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## On the continued fraction expansion of some hyperquadratic functions

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**Abstract:** In this paper, we consider continued fraction expansions for algebraic power series over a finite field. Especially, we are interested in studying the continued fraction expansion of a particular subset of algebraic power series over a finite field, called hyperquadratic. This subset contains irrational elements  $\alpha$  satisfying an equation  $\alpha = f(\alpha^r)$ , where  $r$  is a power of the characteristic of the base field and  $f$  is a linear fractional transformation with polynomial coefficients. The continued fraction expansion for these elements can sometimes be given fully explicitly. We will show this expansion for hyperquadratic power series satisfying certain types of equations.

**Key words:** Finite fields, formal power series, continued fraction

### 1. Introduction

Let  $p$  be a prime number and  $q = p^s$ , where  $s$  is a positive integer. We consider the finite field  $\mathbb{F}_q$  with  $q$  elements. We introduce with an indeterminate  $X$  the ring of polynomials  $\mathbb{F}_q[X]$  and the field of rational functions  $\mathbb{F}_q(X)$ . We also consider the absolute value defined on  $\mathbb{F}_q(X)$  by  $|P/Q| = |X|^{\deg P - \deg Q}$  for  $P, Q \in \mathbb{F}_q[X]$ , where  $|X|$  is a fixed real number greater than 1. By completing  $\mathbb{F}_q(X)$  with this absolute value, we obtain a field, denoted by  $\mathbb{F}_q((X^{-1}))$ , which is the field of formal power series in  $X^{-1}$  with coefficients in  $\mathbb{F}_q$ . Thus, if  $\alpha$  is a nonzero element of  $\mathbb{F}_q((X^{-1}))$ , we have

$$\alpha = \sum_{k \leq k_0} \alpha_k X^k,$$

where  $k_0 \in \mathbb{Z}$ ,  $\alpha_k \in \mathbb{F}_q$ ,  $\alpha_k \neq 0$ , and  $|\alpha| = |X|^{k_0}$ . Observe the analogy between the classical construction of the field of real numbers and this field of formal power series. The rôles of  $\pm 1$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$  are played by  $\mathbb{F}_q^*$ ,  $\mathbb{F}_q[X]$ ,  $\mathbb{F}_q(X)$ , and  $\mathbb{F}_q((X^{-1}))$  respectively.

As in the classical context of real numbers, we have a continued fraction algorithm in  $\mathbb{F}_q((X^{-1}))$ . Then if  $\alpha \in \mathbb{F}_q((X^{-1}))$  we can write

$$\alpha = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots}}}$$

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where  $a_i \in \mathbb{F}_q[X]$ . The  $a_i$  are called the partial quotients and we have  $\deg a_i > 0$  for  $i > 1$ . This continued fraction is finite if and only if  $\alpha \in \mathbb{F}_q(X)$ . We define 2 sequences of polynomial  $(P_n)$  and  $(Q_n)$  by  $P_1 = a_1$ ,  $Q_1 = 1$ ,  $P_2 = a_1a_2 + 1$ ,  $Q_2 = a_2$  and, for any  $n \geq 3$ ,

$$P_n = a_nP_{n-1} + P_{n-2}, \quad Q_n = a_nQ_{n-1} + Q_{n-2}.$$

We easily check that

$$P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n-1}$$

and

$$\frac{P_n}{Q_n} = [a_1, a_2, a_3, \dots, a_n].$$

The rational fraction  $\frac{P_n}{Q_n}$  is called the  $n^{th}$ -convergent of  $\alpha$  and we have for all  $n \geq 2$ :

$$[a_n, a_{n-1}, \dots, a_2] = \frac{Q_n}{Q_{n-1}} \quad \text{and} \quad [a_n, a_{n-1}, \dots, a_1] = \frac{P_n}{P_{n-1}}.$$

Moreover, we have for  $n \geq 1$  the equality:

$$\alpha = [a_1, a_2, \dots, a_n, \alpha_{n+1}] = \frac{P_n\alpha_{n+1} + P_{n-1}}{Q_n\alpha_{n+1} + Q_{n-1}},$$

where  $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \dots]$  is called a complete quotient of  $\alpha$ .

For a general presentation of continued fractions and diophantine approximation in the function field case, the reader may consult Schmidt's work [6].

Finally, we have the notation  $\mathbb{F}_q^+ = \{\alpha \in \mathbb{F}_q((X^{-1})) \text{ with } |\alpha| \geq |X|\}$ .

