

1-1-2014

On minimal Poincaré 4-complexes

ALBERTO CAVICCHIOLI

FRIEDRICH HEGENBARTH

DUSAN REPOVS

Follow this and additional works at: <https://journals.tubitak.gov.tr/math>



Part of the [Mathematics Commons](#)

Recommended Citation

CAVICCHIOLI, ALBERTO; HEGENBARTH, FRIEDRICH; and REPOVS, DUSAN (2014) "On minimal Poincaré 4-complexes," *Turkish Journal of Mathematics*: Vol. 38: No. 3, Article 14. <https://doi.org/10.3906/mat-1211-54>

Available at: <https://journals.tubitak.gov.tr/math/vol38/iss3/14>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

On minimal Poincaré 4-complexes

Alberto CAVICCHIOLI¹, Friedrich HEGENBARTH², Dušan REPOVŠ^{3,*}

¹Department of Mathematics, University of Modena and Reggio Emilia, Modena, Italy

²Department of Mathematics, University of Milan, Milan, Italy

³Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia

Received: 28.11.2012 • Accepted: 25.06.2013 • Published Online: 14.03.2014 • Printed: 11.04.2014

Abstract: We consider 2 types of minimal Poincaré 4-complexes. One is defined with respect to the degree 1-map order. This idea was already present in our previous papers, and more systematically studied later by Hillman. The second type of minimal Poincaré 4-complexes was introduced by Hambleton, Kreck, and Teichner. It is not based on an order relation. In the present paper we study existence and uniqueness questions.

Key words: Poincaré 4-complex, equivariant intersection form, degree 1-map, k -invariant, homotopy type, obstruction theory, homology with local coefficients, Whitehead's quadratic functor, Whitehead's exact sequence

1. Introduction

Minimal objects are usually defined with respect to a partial order. We consider oriented Poincaré 4-complexes (in short, PD_4 -complexes). If X and Y are 2 PD_4 -complexes, we define $X \succ Y$ if there is a degree 1-map $f : X \rightarrow Y$ inducing an isomorphism on the fundamental groups. If also $Y \succ X$, well-known theorems imply that $f : X \rightarrow Y$ is a homotopy equivalence. So " \succ " defines a symmetric partial order on the set of homotopy types of PD_4 -complexes. A PD_4 -complex P is said to be *minimal* for X if $X \succ P$ and whenever $P \succ Q$, Q is homotopy equivalent to P . We also consider special minimal objects called *strongly minimal*. In this paper we study existence and uniqueness questions. It is an interesting problem to calculate homotopy equivalences of X relative to a minimal P : that is, if $f : X \rightarrow P$ is as above, then calculate

$$\text{Aut}(X \succ P) = \{h : X \rightarrow X : h \text{ homotopy equivalence such that } f \circ h \\ \text{is homotopic to } f\}.$$

Self-homotopy equivalences were studied by various authors (see [12] and references there). Pamuk's method can be used to calculate $\text{Aut}(X \succ P)$.

Constructions of minimal objects were indicated by Hegenbarth, Repovš, and Spaggiari in [6] and more recently by Hillman in [8] and [9]. Degree 1-maps can be constructed from Λ -submodules $G \subset H_2(X, \Lambda)$. More precisely, we have the following (cf. Proposition 2.4 below):

Proposition 1.1 *Suppose X is a Poincaré 4-complex, and $G \subset H_2(X, \Lambda)$ is a stably free Λ -submodule such that the intersection form λ_X restricted to G is nonsingular. Then one can construct a Poincaré 4-complex Y*

*Correspondence: dusan.repovs@guest.arnes.si

2010 AMS Mathematics Subject Classification: 57 N 65, 57 R 67, 57 Q 10.

and a degree 1-map $f : X \rightarrow Y$. Moreover, there is an isomorphism

$$K_2(f, \Lambda) = \text{Ker}(H_2(X, \Lambda) \xrightarrow{f_*} H_2(Y, \Lambda)) \cong G$$

and λ_X restricted to $K_2(f, \Lambda)$ coincides with λ_X on G via this isomorphism.

