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On the One-Loop Yukawa Coupling Beta-Function to Order Yg^2 in a General α Gauge and its Gauge Independence

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Abstract

We present the calculation of the one-loop Yukawa coupling beta function to order Yg^2 in a general α gauge, where Y and g are Yukawa and gauge couplings. We show explicitly how the gauge parameter α cancels out from the beta - function and demonstrate how some familiar results are obtained from it.

1. Introduction

In the spontaneously broken gauge field theories, of which the Glashow-Weinberg-Salam model based on the gauge group $SU(2)_L XU(1)$ [1], the so called standard model, is the prototype, the simplest and most elegant way of breaking the symmetry and generating fermion and gauge boson masses is the Higgs mechanism [2, 3]. In this mechanism elementary scalar fields, the Higgs fields, are introduced. The Higgs fields couple gauge invariantly to the gauge bosons through the covariant derivative and to the fermion chiral multiplets

$$\Psi_L = \frac{1}{2}(1-\gamma^5)\Psi \quad \text{and} \quad \Psi_R = \frac{1}{2}(1+\gamma^5)\Psi$$

through the Yukawa couplings of the form

$$L_Y = -Y^a_{jk}[\overline{\Psi}_{Lj}\Psi_{Rk}\Phi^a + h.c.], \tag{1}$$

where repeated indices are summed over and $h.c.$ stands for hermitian conjugate. Here, the indices j and k, and a are group indices in the representation of fermions and scalars, respectively. The $Y^a = (Y^a_{jk})$ are the Yukawa coupling matrices for the scalar field Φ^a, which is a component of a certain Higgs multiplet Φ. Note that Eq. (1) is for a generic family only.

As is well known, in gauge field theories the evolution of the gauge, scalar, and Yukawa coupling constants as a function of some arbitrary scale t are given by their respective beta-functions. It was shown in Ref. [4] for the first time that the evolutions of the gauge couplings are influenced by the presence of quartic scalar couplings, and vice versa. A complete treatment of the Higgs phenomena in asymptotically free gauge field theories at one-loop was presented in Ref. [5], where the Yukawa couplings of the fermions were also included. In Ref. [5] the one - loop beta - functions were all calculated in the Landau gauge in which certain diagrams do not contribute. The calculation of the two - loop contributions to the beta - functions for the Yukawa couplings (which are called Higgs - Yukawa couplings in Ref. [6]) was undertaken in Refs. [6] and [7]. Unfortunately there is no uniformity in the literature on the definition of the gauge parameter α. The gauge field propagator is given by

$$D_{\mu v}(k) = \frac{-i}{k^2}\left(g_{\mu v} - (1-\alpha)\frac{k_\mu k_v}{k^2}\right), \tag{2a}$$

in Ref. [6] and by

$$D_{\mu v}(k) = \frac{-i}{k^2}\left(g_{\mu v} - \alpha\frac{k_\mu k_v}{k^2}\right), \tag{2b}$$

in Ref. [7]. Thus in the definition of Ref. [6] Landau gauge corresponds to $\alpha = 0$, whereas in the definition of Ref. [7] it corresponds to $\alpha = 1$. The issue of local gauge invariance [8], and hence the gauge parameter α independence of all the physical quantities of a gauge theory is an important one. The L_Y in Eq. (1) is locally gauge invariant. Therefore the Yukawa coupling beta-functions β_{Y^a} that follow from it must be gauge independent, i.e. must be independent of the parameter α. At the one - loop level, the problem of gauge parameter independence of β_{Y^a} arises from those diagrams in which a gauge boson is exchanged between two particles (see Fig. 1). The contribution of such diagrams is of order Yg^2, where Y and g are respectively Yukawa and gauge coupling constants. To our knowledge all the previous one-loop calculations of β_{Y^a} [5, 9] were done in a particular gauge, the Landau gauge. The purpose of this paper is to calculate the order Yg^2 contributions to β_{Y^a} in a general α gauge, to show how the α-dependence cancels, and to obtain some results from the requirement of gauge independence of the beta-functions. We will follow the definitions of Ref. [7] in which Landau gauge corresponds to $\alpha = 1$.