In 1976, Baum and Sweet [1] opened up a new field of research on diophantine approximation in the field of formal power series with coefficients in a finite field through the continued fraction expansion. These authors gave an example of a formal power series with coefficients in  $\mathbb{F}_2$ , algebraic of degree 3 over  $\mathbb{F}_2(X)$ , where all partial quotients in its continued fraction expansion are polynomials of degree 1 or 2. Ten years later, Mills and Robbins [5] described an algorithm that allowed them to give explicitly the continued fraction of the cubic series of Baum and Sweet. These studies have identified a subset of algebraic formal power series obtained as fixed points of the composition of a linear fractional transformation with the Frobenius homomorphism. These series are called hyperquadratic. We will denote the set of these elements by  $\mathcal{H}$ . Then an irrational element of  $\mathbb{F}_q((X^{-1}))$ , belonging to  $\mathcal{H}$  satisfies an algebraic equation of the form

$$\alpha = \frac{A\alpha^r + B}{C\alpha^r + D} \tag{1.1}$$

where  $A, B, C, D \in \mathbb{F}_q[X]$  and  $r = p^t$ ,  $t \geq 0$ .

The continued fraction expansion for the hyperquadratic elements could be given explicitly (see [3], [4], [6] for more references) for many examples of power series.

We present the reasoning on which our proofs are based. We start with a Lemma about elementary continued fractions. We recall the following notation. Let  $U/V \in \mathbb{F}_q(X)$  such that  $U/V := [a_1, a_2, \dots, a_n]$ . For all  $x \in \mathbb{F}_q(X)$ , we will note

$$[[a_1, a_2, \dots, a_n], x] := \frac{U}{V} + \frac{1}{x}$$

**Lemma 1.1** *Let  $a_1, \dots, a_n, x \in \mathbb{F}_q(X)$ . We have the following equality:*

$$[[a_1, a_2, \dots, a_n], x] = [a_1, a_2, \dots, a_n, y], \text{ where } y = \frac{(-1)^{n-1}}{Q_n^2}x - \frac{Q_{n-1}}{Q_n}.$$

The proof of this Lemma can be found in Lasjaunias’s article [4].

We now state the general reasoning and notations used in the sequel. Let  $(P, Q, R) \in (\mathbb{F}_q[X])^3$  and let  $l$  be a positive entire. We say that  $\alpha$  satisfies a relation of type  $(r, l, P, Q, R)$  if we have:

$$P\alpha^r = Q\alpha_{l+1} + R.$$

Then, in this case, the continued fraction of  $\alpha$  is determined by the first  $l$  partial quotients and the triple  $(P, Q, R)$ . In some simple cases the expansion can be completely and explicitly described and it has a very regular pattern. It is likely that this relationship is true for almost all hyperquadratic formal series, but, to our knowledge, no general result is confirmed (the reader may consult [4] for some comment on that).

The aim of this work is to describe the result of expansions of a subset of formal series satisfying an equation of type (1.1). We are concerned on the continued fraction expansion with a regular pattern of formal power series satisfying (1.1): with  $B = 1$  and  $D = 0$  in the first part, and with  $B = 0$  and  $D = 1$  in the second part. To describe the continued fraction expansion of these formal series, we use a technique that has been used by Lasjaunias in [4]. We will give certain continued fraction expansions for power series with all partial quotients of degree one, which are nonquadratic algebraic elements over the field of rational functions.

**2. Results**

**Lemma 2.1** *Let  $r = p^t, t \geq 1$ . The equation*

$$x = (Ax^r + 1)/Cx^r \quad (1)$$

where  $(A, C) \in \mathbb{F}_q[X] \times \mathbb{F}_q^*[X]$  such that  $\deg A > \deg C$ , admits a unique irrational solution in  $\mathbb{F}_q^+$ . Moreover, this solution admits unbounded partial quotients if  $(r - 1) \deg A > r \deg C$ .

**Proof** We denote by  $f$  the map defined on  $\mathbb{F}_q^+$  by

$$f(x) = \frac{Ax^r + 1}{Cx^r}.$$

Then  $f(x) = \frac{A}{C} + \frac{1}{Cx^r}$  and since  $\deg A > \deg C$  then  $|f(x)| \geq |X|$ . Hence,  $f$  is a map from  $\mathbb{F}_q^+$  to  $\mathbb{F}_q^+$ . For  $x, y \in \mathbb{F}_q^+$ , by straightforward calculation and using the Frobenius homomorphism if  $r > 1$ , we obtain

$$f(x) - f(y) = \frac{(y - x)^r}{Cx^r y^r}.$$

Since  $x, y \in \mathbb{F}_q^+$ , we have  $|1/x - 1/y|^{r-1} < 1$ . So

$$|f(x) - f(y)| = \frac{|x - y||x - y|^{r-1}}{|C||x||y||xy|^{r-1}} < \frac{|x - y|}{|C||X|^2}.$$

