Proof Let $G = F - d$. It is clear that G is an additive mapping and for all $x, y \in R$, we have

$$\begin{aligned} G(xy) &= (F - d)(xy) = F(xy) - d(xy) \\ &= F(x)y + xd(y) - d(x)y - xd(y) \\ &= (F(x) - d(x))y = G(x)y. \end{aligned} \tag{2.3}$$

Therefore, G is a left centralizer on R. □

Lemma 2.8 *Let R be a prime ring and J a nonzero Jordan ideal of R. If G is a left centralizer of R such that $G(u) = 0$ for all $u \in J$, then $G(r) = 0$ for all $r \in R$.*

Proof Since J is a Jordan ideal of R, $ur+ru \in J$ for all $u \in J$ and $r \in R$. By hypotheses,

$$F(u) = 0 \text{ for all } u \in J. \tag{2.4}$$

Replacing u by $ur+ru$, $r \in R$, in (2.4) and using (2.4), we get $G(r)u = 0 \quad \forall u \in J, r \in R$, and hence $G(r)J = (0)$ for all $r \in R$. Thus, by Lemma 2.2, we get $G(r) = (0)$ for all $r \in R$. □

Remark 2.9 The assumption that J is both a Jordan ideal and a subring of R seems close to assuming that J is an ideal of the ring. However, we can see that there exists a Jordan ideal and a subring of R, which is not an ideal of R.

Example 2.10 [2]. *Let R be a ring of all 2×2 matrices with entries form $GF(2)$. Consider $J = \{\begin{pmatrix} a & b \\ b & a \end{pmatrix} | a,b \in GF(2)\}$ we can verify that J is both a Jordan ideal and a subring of R, but it is not an ideal of R.*

3. Main results

We start by the following theorem, which is the proposition 3.1 in [3] neglecting the condition subring on a subset.

Theorem 3.1 *Let R be a 2-torsion free semiprime ring, J a nonzero Jordan ideal, and F an additive mapping on R. If F is centralizing on J, then F is commuting on J.*

Proof A linearization of $[F(u),u] \in Z(R)$ gives $[F(u),v]+[F(v),u] \in Z(R)$ for all $u,v \in J$. In particular, replacing v by $2u^2$, we get $2[F(u),u^2]+2[F(u^2),u] \in Z(R)$.
Since $[F(u),u] \in Z(R)$, we have $[F(u),u^2] = 2[F(u),u]u$. Thus

$$4[F(u),u]u + 2[F(u^2),u] \in Z(R) \text{ for all } u \in J. \tag{3.1}$$

By assumption, $4[F(u^2),u^2] \in Z(R)$ for all $u \in J$. That is,

$$4[F(u^2),u]u + 4u[F(u^2),u] \in Z(R) \text{ for all } u \in J. \tag{3.2}$$

Now fix $u \in J$ and let $z = [F(u),u] \in Z(R)$, $s = [F(u^2),u]$. By (3.1) we have $0 = [F(u), 4zu+2s] = 2(2z^2+[F(u),s])$. Thus

$$[F(u),s] = -2z^2 \tag{3.3}$$

According to (3.2) we have $0 = [F(u),4su+4us] = 4([F(u),s]u + s[F(u),u] + [F(u),u]s + u[F(u),s])$, and applying (3.3), we get $-4z^2u + 2zs = 0$. Multiplying (3.3) by z from the left and using the last relation we obtain $-2z^3 = z[F(u),s] = [F(u),zs] = [F(u),2z^2u] = 2z^3$. Hence $z^3 = 0$. Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that $z=0$. This proves the theorem. □

Theorem 3.2 *Let R be a prime ring with $\operatorname{char} R \neq 2$, and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that F is centralizing on J, then $J \subseteq Z(R)$.*

Proof By Theorem 3.1 we have

$$[F(u),u]=0\,\text{for all}\,u\in J. \tag{3.4}$$

Linearizing (3.4) and using (3.4), we obtain

$$[F(u),v]+[F(v),u]=0\,\text{for all}\,u,v\in J. \tag{3.5}$$

Replacing v by vu in (3.5) and using (3.5) we obtain

$$\begin{aligned}&[F(u),v]u+[F(v),u]u+v[d(u),u]+[v,u]d(u)\\&=v[d(u),u]+[v,u]d(u)=0\,\text{for all}\,u,v\in J.\end{aligned} \tag{3.6}$$