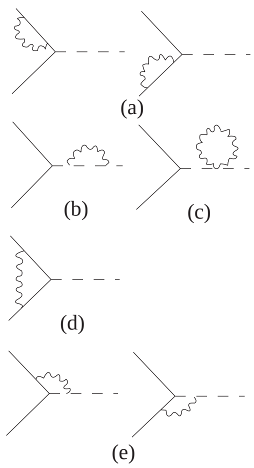

Figure 1. One - loop diagrams contributing to the Yukawa coupling constant renormalization to order Yg^2.

To avoid any possible confusion about the meaning of Eq. (1), we quote the Yukawa couplings in the standard model. Denoting the lepton, quark, and the Higgs doublets by

$$\Psi_L^{(\ell)} = \begin{pmatrix} \Psi_1^{(\ell)} \\ \Psi_2^{(\ell)} \end{pmatrix}_L = \begin{pmatrix} \ell^U = \nu \\ \ell^D \end{pmatrix}_L , \qquad \Psi_L^{(q)} = \begin{pmatrix} \Psi_1^{(q)} \\ \Psi_2^{(q)} \end{pmatrix}_L = \begin{pmatrix} q^U \\ q^D \end{pmatrix}_L ,$$

$$\Phi = \begin{pmatrix} \Phi^I \\ \Phi^{II} \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad \Phi^C = i\sigma_2 \Phi^* = \begin{pmatrix} \Phi^{CI} \\ \Phi^{CII} \end{pmatrix} = \begin{pmatrix} \overline{\phi^0} \\ -\phi^- \end{pmatrix}. \tag{3}$$

The matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

is the usual Pauli matrix and thus Φ^C is the charge conjugate of Φ and enables the "up" quarks to acquire mass. Before spontaneous symmetry breaking we have for the leptons of a generic family

$$L_Y(\ell) = -Y^I_{12}(\ell)\overline{\nu}_L \ell^D_R \phi^+ - Y^{II}_{22}(\ell)\overline{\ell}^D_L \overline{\ell}^D_R \phi^0 + h.c., \tag{4a}$$

and for the quarks of a generic family (suppressing the color indices)

$$L_Y(q^D) = -Y^I_{12}(q^D)\overline{q}^U_L q^D_R \phi^+ Y^{II}_{22}(q^D)\overline{q}^D_L q^D_R \phi^0 + h.c.\,, \tag{4b}$$

$$L_Y(q^U) = -Y^I_{11}(q^U)\overline{q}^U_L q^U_R \overline{\phi}^0 + Y^{II}_{21}(q^U)\overline{q}^D_L q^U_R \phi^- + h.c.\,. \tag{4c}$$

Note that right-handed neutrinos are assumed not to exist in the standard model. Each Lagrangian in equations (4a), (4b), and (4c) conserves electric charge, baryon and lepton numbers, and must be gauge invariant under $SU(2)_L$ and $U(1)$ (as well as $SU(3)_C$) transformations. Gauge invariance requires that

$$Y_{12}^{I}(\ell) = Y_{22}^{II}(\ell), \quad Y_{12}^{I}(q^D) = Y_{22}^{II}(q^D), \quad \text{and} \quad Y_{11}^{I}(q^U) = Y_{21}^{II}(q^U).$$

Each of these Lagrangians gives mass to a different fermion after spontaneous symmetry braking. As a reminder of this we have written the type of the fermion in parenthesis in the L_Y and the Yukawa couplings. In textbooks these Yukawa couplings are usually denoted by

$$Y_{22}^{II}(\ell) = G_\ell, \quad Y_{12}^{I}(q^D) = Y_{22}^{II}(q^D), \quad \text{and} \quad Y_{11}^{I}(q^U) = Y_{21}^{II}(q^U) = G_U,$$

where it should be understood that for the first family $\ell = e$, $D = d$, and $U = u$, for the second family $\ell = \mu$, $D = s$, and $U = c$, and for the third family $\ell = \tau$, $D = b$, and $U = t$. Ref. [6], however, denotes G_D and G_U by g_B and g_T, respectively, in which B stands for "bottom" and T for "top".