This shows that  $f$  is a contracting map from  $\mathbb{F}_q^+$  to  $\mathbb{F}_q^+$ . Thus, as  $\mathbb{F}_q^+$  is a complete metric subspace of  $\mathbb{F}_q((X^{-1}))$ , the equation  $x = f(x)$  has a unique solution in  $\mathbb{F}_q^+$ . Finally we shall prove that this solution is irrational. We assume that it is rational and we shall obtain a contradiction. So we can write  $\alpha = R/S$  with  $R, S \in \mathbb{F}_q[X]$ ,  $\gcd(R, S) = 1$  and  $|R| > |S|$ . From the equality  $R/S = f(R/S)$  we obtain

$$R/S = U/V$$

with  $U = AR^r + S^r$  and  $V = CR^r$ . Then we have

$$CU - AV = CS^r \text{ and } V = CR^r.$$

So if we consider  $W = \gcd(U, V)$  and since  $\gcd(R, S) = 1$  then  $|W| < |C|$ . Consequently we have

$$|V/W| > |V/C| > |R| > |S|.$$

Hence, we have  $V/W \neq S$  and this gives a contradiction.

Now, in relation with the equation defining elements of  $\mathcal{H}$ , we have here  $|\Delta| = |(AD - BC)| = |C|$ . Consequently, if  $\alpha$  has a partial quotient, other than the first, with degree  $> \frac{\deg(C)}{r-1}$ , then  $\alpha$  has unbounded partial quotients (see Lemma 3 in [2]).

Set  $|A| = |X|^{\deg A}$  and  $|C| = |X|^{\deg C}$ . It is clear that  $|\alpha| = |X|^{\deg A - \deg C}$ . Furthermore,

$$\left| \alpha - \frac{A}{C} \right| = \frac{1}{|C||\alpha|^r} = \frac{1}{|X|^{\deg C}|X|^{r(\deg A - \deg C)}} = \frac{1}{|X|^{2 \deg C}|X|^{r(\deg A - \deg C) - \deg C}}.$$

We have  $r \deg A - r \deg C - \deg C > \frac{\deg C}{r-1}$  equivalent to  $(r-1) \deg A > r \deg C$ . So we conclude that if  $(r-1) \deg A > r \deg C$  then  $\alpha$  admits unbounded partial quotients. □

**Remark 2.2** *i) It is easy to see that if  $\deg C < r-1$  then the solution of the equation (1) admits unbounded partial quotients, and if  $\deg C < r$  then  $A/C$  is always a convergent for the solution of (1).*

*ii) The previous Lemma give us a necessary condition on the degree of the coefficients  $A$  and  $C$  of the equation (1) to obtain a solution having bounded partial quotients in its continued fraction expansion.*

**Lemma 2.3** *Let  $\alpha \in \mathbb{F}_q((X^{-1}))$  be an irrational formal power series that satisfies the equation (1) and  $(\frac{P_n}{Q_n})_{n \geq 1}$  be the sequence of convergent of  $\alpha$ . Suppose that there exists  $n \geq 1$  such that  $A/C = P_n/Q_n$ . Then  $\alpha$  is an expansion of the type  $(r, n, \gcd(A, C), (-1)^n Q_n, (-1)^n Q_{n-1})$ .*

**Proof** We have  $\alpha$  is a solution of (1); then with a simple transformation on (1), we get that  $\alpha$  satisfies

$$\alpha^r = \frac{1}{-A + C\alpha}.$$

Let  $D = \gcd(A, C)$ . Suppose that  $A/C = P_n/Q_n$  is a convergent of  $\alpha$  ( $P_n = A/D$  and  $Q_n = C/D$ ). We recall that if  $P_n/Q_n = [a_1, a_2, \dots, a_n]$  then we have the equality

$$\alpha = [a_1, a_2, \dots, a_n, \alpha_{n+1}] = \frac{P_n \alpha_{n+1} + P_{n-1}}{Q_n \alpha_{n+1} + Q_{n-1}}.$$

So we obtain

$$D\alpha^r = \frac{1}{-P_n + Q_n \left( \frac{P_n \alpha_{n+1} + P_{n-1}}{Q_n \alpha_{n+1} + Q_{n-1}} \right)} = (-1)^n Q_n \alpha_{n+1} + (-1)^n Q_{n-1}$$

and  $\deg Q_{n-1} < \deg Q_n$ . This gives that  $\alpha$  is an expansion of the type  $(r, n, D, (-1)^n Q_n, (-1)^n Q_{n-1})$ .

This Lemma will allow us to determine explicitly the continued fraction expansion of power series satisfying an equation of the type (1). □

**Theorem 2.4** *Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (1) such that:*

- 1)  $A/C = [a_1, C]$  is a convergent of  $\alpha$
- 2)  $C$  divides  $a_1^{r-1}$ .