Corollary 1.2 *Given any Poincaré 4-complex X , there exists a minimal Poincaré 4-complex P for X .*

The above proposition is useful to answer the following 2 basic questions about the minimal objects:

- (1) Existence; and
- (2) Uniqueness.

A Poincaré 4-complex P is called *strongly minimal* for π if the adjoint map $\hat{\lambda}_P : H_2(P, \Lambda) \rightarrow \text{Hom}_\Lambda(H_2(P, \Lambda), \Lambda)$ of the intersection form λ_P vanishes [8]. Proposition 1.1 implies that P is minimal. The same questions arise if we consider the originally defined minimal objects in [5].

Existence of strongly minimal models P is known only for few fundamental groups π (see [5] and [8]). All these examples satisfy $H^3(B\pi, \Lambda) \cong 0$, and hence $\text{Hom}_\Lambda(H_2(P, \Lambda), \Lambda) \cong 0$ (see below). So all are “trivial” in the sense that λ_P is zero because its adjoint $\hat{\lambda}_P : H_2(P, \Lambda) \rightarrow \text{Hom}_\Lambda(H_2(P, \Lambda), \Lambda)$ maps to the trivial Λ -module. An interesting question is therefore: *Do there exist strongly minimal models P such that $H^3(B\pi_1(P), \Lambda) \neq 0$?*

We prove the following:

Theorem 1.3 *Let π be a finitely presented group such that $H^2(B\pi, \Lambda)$ is not a torsion group. Let P and P' be strongly minimal models for π . Then P and P' are homotopy equivalent if the map $G : H_4(D, \mathbb{Z}) \rightarrow \text{Hom}_\Lambda(H^2(D, \Lambda), \overline{H}_2(D, \Lambda))$ is injective, and if the k -invariants of P and P' correspond appropriately.*

Here D is a 2-stage Postnikov space and G is defined via cap-products. Apart from the k -invariant, the injectivity of the map G is an essential condition for uniqueness of strongly minimal models. In Section 4 we consider groups π such that $B\pi$ is homotopy equivalent to a 2-complex and prove that for any element of $\text{Ker } G$ one can construct a strongly minimal model. More precisely, we obtain:

Theorem 1.4 *Suppose $B\pi$ is homotopy equivalent to a 2-complex, and $\pi_2 = H^2(B\pi, \Lambda)$ is not a torsion group. Then $\text{Ker } G \cong \Gamma(\pi_2) \otimes_\Lambda \mathbb{Z}$. Moreover, for any strongly minimal model P and any $\xi \in \Gamma(\pi_2)$, another strongly minimal model X can be constructed.*

Examples are given by solvable Baumslag–Solitar groups (see [5]), or by surface fundamental groups. In Section 5 we construct non-homotopy equivalent strongly minimal models for these fundamental groups.

2. Construction of degree 1-maps

In this section we are going to prove Proposition 1.1 announced in Section 1. First we mention a result of Wall [14].

Lemma 2.1 *Let $f : X \rightarrow Y$ be a degree 1-map between Poincaré 4-complexes and suppose that $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is an isomorphism. Then $K_2(f, \Lambda) = \text{Ker}(H_2(X, \Lambda) \rightarrow H_2(Y, \Lambda))$ is a stably Λ -free submodule of $H_2(X, \Lambda)$ and λ_X restricted to $K_2(f, \Lambda)$ is nonsingular. Also, $K_2(f, \Lambda) \subset H_2(X, \Lambda)$ is a direct summand.*

This section is devoted to proving a converse statement to Lemma 2.1.

First we will show Proposition 2.2. Before that, let us note that Λ has an anti-involution that permits a switch from Λ -left to Λ -right modules and to introduce compatible Λ -module structures on Hom-duals, etc. We follow Wall's convention and consider Λ -right modules.