2. Local Gauge Invariance

Let the $T^A_{L(R)}$ represent the generators of the gauge group acting on the left-handed (right-handed) fermions. The $T^A_{L(R)}$ are hermitian. Similarly, let the θ^A represent the generators of the gauge group acting on the scalars which we assume to be real, so that the θ^A are imaginary and asymmetric. The superscript A runs over the generators of the gauge group. Local gauge invariance requires that the gauge transformed Yukawa Lagrangian

$$L'_Y = -Y^a_{jk}\overline{(\exp[i\xi_A T^A_L]\Psi_L)_j}(\exp[i\xi_A T^A_R]\Psi_R)_k(\exp[i\xi_A\theta^A]\Phi)^a + h.c. \tag{5}$$

must be equal to the original one in Eq. (1). In Eq. (5) the ξ_A are the (real) gauge transformation parameters whose number equals the number of generators of the gauge group. Carrying out the transformations in Eq. (5) infinitesimally (i.e. keeping only those terms that are linear in the ξ_A) yields

$$L'_Y = -Y^a_{jk}[\overline{\Psi}_{Lj} + \overline{\Psi}_{Lm}(-i\xi_A T^A_L)_{mj}][\Psi_{Rk} + i(\xi_A T^A_R)_{km}\Psi_{Rm}][\Phi^a + i(\xi_A\theta^A)_{ab}\Phi^b] + h.c.\ . \tag{6}$$

After a little algebra the extra piece in Eq. (6) that is required to vanish can be cast into

$$T^A_L Y^a - Y^a T^A_R = \theta^A_{ab} Y^b. \tag{7}$$

This is the gauge invariance condition. This relation will prove to be essential in showing gauge parameter α independence of the beta function β_{Y^a}.

3. The One-Loop Calculation

The order Yg^2 diagrams that contribute to the Yukawa coupling constant renormalization are depicted in Fig. 1. The renormalization constants Z_{Ψ_L}, Z_{Ψ_R}, Z_Φ, $Z_{coupling}$ are defined by

$$\begin{aligned} \Psi^{un}_{L,R} &= Z^{1/2}_{\Psi_{L,R}}\Psi^{ren}_{L,R}, \\ \Phi^{un}_a &= Z^{1/2}_\Phi \Phi^{ren}_a, \\ Y^a_{un}\overline{\Psi}^{un}_L\Psi^{un}_R\Phi^{un}_a &= Y^a_{ren}Z_{coupling}\overline{\Psi}^{ren}_L\Psi^{ren}_R\Phi^{ren}_a, \end{aligned} \tag{8}$$

where "un" and "ren" stand for "unrenormalized" and "renormalized", respectively. It follows from (8) that

$$\begin{aligned} Y^a_{un} &= Z_{coupling}Z^{-1/2}_{\Psi_L}Z^{-1/2}_{\Psi_R}Z^{-1/2}_\Phi Y^a_{ren} \\ &= Z_{Y^a}Y^a_{ren}. \end{aligned} \tag{9}$$

We have calculated the Yukawa coupling renormalization constants using dimensional regularization in $n = 4 - \epsilon$ dimensions. The Feynman rules relevant for this calculation are given in Fig. 2. Of course, the renormalized coupling constants, fermion and scalar fields, propagators, etc. must be free of any divergences. Any such divergent expression is rendered finite by subtracting the singularities in some way. The most elegant and simplest subtraction scheme for our calculation is the minimal subtraction (MS) scheme. We refer the reader to Ref. [10] for a quick review of the subtraction schemes and references therein for the details. The calculation of the renormalization constants is similar to the examples worked out in Chapter 3 of Ref. [11] in which a collection of integral identities in n dimensions required for such a calculation are also given. The contribution of each diagram in Fig. 1 is as follows:

$$Z^{(a)}_{\Psi_L} = 1 - \frac{g^2}{8\pi^2}(T^A_L T^A_L)(1-\alpha)\frac{1}{\epsilon}, \tag{10a}$$