Then

$$\alpha = [a_1; C, a_3, \dots, a_n, \dots]$$

where

$$a_{4k+1} = -a_{2k}^r/C \quad k \geq 1, \quad a_{4k+2} = C \quad k \geq 0$$

$$a_{4k+3} = a_{2k+1}^r/C \quad k \geq 0, \quad a_{4k} = -C \quad k \geq 1$$

**Proof** Following the Lemma 2.3, we have in this case  $n = 2$  and  $\alpha^r = C\alpha_3 + 1$ .

So  $[a_1^r, \alpha_2^r] = C\alpha_3 + 1$  and then  $[[a_1^r/C, -C], C\alpha_2^r] = \alpha_3$ . Thus by Lemma 1.1 and since  $C$  divides  $a_1^r$  we get that

$$\alpha_3 = [a_1^r/C, -C, \alpha'], \quad \text{with } \alpha' = -\frac{\alpha_2^r}{C} + \frac{1}{C}.$$

Since  $|\alpha'| > 1$  then  $a_3 = a_1^r/C$ ,  $a_4 = -C$  and  $\alpha' = \alpha_5$ .

Applying the same reasoning again, we obtain

$$\alpha_5 = [[a_2^r/C, C], -C\alpha_3^r] = [a_2^r/C, C, \alpha''], \quad \text{with } \alpha'' = \frac{\alpha_3^r}{C} - \frac{1}{C}$$

We deduce that  $a_5 = -\alpha_2^r/C$ ,  $a_6 = C$  and since  $|\alpha''| > 1$  then  $\alpha'' = \alpha_7$ . So we have

$$\alpha_3 = [a_1^r/C, -C, -a_2^r/C, C, \alpha_7].$$

In general, by an easy recurrence on  $k$  we obtain that:

$$\alpha_{2k}^r = -C\alpha_{4k+1} + 1$$

$$\alpha_{2k+1}^r = C\alpha_{4k+3} + 1,$$

and

$$\alpha_{4k+3} = [a_{2k+1}^r/C, -C, -a_{2k+2}^r/C, C, \alpha_{4k+7}].$$

So we obtain the desired result. □

**Corollary 2.5** Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (1) with  $r = 2$  and  $q$  is a power of 2. Suppose that  $A/C = [a_1, C]$  with  $a_1 = u_1X$  and  $C = u_2X$  where  $(u_1, u_2) \in (\mathbb{F}_q^*)^2$ . If  $a \in \mathbb{F}_q[X]$  and  $n \geq 0$  is an integer,  $[a]^n$  denotes the sequence  $a, a, \dots, a$ , where  $a$  is repeated  $n$  times and  $[a]^0$  is the empty sequence. Define  $H_n$  a finite sequence of elements of  $\mathbb{F}_q[X]$ , for  $n \geq 1$ , by

$$H_n = u_1^{2n}X/u_2^{2n-1}, [u_2X]^{2n+1}.$$

Let  $H_\infty$  be the infinite sequence defined by:

$$H_\infty = H_1, H_2, \dots, H_n, \dots$$

Then the continued fraction expansion of  $\alpha$  is

$$\alpha = [u_1X, u_2X, H_\infty].$$

**Proof** According to the previous Theorem we get that if we put  $\alpha = [a_1, a_2, \dots, a_n, \dots]$  then

$$a_1 = u_1X \quad \text{and} \quad a_2 = u_2X$$

$$a_3 = u_1^2X/u_2 \quad \text{and} \quad a_4 = a_5 = a_6 = u_2X,$$

$$a_7 = u_1^4X/u_2^3 \quad \text{and} \quad a_8 = a_9 = a_{10} = a_{11} = a_{12} = a_{13} = a_{14} = u_2X,$$

$$a_{15} = u_1^8X/u_2^5, \dots$$

Thus, we built by recurrence a sequence of rational functions  $(H_n)_{n \geq 1}$  such that  $H_1 = u_1^2X/u_2, [u_2X]^3$  and for  $n \geq 2$ ,

$$H_n = u_1^{2^n}X/u_2^{2^{n-1}+1}, [u_2X]^{2^{n+1}-1}.$$

Hence we get the explicit continued fraction expansion of  $\alpha$ . □

**Theorem 2.6** Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (1) such that:

- 1)  $A/C = [a_1, C]$  is a convergent for  $\alpha$
- 2)  $C^2$  divides  $a_1^r - 1$ .