Proposition 2.2 *Let X be a Poincaré 4-complex and $G \subset H_2(X, \Lambda)$ a Λ -free submodule so that λ_X restricts to a nonsingular Hermitian pairing on G . Then there exist a Poincaré 4-complex P and a degree 1-map $f : X \rightarrow P$ such that $f_* : \pi_1(X) \rightarrow \pi_1(P)$ is an isomorphism and $K_2(f, \Lambda) \cong G$.*

Proof We recall that λ_X is defined as the composite map

$$\begin{array}{ccc} H^2(X, \Lambda) \times H^2(X, \Lambda) & \xrightarrow{\cup} & H^4(X, \Lambda \otimes_{\mathbb{Z}} \Lambda) \cong H_0(X, \Lambda \otimes_{\mathbb{Z}} \Lambda) \cong \mathbb{Z} \otimes_{\Lambda} (\Lambda \otimes_{\mathbb{Z}} \Lambda) \\ \cong \uparrow & & \uparrow \cong \\ H_2(X, \Lambda) \times H_2(X, \Lambda) & \xrightarrow{\lambda_X} & \Lambda \cong \Lambda \otimes_{\Lambda} \Lambda \end{array}$$

and

$$\hat{\lambda}_X : H_2(X, \Lambda) \rightarrow \text{Hom}_{\Lambda}(H_2(X, \Lambda), \Lambda)$$

is the adjoint map of λ_X .

To construct P , we consider a Λ -base a_1, \dots, a_r of $G \subset H_2(X, \Lambda) \cong \pi_2(X)$, and

$$\varphi_1, \dots, \varphi_r : \mathbb{S}^2 \rightarrow X$$

representatives of a_1, \dots, a_r , respectively. Then P is obtained from X by adjoining 3-cells along $\varphi_1, \dots, \varphi_r$. So $X \subset P$, and

$$H_p(P, X, \Lambda) \cong \begin{cases} G & p = 3 \\ 0 & \text{otherwise} \end{cases} \quad H^p(P, X, \Lambda) \cong \begin{cases} G^* = \text{Hom}_{\Lambda}(G, \Lambda) & p = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the sequence

$$0 \longrightarrow H_3(P, X, \Lambda) \xrightarrow{\partial_*} H_2(X, \Lambda) \longrightarrow H_2(P, \Lambda) \longrightarrow 0$$

is exact because $\partial_* : H_3(P, G, \Lambda) \rightarrow G \subset H_2(X, \Lambda)$ is an isomorphism.

Note that there is a natural homomorphism

$$\mu : H^2(X, \Lambda) \rightarrow \text{Hom}_{\Lambda}(H_2(X, \Lambda), \Lambda)$$

such that the diagram

$$\begin{array}{ccc} H^2(X, \Lambda) & \xrightarrow{\mu} & \text{Hom}_{\Lambda}(H_2(X, \Lambda), \Lambda) \\ \cap[X] \downarrow & & \parallel \\ H_2(X, \Lambda) & \xrightarrow{\hat{\lambda}_X} & \text{Hom}_{\Lambda}(H_2(X, \Lambda), \Lambda) \end{array}$$

commutes. Let $[P] = f_*[X]$, where $f : X \subset P$ is the inclusion. Consider the diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(P, \Lambda) & \xrightarrow{f^*} & H^2(X, \Lambda) & \xrightarrow{\delta^*} & H^3(P, X, \Lambda) & \xrightarrow[\cong]{\mu} & \text{Hom}_{\Lambda}(H_3(P, X, \Lambda), \Lambda) = G^* \\ & & \cap[P] \downarrow & & \downarrow \cap[X] & & & & \uparrow \hat{\lambda}_G \\ 0 & \longleftarrow & H_2(P, \Lambda) & \xleftarrow{f_*} & H_2(X, \Lambda) & \xleftarrow{\partial_*} & H_3(P, X, \Lambda) = G & \longleftarrow & G & \longleftarrow & 0 \end{array}$$

and

$$\begin{array}{ccc}
 H^2(X, \Lambda) & \xrightarrow{\delta^*} & H^3(P, X, \Lambda) \\
 \mu \downarrow & & \cong \downarrow \mu \\
 \text{Hom}_\Lambda(H_2(X, \Lambda), \Lambda) & \longrightarrow & \text{Hom}_\Lambda(H_3(P, X, \Lambda), \Lambda) = G^* \\
 \uparrow \hat{\lambda}_X & & \uparrow \hat{\lambda}_G \\
 H_2(X, \Lambda) & \xleftarrow{\partial_*} & H_3(P, X, \Lambda) = G.
 \end{array}$$