$$Z^{(a)}_{\Psi_R} = 1 - \frac{g^2}{8\pi^2}(T^A_R T^A_R)(1-\alpha)\frac{1}{\epsilon}, \tag{10b}$$

$$Z^{(b)}_\Phi = 1 + \frac{g^2}{8\pi^2}(\theta^A\theta^A)(2+\alpha)\frac{1}{\epsilon}, \tag{11}$$

$$Z^{(c)}_\Phi = 0, \tag{12}$$

$$Z^{-1}_{coupling}Y^a = (1 + \Delta Z^{(d)}_1 + \Delta Z^{(e)}_1)Y^a, \tag{13}$$

where

$$\Delta Z^{(d)}_1 Y^a = \frac{g^2}{8\pi^2}(T^A_L Y^a T^A_R)(4-\alpha)\frac{1}{\epsilon}, \tag{14}$$

$$\Delta Z_1^{(e)} Y^a = \frac{g^2}{8\pi^2}\theta^A_{ba}(T^A_L Y^b - Y^b T^A_R)(1-\alpha)\frac{1}{\epsilon}. \tag{15a}$$

It should be kept on mind that to simplify the notation we have omitted the fermion indices j and k in these equations. Thus a term like $T^A_L Y^a T^A_R$ actually means $T^A_{Ljm} Y^a_{mn} T^A_{Rnk}$. Eq. (15a) should be simplified by using the gauge invariance condition, Eq. (7). Noting that $\theta^A_{cb}\theta^A_{ba} Y^c = (\theta^A\theta^A)_{ca} Y^c = C_2(S)\delta^{ca} Y^a$, where $C_2(S)\delta^{ca} = (\theta^A\theta^A)_{ca}$ is the quadratic Casimir invariant in the representation of the scalars. Equation (15a) then becomes

$$\Delta Z_1^{(e)} Y^a = \frac{g^2}{8\pi^2}\theta^A\theta^A Y^a (1-\alpha)\frac{1}{\epsilon}. \tag{15b}$$

The reader should pay attention to the appearance of the matrix Y^a in Eqs. (13)-(15). Had we done this calculation for an "up" or "down" quark, in other words for a diagonal coupling h^a (such as g_B or g_T) of the matrix Y^a as in Ref. [6], Y^a would not have appeared in these equations. The general case, as treated here, is more demanding and requires careful handling of the matrices. Now expanding the Z's as $Z = 1 + Z^{(1)}\frac{1}{\epsilon} + \cdots$ and noting that

$$\beta_{Y^a} = \frac{d}{d(1/\epsilon)} Z_{Y^a} Y^a, \tag{16}$$

we get

$$\beta_{Y^a} = Z^{(1)}_{coupling} Y^a - \frac{1}{2}(Z^{(1)}_{\Psi_L} + Z^{(1)}_{\Psi_R} + Z^{(1)}_{\Phi}) Y^a. \tag{17}$$

Generalizing the gauge group G to a product of G_i factors, the order $Y^a g_i^2$ contribution to the β_{Y^a} is

$$\begin{aligned} 16\pi^2\frac{dY^a}{dt} = 16\pi^2\beta_{Y^a} \quad &= \sum_{i,A_i} g_i^2[-6T^{A_i}_L Y^a T^{A_i}_R - 3\theta^{A_i}\theta^{A_i} Y^a] \\ &+ \sum_{i,A_i} g_i^2[-2T^{A_i}_L Y^a T^{A_i}_R + T^{A_i}_L T^{A_i}_L Y^a + \\ &Y^a T^A_R T^{A_i}_R - \theta^{A_i}\theta^{A_i} Y^a_k](1-\alpha). \end{aligned} \tag{18}$$

In Eq. (18) we have resumed the summation symbol to emphasize the contribution of each group factor in the product $G = G_1 \times G_2 \times G_3 \cdots$.