Then

$$\alpha = [a_1; C, a_3, \dots, a_n, \dots]$$

where  $a_3 = \frac{a_1^r - 1}{C}$  and for all  $k \geq 2$

$$a_{2k} = Ca_{2k-2}^r, \quad a_{2k+1} = a_{2k-1}^r/C.$$

**Proof** Following the Lemma 2.3, we have in this case  $n = 2$  and  $a^r = C\alpha_3 + 1$ . This relationship can be written in the form  $[a_1^r, \alpha_2^r] = C\alpha_3 + 1$ , then  $[\frac{a_1^r - 1}{C}, C\alpha_2^r] = \alpha_3$ . This gives that  $a_3 = \frac{a_1^r - 1}{C}$  and

$$\alpha_4 = C\alpha_2^r = [Ca_2^r, \alpha_3^r/C].$$

Hence we get  $a_4 = Ca_2^r$  and since  $C$  divides  $a_3$  then

$$\alpha_5 = \alpha_3^r/C = [a_3^r/C, C\alpha_4^r].$$

In general, by an easy recurrence on  $k \geq 2$  we obtain

$$\begin{aligned} \alpha_{2k-2}^r &= \alpha_{2k}/C \\ \alpha_{2k-1}^r &= C\alpha_{2k+1}, \end{aligned}$$

which ends the proof. □

**Theorem 2.7** *Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (1) such that  $q$  is a power of 2 and  $r = 4$ . Suppose that  $A/C = [\lambda X; X, X, X] = \lambda X, [X]^3$ , where  $\lambda \in \mathbb{F}_q^*$ . Then*

$$\alpha = [H_\infty].$$

where  $H_n = \lambda^{4^{n-1}}X, [X]^{4^n-1}$  for  $n \geq 1$ .

**Proof** Note that in this case  $A = \lambda X^4 + X^2 + 1$  and  $C = X^3$ . Since  $A/C = [\lambda X; X, X, X] = P_4/Q_4$  is a convergent of  $\alpha$  then  $Q_3/Q_4 = [X, X, X]$  and  $A/C = H_1$ . Furthermore, we have in this case  $n = 4$  and  $\alpha^4 = C\alpha_5 + Q_3$ . So  $[a_1^4, \alpha_2^4] = C\alpha_5 + Q_3$  then

$$\frac{a_1^4}{C} + \frac{Q_3}{C} + \frac{1}{C\alpha_2^4} = \alpha_5.$$

Hence  $[[a_1^4/C, X, X, X], C\alpha_2^4] = \alpha_5$ . Following the Lemma 1.1 and since  $C$  divides  $a_1^4$  we get

$$[a_1^4/C, X, X, X, \alpha'] = \alpha_5, \quad \text{with } \alpha' = \frac{\alpha_2^4}{C} + \frac{Q_3}{C}.$$

Since  $|\alpha'| > 1$  then  $\alpha' = \alpha_9$ . So we obtain:

$$a_5 = \frac{a_1^4}{C} = \lambda^4 X, a_6 = a_7 = a_8 = X, \alpha_9 = \frac{\alpha_2^4}{C} + \frac{Q_3}{C}.$$

We apply again the same reasoning and we get the following relation:

$$[a_2^4/C, X, X, X], \alpha'' = \alpha_9.$$

This gives that  $a_9 = a_{10} = a_{11} = a_{12} = X$ , and since  $|\alpha''| > 1$  then  $\alpha'' = \alpha_{13} = \frac{\alpha_3^4}{C} + \frac{Q_3}{C}$ .

In general, by an easy recurrence on  $k \geq 1$  we obtain:

$$\alpha_{4k+1} = \frac{\alpha_k^4}{C} + \frac{Q_3}{C} = [\frac{a_k^4}{C}, X, X, X, \alpha_{4k+5}].$$

Using this relation of recurrence, we obtain  $a_{13} = \dots = a_{20} = X$  and  $a_{21} = \lambda^{16}X$ . Thus  $H_1 = \lambda^4 X, [X]^{15}$  and  $H_2$  will begin with  $\lambda^{16}X$ . It is clear that the sequence  $(H_n)_{n \geq 1}$  begins with the first partial quotient  $a_n$  obtained by the process, which has a power of  $\lambda$  as coefficient. We get that for all  $n \geq 1$ ,  $H_{n+1}$  is obtained from  $H_n$  after  $4^n - 1$  iteration. So we obtain the desired result. □

We exhibit now some results concerning another family of hyperquadratic power series.



**Lemma 2.8** Let  $r = p^t$ ,  $t \geq 1$  such that  $r > 2$ . The equation

$$x = Ax^r / (Cx^r + 1) \quad (2)$$

where  $(A, C) \in \mathbb{F}_q[X] \times \mathbb{F}_q^*[X]$  such that  $\deg A > \deg C$ , admits a unique irrational solution  $\alpha \in \mathbb{F}_q^+$ . Moreover, we have the following result:

i)  $\alpha$  admits unbounded partial quotients if  $(r - 2) \deg A > (r - 1) \deg C$ .

ii) If  $A/C$  is a convergent of  $\alpha$ , i.e. there exists  $n \geq 1$  such that  $A/C = P_n/Q_n$  (where  $(\frac{P_n}{Q_n})_{n \geq 1}$  is the sequence of convergent of  $\alpha$ ), then  $\alpha$  is an expansion of the type

$$(r - 1, n, \gcd(A, C), (-1)^n Q_n, (-1)^n Q_{n-1}) \text{ and } (r, n, \gcd(A, C), (-1)^n P_n, (-1)^n P_{n-1}).$$

**Proof** The proof of the existence and the uniqueness of the irrational solution of (2) and the property i) is the same as in the Lemma 2.1.