Again replacing v by wv in (3.6) and using (3.6), we get $[w,u]vd(u)=0$ for all $u,v,w\in J$, and hence $[w,u]Jd(u)=(0)$. Thus, by Lemma 2.3, we find that for each $u\in J$ either $[w,u]=0$ or $d(u)=0$. Now let $J_1=\{u\in J\mid d(u)=0\}$ and $J_2=\{u\in J\mid [w,u]=0,\,\text{for all}\,w\in J\}$. Thus, J_1 and J_2 are additive subgroups of J and $J=J_1\bigcup J_2$. However, a group cannot be the union of 2 of its proper subgroups; hence $J_1=J$ or $J_2=J$. If $J_1=J$, then $d(u)=0$ for all $u\in J$. Thus, by Lemma 2.4, we get $J\subseteq Z(R)$. On the other hand, if $[w,u]=0$ for all $w,u\in J$, then, by Lemma 2.1, we get $J\subseteq Z(R)$. □

Theorem 3.3 *Let R be a prime ring with $\operatorname{char}R\neq 2$ and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d\neq 0$ such that $F(u)u=ud(u)$ for all $u\in J$, then $J\subseteq Z(R)$.*

Proof By hypothesis we have

$$F(u)u=ud(u)\,\text{for all}\,u\in J. \tag{3.7}$$

Linearizing the above equation gives

$$F(u)v+F(v)u=ud(v)+vd(u)\,\text{for all}\,u,v\in J. \tag{3.8}$$

Replace v by vu and use (3.8) to get

$$2vd(u)u=(u\circ v)d(u)\,\text{for all}\,u,v\in J. \tag{3.9}$$

Replacing v by wv in (3.9) and using (3.9), we have $[u,w]vd(u)=(0)$ for all $u,v,w\in J$, so $[u,w]Jd(u)=(0)$. Thus by Lemma 2.3, we find that for each $u\in J$ either $[u,w]=0$ or $d(u)=0$ for all $w\in J$. Now using similar arguments as used in the proof of Theorem 3.2, we get $J\subseteq Z(R)$. □

Theorem 3.4 *Let R be a prime ring with $\operatorname{char}R\neq 2$ and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d\neq 0$ such that $F(u^2)-2uF(u)=d(u^2)-2ud(u)$ for all $u\in J$, then either $J\subseteq Z(R)$ or $F=d$.*

Proof By hypothesis we have

$$F(u^2)-2uF(u)=d(u^2)-2ud(u)\,\text{for all}\,u\in J. \tag{3.10}$$

Since F and d are additive mappings, (3.10) could be rewritten as

$$(F-d)(u^2) = 2u(F-d)(u) \text{ for all } u \in J. \tag{3.11}$$

Let $G = F - d$ we get $G(u^2) = 2uG(u)$ for all $u \in J$. By Lemma 2.5 (taking $\theta = I$), either $J \subseteq Z(R)$ or $G(J) = 0$. If $G(J) = 0$, by Lemma 2.7 G is a left centralizer. Using Lemma 2.8 we get $G(r) = F(r) - d(r) = 0$ for all $r \in R$; thus $F(r) = d(r)$ for all $r \in R$. □

Theorem 3.5 *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F^2(u)+3d^2(u) = 2Fd(u)+2dF(u)$ for all $u \in J$, then either $J \subseteq Z(R)$ or $F = d$.*

Proof By hypothesis we have

$$F^2(u) + 3d^2(u) = 2Fd(u) + 2dF(u) \text{ for all } u \in J. \tag{3.12}$$

Replacing u by uv in (3.12) we get

$$F(F(u)v + ud(v)) + 3d(d(u)v + ud(v)) = 2F(d(u)v + ud(v)) + 2d(F(u)v + ud(v)) \text{ for all } u, v \in J. \tag{3.13}$$