4. The Proof of Gauge Independence

Note that in the beta-function in Eq. (18) the gauge parameter α does not cancel automatically! For $\alpha = 1$ (Landau gauge) Eq. (18) reduces to the one quoted in the literature [5, 9]. The sum of the α-terms, or equivalently the coefficient of the $(1-\alpha)$, in Eq. (18) is

$$A_{\beta_{Y^a}} = \sum_{i,A_i} g_i^2[-2T^{A_i}_L Y^a T^{A_i}_R + T^{A_i}_L T^{A_i}_L Y^a + Y^a T^{A_i}_R T^{A_i}_R - \theta^{A_i}\theta^{A_i} Y^a]. \tag{19}$$

It is necessary that $A_{\beta_{Y^a}}$ vanishes, so that the beta - function in Eq. (18) is independent of the gauge parameter α. This is indeed the case. To see this, we apply the gauge invariance condition, Eq. (7), to the last term of $A_{\beta_{Y^a}}$ twice. We get

$$\theta^{A_i}\theta^{A_i}Y^a = T_L^{A_i}T_L^{A_i}Y^a + Y^a T_R^{A_i}T_R^{A_i} - 2T_L^{A_i}Y^a T_R^{A_i}, \tag{20}$$

which cancels the other terms in $A_{\beta_{Y^a}}$. Therefore the β_{Y^a} are gauge-independent and given, to this order, by

$$16\pi^2\beta_{Y^a} = -3\sum_{i,A_i} g_i^2[2T_L^{A_i}Y^a T_R^{A_i} + \theta^{A_i}\theta^{A_i}Y^a]. \tag{21}$$

This is in the form given in references [5] and [9]. Using the gauge invariance relation as given in Eq. (20) Eq. (21) reduces to

$$16\pi^2\beta_{Y^a} = -3\sum_{i,A_i} g_i^2[T_L^{A_i}T_L^{A_i}Y^a + Y^a T_R^{A_i}T_R^{A_i}], \tag{22}$$

which is in the form given in Ref. [12]. Since $\Sigma_A T^A T^A$ is proportional to the unit matrix, Eq. (22) becomes

$$16\pi^2\beta_{Y^a} = -3Y^a\sum_i g_i^2\sum_{A_i}[T_L^{A_i}T_L^{A_i} + T_R^{A_i}T_R^{A_i}]. \tag{23}$$

For $SU(N)$ groups the Casimir invariants are given by

$$\sum_A (T^A T^A)_{jk} = \frac{N^2-1}{2N}\delta^{jk}. \tag{24}$$

It is remarkable that Eqs. (22) and (23) have no reference to the scalars involved in the Yukawa couplings.

5. Implications of the Gauge Independence

Even though the $A_{\beta_{Y^a}}$ of Eq. (19) vanishes due to the gauge invariance of the Yukawa couplings, we can still obtain interesting, though familiar, information from it. For simplicity, let us consider flavor groups in fermion and scalar couplings. Furthermore, let us consider the couplings of a neutral Φ^a. Due to electric charge conservation and the fact that such a scalar gives mass to either "up" or "down" fermions, depending on the dimension of the flavor group the Yukawa coupling matrix for such a case takes a form similar to (for each color)

$$Y^a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h^a & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

Then $A_{\beta_{Y^a}}$ reduces to (dropping the index i and with no summation over j and a)

$$A_{\beta_{Y^a}} = h^a \sum_A \left[-2(T_L^A T_R^A)_{jj} + (T_L^A T_L^A)_{jj} + (T_R^A T_R^A)_{jj} - (\theta^A \theta^A)_{aa}\right] = 0, \tag{26}$$

from which we arrive at the following conclusions (which can be shown to be true for color groups too):

1) For a vectorlike group G_V (in which fermions have vectorial couplings to gauge bosons) whose generators satisfy $T_L^A = T_R^A = T^A$, Eq. (26) reduces to

$$\sum_A (\theta^A \theta^A)_{aa} = 0, \tag{27}$$

which implies that the scalars must be singlets -i.e. just constant numbers- under the vectorlike group G_V. This is the case for the color group $SU(3)_c$.