Now, we can write the equation (2) in 2 forms:

$$\alpha^{r-1} = \frac{1}{-A + C\alpha}, \quad (2.2)$$

or

$$\alpha^r = \frac{\alpha}{-A + C\alpha}. \quad (2.3)$$

So if  $A/C$  is a convergent of  $\alpha$  then there exists  $n \geq 1$  such that  $A/C = P_n/Q_n$ . Then as Lemma 2.3, the equation (2.2) implies that  $\alpha$  is of the type  $(r - 1, n, \gcd(A, C), (-1)^n Q_n, (-1)^n Q_{n-1})$ .

The equation (2.3) implies that:

$$D\alpha^r = \frac{\frac{P_n\alpha_{n+1} + P_{n-1}}{Q_n\alpha_{n+1} + Q_{n-1}}}{-P_n + Q_n\left(\frac{P_n\alpha_{n+1} + P_{n-1}}{Q_n\alpha_{n+1} + Q_{n-1}}\right)} = (-1)^n P_n\alpha_{n+1} + (-1)^n P_{n-1},$$

where  $D = \gcd(A, C)$ . So we get that  $\alpha$  is of the type  $(r, n, D, (-1)^n A, (-1)^n P_{n-1})$ . □

Now, using this Lemma, we will determine the continued fraction expansion of some power series of strictly positive degree satisfying an equation of type (2). All given examples are with a regular pattern. Some of them are general (i.e. with arbitrary coefficients  $A$  and  $C$ ) and other are well chosen to obtain power series with all partial quotients of degree one. Note that the part (i) of the previous Lemma allows us to chose  $\deg A$  and  $\deg C$  in such a way that we could obtain power series with bounded partial quotients.

**Theorem 2.9** Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (2) such that  $A/C = [a_1, C]$  is a convergent of  $\alpha$ .

Then

i) If  $C$  divides  $a_1^{r-2}$ .

Then

$$\alpha = [a_1; C, a_3, \dots, a_n, \dots]$$

where

$$a_{4k+1} = -a_{2k}^{r-1}/C \quad k \geq 1, \quad a_{4k+2} = C \quad k \geq 0$$

$$a_{4k+3} = a_{2k+1}^{r-1}/C \quad k \geq 0, \quad a_{4k} = -C \quad k \geq 1$$

ii) If  $C^2$  divides  $a_1^{r-1} - 1$ .

Then

$$\alpha = [a_1; C, a_3, \dots, a_n, \dots]$$

where  $a_3 = \frac{a_1^{r-1} - 1}{C}$  and for all  $k \geq 2$

$$a_{2k} = Ca_{2k-2}^{r-1}, \quad a_{2k+1} = a_{2k-1}^{r-1}/C.$$

**Corollary 2.10** Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (2) with  $r = 3$  and  $q$  is a power of 3. Suppose that  $A/C = [a_1, C]$  with  $a_1 = u_1X$  and  $C = u_2X$ , where  $(u_1, u_2) \in (\mathbb{F}_q^*)^2$ . Define the sequence of integers  $(v_n)_{n \geq 1}$  by

$$v_1 = 1 \quad \text{and} \quad v_n = 2v_{n-1} + 1.$$

Then the continued fraction expansion of  $\alpha$  is

$$\alpha = [u_1X, u_2X, u_1^2X/u_2, \dots, a_n, \dots],$$

such that for all  $k \geq 1$ :

$$a_{4k} = a_{4k+1} = 2u_2X, \quad a_{4k+2} = u_2X,$$

and

$$a_{4k+3} = \begin{cases} u_1^{2^{n+1}}X/u_2^{2^{n+1}-1} & \text{if there exists } n \text{ such that } k = v_n \\ u_2X & \text{else.} \end{cases}$$

The proof of the Theorem 2.9 and Corollary 2.10 is the same as that of the Theorem 2.4 and 2.6 and Corollary 2.4.

**Theorem 2.11** Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (2) such that  $q$  is a power of 2 and  $r = 4$ . Suppose that  $A = X^3 + X^2 + 1$  and  $C = X^2 + X + 1$ . Then

$$\alpha = [X; X, X + 1, \dots, a_n, \dots]$$

such that for all  $k \geq 1$ :

$$a_{3k+1} \in \{X, X + 1\}, \quad a_{3k+2} = X \quad \text{and} \quad a_{3k} = X + 1.$$