The above equation gives

$$2F(u)d(v) = 2d(u)d(v) \text{ for all } u, v \in J. \tag{3.14}$$

However, char$R \neq 2$; hence

$$(F(u) - d(u))d(v) = 0 \text{ for all } u, v \in J. \tag{3.15}$$

Again replacing v by vw and using (3.15) we get

$$(F(u) - d(u))vd(w) = 0 \text{ for all } u, v, w \in J. \tag{3.16}$$

Thus, we get $(F(u)-d(u))Jd(w) = (0)$ for all $u, w \in J$. By Lemma 2.3 we have either $d(w) = 0$ for all $w \in J$ or $F(u)-d(u) = 0$ for all $u \in J$. If $d(w) = 0$ for all $w \in J$, hence $d(J) = 0$. Thus by Lemma 2.4 we get $J \subseteq Z(R)$. On the other hand, if $F(u)-d(u) = 0$. Using the same steps in Theorem 3.4 we get $F(r) = d(r)$ for all $r \in R$. □

Theorem 3.6 *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $d(u^2) = 2F(u)u$ for all $u \in J$, then $J \subseteq Z(R)$.*

Proof By hypothesis we have

$$d(u^2) = 2F(u)u \text{ for all } u \in J. \tag{3.17}$$

This gives

$$d(u)u + ud(u) = 2F(u)u \text{ for all } u \in J. \tag{3.18}$$

Linearizing the above equation gives

$$d(u)v + d(v)u + ud(v) + vd(u) = 2F(u)v + 2F(v)u \text{ for all } u, v \in J. \tag{3.19}$$

Replace v by vu and use (3.19) to get

$$uvd(u) + vud(u) = 2vd(u)u \text{ for all } u, v \in J. \tag{3.20}$$

Thus

$$(u \circ v)d(u) = 2vd(u)u, \text{ for all } u, v \in J. \tag{3.21}$$

Replacing v by wv in (3.21) and using (3.21), we have $[u,w]vd(u) = (0)$ for all $u, v, w \in J$; hence $[u,w]Jd(u) = (0)$. Thus, by Lemma 2.3, we find that for each $u \in J$ either $[u,w] = 0$ or $d(u) = 0$ for all $w \in J$. Now, using similar arguments as used in the proof of Theorem 3.2, we get $J \subseteq Z(R)$. □

Theorem 3.7 *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(u^2) = 2uF(u)$ for all $u \in J$, then $J \subseteq Z(R)$.*

Proof By hypothesis we have

$$F(u^2) = 2uF(u) \text{ for all } u \in J. \tag{3.22}$$

Using Lemma 2.5 with $\theta = I$, we get either $J \subseteq Z(R)$ or $F(J) = 0$. If $F(J) = 0$, then

$$F(u) = 0 \text{ for all } u \in J. \tag{3.23}$$

Replacing u by uv in (3.23) we get

$$F(u)v + ud(v) = 0 \text{ for all } u \in J. \tag{3.24}$$

Using (3.23) we have $ud(v) = 0$ for all $u, v \in J$; thus $Jd(v) = 0$ for all $v \in J$. By Lemma 2.2 we get $d(v) = 0$ for all $v \in J$, and by Lemma 2.4 we get $J \subseteq Z(R)$. □

Corollary 3.8 *Let R be a prime ring with char$R \neq 2$ and I a nonzero ideal of R. Suppose that R admits a generalized derivation F associated with a nonzero derivation d such that any one of the following holds:*
(i) $F(u)u = ud(u)$ for all $u \in I$;
(ii) $d(u^2) = 2F(u)u$ for all $u \in I$;
(iii) $F(u^2) = 2uF(u)$ for all $u \in I$;
then R is commutative.
Moreover, if any one of the following holds:
(iv) $F(u^2) - 2uF(u) = d(u^2) - 2ud(u)$ for all $u \in I$;
(v) $F^2(u) + 3d^2(u) = 2Fd(u) + 2dF(u)$ for all $u \in I$;
then either R is commutative or $F = d$.

In Theorem 3.2, if we assume that J is only a subring of R, then J is not central. This can be shown by the following example.

Example 3.9 *Let R be the prime ring of all 2×2 matrices over a noncommutative prime ring S with char$S \neq 2$. Consider $J = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} | a, b \in S\}$. Hence U is a subring, but not a Jordan ideal of R.*

Let us define mappings $F : R \to R$ *and* $d : R \to R$ *as follows:*

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, \tag{3.25}$$

$$d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}. \tag{3.26}$$

Therefore, d *is a nonzero derivation on* R*, and* F *is a generalized derivation on* R *satisfying the condition* $[F(u), u] \in Z(R)$ *for all* $u \in J$*. But* $J \nsubseteq Z(R)$*.*

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