2) For a chiral group $G_{L(R)}$ (in which fermions have chiral couplings to gauge bosons) for which $T_{R(L)}^A = 0$, Eq. (26) reduces to

$$\sum_A (\theta^A \theta^A)_{aa} = \sum_A (T_{L(R)}^A T_{L(R)}^A)_{jj}, \tag{28}$$

which implies that the scalars which have gauge invariant Yukawa couplings must be in the same representation as the left-handed (right-handed) fermions. Again this is the case for the $SU(2)_L$ group of the standard model. We should warn the reader, however, that these results are not general. They are valid only for the type of Yukawa couplings given in Eq. (1).

6. Discussion

The Yukawa couplings given in Eq. (1) are not the only allowed ones. There are other possibilities. For example, written in matrix form

$$L_Y = -\frac{1}{2} Y \overline{(\Psi_L)^c} \chi \Psi_L + h.c. = -\frac{1}{2} Y \Psi_L^T C \chi \Psi_L + h.c. \tag{29}$$

is another possibility. Here, $\Psi^c = C\overline{\Psi^T}$ and C is the charge conjugation operator. The factor of $\frac{1}{2}$ is put to avoid double counting. (However, see later for vectorlike groups). For $SU(N)$ groups, the fermion multiplet Ψ_L is usually in the fundamental representation $\underline{n}$. The dimensions of the allowed Higgs multiplets are given by the decomposition of $\underline{n} \times \underline{n}$:

$$n \times n = \frac{n(n-1)}{2} + \frac{n(n+1)}{2}.$$

When the Higgs multiplet χ is in the symmetric representation conjugate to the one whose dimension is $\frac{n(n+1)}{2}$, the neutrinos acquire Majorana masses. The Yukawa couplings in Eq. (29) will be gauge invariant provided

$$T_L^{A_i} Y^a + Y^a (T_L^{A_i})^T = \theta_{ab}^{A_i} Y^b. \tag{30}$$

To order Yg^2, the Yukawa coupling beta-function for this case becomes

$$\beta_{Y^a} = \frac{3}{2}\sum_{i,A_i} g_i^2 [2T_L^{A_i} Y^a (T_L^{A_i})^T - \theta^{A_i}\theta^{A_i} Y^a], \tag{31}$$

which should be compared with Eq. (21). Using the gauge invariance condition, Eq. (30), this reduces to

$$\begin{aligned} \beta_{Y^a} &= -\frac{3}{2}\sum_{i,A_i} g_i^2 [T_L^{A_i} T_L^{A_i} Y^a + Y^a (T_L^{A_i})^T (T_L^{A_i})^T] \\ &= -3Y^a \sum_{i,A_i} g_i^2 [T_L^{A_i} T_L^{A_i}], \end{aligned} \tag{32}$$

where

$$\sum_A T^A T^A = \sum_A (T^A)^T (T^A)^T$$

has been used. This should be compared with Eq. (23) in which either L - handed or R - handed generators contribute for chiral gauge groups. For vectorlike groups like $SU(3)_c$ Eq. (32) seems to be contributing half of Eq. (23). This is due to the factor of $\frac{1}{2}$ introduced in Eq. (29). Therefore this factor should be omitted for vectorlike groups.

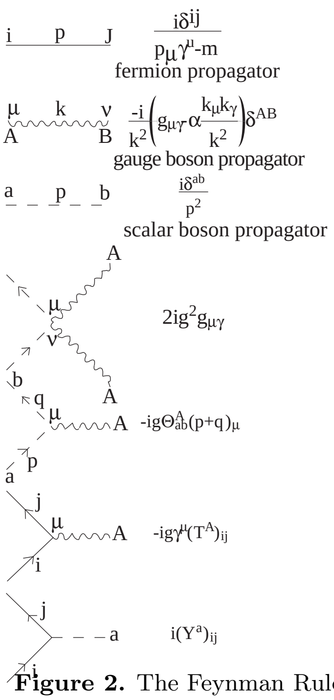

Figure 2. The Feynman Rules for fermion - gauge boson - scalar (Higgs) boson interactions.

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CONTENTS