**Proof** We have  $A/C = [X, X, X + 1]$  and we can verify that  $A/C$  is the  $3^{rd}$ -convergent of  $\alpha$ . Following the Lemma 2.8 we have that  $\alpha$  is of the type  $(3, 3, 1, C, Q_2)$  and it satisfies the equation  $\alpha^3 = C\alpha_4 + Q_2$ . So  $[a_1^3, \alpha_2^3] = C\alpha_4 + Q_2$  and then

$$\frac{a_1^3}{C} + \frac{Q_2}{C} + \frac{1}{C\alpha_2^2} = \alpha_4,$$

hence

$$\frac{X^3 + X}{X^2 + X + 1} + \frac{1}{C\alpha_2^2} = \alpha_4.$$

We have  $\frac{X^3 + X}{X^2 + X + 1} = [X + 1, X, X + 1]$  then

$$\alpha_4 = [X + 1, X, X + 1, \alpha'], \text{ with } \alpha' = \frac{\alpha_2^3}{C} + \frac{Q_2}{C}.$$

Thus  $a_4 = X + 1, a_5 = X, a_6 = X + 1$ , and  $\alpha' = \alpha_7$ . We apply again the same reasoning and we get

$$\frac{a_2^3}{C} + \frac{Q_2}{C} + \frac{1}{C\alpha_3^3} = \alpha_7.$$

Then since  $a_2 = a_1 = X$ , we get

$$\alpha_7 = [X + 1, X, X + 1, \alpha''], \text{ with } \alpha'' = \frac{\alpha_3^3}{C} + \frac{Q_2}{C}.$$

Thus  $a_7 = X + 1, a_8 = X, a_9 = X + 1$ , and  $\alpha'' = \alpha_{10}$ . Now, we have

$$\frac{a_3^3}{C} + \frac{Q_2}{C} + \frac{1}{C\alpha_4^2} = \alpha_{10},$$

then

$$\frac{(X + 1)^3 + X}{X^2 + X + 1} + \frac{1}{C\alpha_4^3} = \frac{X^3 + X^2 + 1}{X^2 + X + 1} + \frac{1}{C\alpha_4^3} = \alpha_{10}.$$

Since  $\frac{X^3 + X^2 + 1}{X^2 + X + 1} = [X, X, X + 1]$  then

$$\alpha_{10} = [X, X, X + 1, \alpha'''], \text{ with } \alpha''' = \frac{\alpha_4^3}{C} + \frac{Q_2}{C}.$$

So  $a_{10} = X, a_{11} = X, a_{12} = X + 1$ , and  $\alpha''' = \alpha_{13}$ . In general, by an easy recurrence on  $k$  we obtain:

$$\alpha_{3k+1} = \frac{a_k^3}{C} + \frac{Q_2}{C} + \frac{1}{C\alpha_{k+1}^2}.$$

Since  $a_k \in \{X, X + 1\}$ , then  $\frac{a_k^3}{C} + \frac{Q_2}{C} \in \{[X, X, X + 1], [X + 1, X, X + 1]\}$ . So  $a_{3k+1} \in \{X, X + 1\}$ ,  $a_{3k+2} = X$ , and  $a_{3k+3} = X + 1$ . □

**Theorem 2.12** Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (2). Suppose that  $A/C = [a_1, a_2, a_3]$  is a convergent of  $\alpha$  such that  $A$  divides  $a_1^{r-1}$ ,  $a_2^{r-1}$ , and  $a_3^{r-1}$ . Then

$$\alpha = [a_1; a_2, a_3, \dots, a_n, \dots],$$

where for all  $k \geq 1$ :

$$a_{4k} = -a_k^r/A, \quad a_{4k+1} = -a_3, \quad a_{4k+2} = -a_2, \quad \text{and} \quad a_{4k+3} = -a_1.$$

**Proof** In this case we have  $A/C = P_3/Q_3$  and  $\alpha^r = -A\alpha_4 - P_2$ . Then we have:

$$\frac{-a_1^r}{A} - \frac{P_2}{P_3} - \frac{1}{A\alpha_2^r} = \alpha_4.$$

Since  $\frac{P_2}{P_3} = [0, a_3, a_2, a_1]$  and taking into account that  $A$  divides  $a_1^{r-1}$ ,  $a_2^{r-1}$ , and  $a_3^{r-1}$  then:

$$[-a_1^r/A, -a_3, -a_2, -a_1, \alpha'] = \alpha_4, \quad \text{with } \alpha' = \frac{-\alpha_2^r}{A} - \frac{P_2}{P_3}.$$

Hence  $a_4 = -a_1^r/A$ ,  $a_5 = -a_3$ ,  $a_6 = -a_2$ ,  $a_7 = -a_1$ , and  $\alpha' = \alpha_8$ . We apply again the same reasoning and we obtain

$$[-a_2^r/A, -a_3, -a_2, -a_1, \alpha''] = \alpha_8, \quad \text{with } \alpha'' = \frac{-\alpha_3^r}{A} - \frac{P_2}{P_3}.$$