Here $\hat{\lambda}_G = \hat{\lambda}_X|_G$. The left-hand square of the first diagram commutes. Combining the right-hand square of the first diagram with the second diagram gives only

$$\mu \circ \delta^* \circ (\cap[X])^{-1} \circ \partial_* = \hat{\lambda}_G.$$

However, this is sufficient to deduce that $\cap[P] : H^2(P, \Lambda) \rightarrow H_2(P, \Lambda)$ is an isomorphism. It follows from the above short exact sequence that

$$f_* : H_3(X, \Lambda) \xrightarrow{\cong} H_3(P, \Lambda) \quad f^* : H^3(P, \Lambda) \xrightarrow{\cong} H^3(X, \Lambda)$$

hence we obtain that

$$\cap[P] : H^*(P, \Lambda) \xrightarrow{\cong} H_{4-*}(P, \Lambda)$$

for all $*$. The map f is obviously of degree 1. □

In the sequel we shall need another result of Wall about Poincaré complexes (see for instance [14]).

Lemma 2.3 *Any Poincaré 4-complex X is homotopy equivalent to a CW-complex of the form $K \cup_\varphi D^4$, where K is a 3-complex and $\varphi : \mathbb{S}^3 \rightarrow K$ is an attaching map of the single 4-cell D^4 .*

Proposition 2.2 can be improved so that together with Lemma 2.1, we obtain the following:

Proposition 2.4 *Let X be a Poincaré 4-complex. There exists a degree 1-map $f : X \rightarrow Q$ if and only if there exists a stably free Λ -submodule $G \subset H_2(X, \Lambda)$ so that λ_X restricts to a nonsingular Hermitian form on G . In this case, $G \cong K_2(f, \Lambda)$.*

Proof By Lemma 2.3 we can identify $X = K \cup_\varphi D^4$. The submodule G is stably free, so $G \oplus H \cong \oplus_1^\ell \Lambda$, where H is Λ -free. We may assume $H = \oplus_1^{2m} \Lambda$. Let $Z = X \# (\#_1^m(\mathbb{S}^2 \times \mathbb{S}^2))$ be the Poincaré 4-complex formed from X by connected sum inside the 4-cell with $\#_1^m(\mathbb{S}^2 \times \mathbb{S}^2)$. Then $G \oplus H \subset H_2(Z, \Lambda)$ and λ_Z restricted to H is the canonical hyperbolic form. If $a_1, \dots, a_\ell \in G \oplus H$ is a Λ -base, we attach 3-cells to Z along representatives $\varphi_1, \dots, \varphi_\ell : \mathbb{S}^2 \rightarrow X$ as in Proposition 2.2. We obtain a Poincaré 4-complex Q and a degree 1-map $g : Z \rightarrow Q$ with $K_2(g, \Lambda) = G \oplus H$. We are going to show that g factors over the collapsing map

$$c : Z = X \# (\#_1^m(\mathbb{S}^2 \times \mathbb{S}^2)) \rightarrow X$$

giving a degree 1-map $f : X \rightarrow Q$. Note that

$$X \# (\#_1^m(\mathbb{S}^2 \times \mathbb{S}^2)) \setminus 4\text{-cell} \simeq K \vee \{\vee_1^m(\mathbb{S}^2 \vee \mathbb{S}^2)\}$$