So  $a_8 = -a_2^r/A$ ,  $a_9 = -a_3$ ,  $a_{10} = -a_2$ ,  $a_{11} = -a_1$ , and  $\alpha'' = \alpha_{12}$ . Thus, by recurrence we show that

$$\alpha_{4k} = \frac{-\alpha_k^r}{A} - \frac{P_2}{P_3}$$

for all  $k \geq 1$  and then

$$a_{4k} = -a_k^r/A, \quad a_{4k+1} = -a_3, \quad a_{4k+2} = -a_2, \quad \text{and } a_{4k+3} = -a_1.$$

□

**Corollary 2.13** Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (2) with  $r = 4$  and  $q$  is a power of 2. Suppose that  $A = \delta X^3$  and  $C = \delta X^2 + 1$ , where  $\delta \in \mathbb{F}_q^*$ . Then

$$\alpha = [\delta X; X, \delta X, \dots, a_n, \dots],$$

where for all  $k \geq 1$ :

$$a_{4k} = -a_k^4/A, \quad a_{4k+1} = \delta X, \quad a_{4k+2} = X, \quad \text{and } a_{4k+3} = \delta X.$$

**Proof** This corollary is a direct application of the previous Theorem. In fact, we have that  $A/C = [\delta X, X, \delta X] = [a_1, a_2, a_3]$  is a convergent of  $\alpha$  and  $A$  divides  $a_1^3$ ,  $a_2^3$ , and  $a_3^3$ . □

We conclude this paper by seeing the behavior of the partial quotients of the solution of the equations (1) and (2) when  $C$  divides  $A$ .

**Theorem 2.14** Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (1) such that  $C$  divides  $A$ . Then

$$\alpha = [a_1, \dots, a_n, \dots]$$

where for all  $n \geq 1$

$$a_n = \left(\frac{A}{C}\right)^{r^{n-1}} (C)^{\frac{r^{n-1} + (-1)^n}{r+1}}.$$

**Proof** It is clear that if  $C$  divides  $A$  then the first partial quotient of  $\alpha$  is  $a_1 = A/C$  and  $\alpha = \frac{A}{C} + \frac{1}{\alpha_2}$ . But if  $\alpha$  is a solution of (1) then we have

$$\alpha^r = \frac{1}{-A + C\alpha} = \frac{\alpha_2}{C}.$$

Then  $C\alpha^r = \alpha_2$ . So  $Ca_1^r + \frac{C}{\alpha_2^r} = \alpha_2$ . Hence  $a_2 = Ca_1^r$  and  $\alpha_3 = \alpha_2^r/C$ . We apply again the same reasoning and

we obtain  $\alpha_3 = \frac{a_2^r}{C} + \frac{1}{C\alpha_3^r}$ , so  $a_3 = a_2^r/C$  and  $\alpha_4 = C\alpha_3^r$ . Then  $a_3 = C^{r-1}a_1^{r^2}$  and  $a_4 = Ca_3^r = C^{r^2-r+1}a_1^{r^3}$ .

We remark that  $a_3 = C^{\frac{r^2-1}{r+1}}a_1^{r^2}$  and  $a_4 = C^{\frac{r^3+1}{r+1}}a_1^{r^3}$ . By recurrence on  $k$  we prove easily that  $\alpha_{2k} = C\alpha_{2k-1}^r$ ,  $\alpha_{2k+1} = \alpha_{2k}^r/C$ , and  $a_k = C^{\frac{r^{k-1}+(-1)^k}{r+1}}a_1^{r^{k-1}}$ . □

We obtain with the same method as the previous Theorem the following result.

**Theorem 2.15** *Let  $\alpha \in \mathbb{F}_q^+$  be the solution of the equation (2) such that  $C$  divides  $A$ . Then*

$$\alpha = [a_1, \dots, a_n, \dots],$$

where for all  $n \geq 1$

$$a_n = \left(\frac{A}{C}\right)^{(r-1)^{n-1}} (C)^{\frac{(r-1)^{n-1}+(-1)^n}{r}}.$$

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**References**

[1] Baum L, Sweet M. Continued fractions of algebraic power series in characteristic 2. *Ann of Math* 1976; 103: 593–610.  
 [2] Lasjaunias A. Continued fractions for algebraic power series over finite field. *Finite Fields Appl* 1999; 5: 46–56.  
 [3] Lasjaunias A. A survey of diophantine approximation in fields of power series. *Monatsh Math* 2000; 130: 211–229.  
 [4] Lasjaunias A. Continued fractions for hyperquadratic power series over finite field. *Finite Fields Appl* 2008; 14: 329–350.  
 [5] Mills M, Robbins D. Continued fractions for certain algebraic power series. *Journal of Number Theory* 1986; 23: 388–404.  
 [6] Schmidt W. On continued fractions and diophantine approximation in power series fields. *Acta Arith* 2000; 95: 139–166.