and the attaching map of the 4-cell of Z is of the following type

$$a \oplus b \in \pi_3(K) \oplus [\pi_3(\vee_1^m(\mathbb{S}^2 \vee \mathbb{S}^2)) \otimes \Lambda] \subset \pi_3(Z \setminus (4\text{-cell})),$$

where $a = [\varphi]$ and $b = [\psi] \otimes 1$ with $\psi : \mathbb{S}^3 \rightarrow \vee_1^m(\mathbb{S}^2 \vee \mathbb{S}^2)$ the attaching map of the 4-cell of $\#_1^m(\mathbb{S}^2 \times \mathbb{S}^2)$. Obviously, $a \oplus b$ maps to zero in $\pi_3(Q)$.

Now we apply Whitehead's Γ -functor to

$$\pi_2(Z) \cong \pi_2(K) \oplus H \cong \pi_2(Z \setminus (4\text{-cell})) : \Gamma(\pi_2(Z)) \cong \Gamma(\pi_2(K)) \oplus \Gamma(H) \oplus \pi_2(K) \otimes H.$$

The Γ -functor fits into a certain Whitehead's exact sequence (see [1] and [15]) and by naturality one has the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\pi_2(K) \oplus H) & \longrightarrow & \pi_3(Z \setminus (4\text{-cell})) & \longrightarrow & H_3(Z \setminus (4\text{-cell}), \Lambda) & \longrightarrow & 0 \\ & & \parallel & & & & \parallel & & \\ 0 & \longrightarrow & \Gamma(\pi_2(K)) \oplus \Gamma(H) \oplus \pi_2(K) \otimes H & \longrightarrow & \pi_3(Z \setminus (4\text{-cell})) & \longrightarrow & H_3(K, \Lambda) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ H_4(Q, \Lambda) & \longrightarrow & \Gamma(\pi_2(Q)) & \longrightarrow & \pi_3(Q) & \longrightarrow & H_3(Q, \Lambda) & \longrightarrow & 0. \end{array}$$

Obviously, $b \in \Gamma(H) \subset \pi_3(Z \setminus (4\text{-cell}))$, and hence $b = \sum \lambda_{ij}[e_i, e_j]$, where the set $\{e_1, \dots, e_{2m}\} \subset H$ is the standard base and $[\cdot, \cdot]$ denotes the Whitehead product. Now $H \subset G \oplus H \subset \pi_2(Z)$ maps to zero under $g_* : \pi_2(Z) \rightarrow \pi_2(Q)$, so $b \in \Gamma(\pi_2(K) \oplus H)$ maps to zero in $\Gamma(\pi_2(Q))$, and hence it is zero in $\pi_3(Q)$. Because $a \oplus b$ is zero in $\pi_3(Q)$, $a \in \pi_3(K)$ also maps to zero under $\pi_3(K) \rightarrow \pi_3(Q)$. Therefore, the inclusion map $K \subset Q$ extends to $f : X \rightarrow Q$, and f induces a map

$$(X, K) \rightarrow (Q, Q \setminus (4\text{-cell})).$$

We also have

$$g : (Z, Z \setminus (4\text{-cell})) \rightarrow (Q, Q \setminus (4\text{-cell}))$$

and a collapsing map

$$c : (Z, Z \setminus (4\text{-cell})) \rightarrow (X, K).$$

Since Q is obtained from Z by adding 3-cells attached away from the 4-cell, the following diagram commutes:

$$\begin{array}{ccc} H_4(Z, Z \setminus (4\text{-cell}), \mathbb{Z}) & \xrightarrow{c_*} & H_4(X, K, \mathbb{Z}) \\ g_* \downarrow & & \downarrow f_* \\ H_4(Q, Q \setminus (4\text{-cell}), \mathbb{Z}) & \xlongequal{\quad} & H_4(Q, Q \setminus (4\text{-cell}), \mathbb{Z}). \end{array}$$

Because c_* and g_* map the fundamental class to the fundamental class, the degree of f is 1. □

Proof of Corollary 1.2 We observe that for any degree 1-map $f : X \rightarrow Y$ with $f_* : \pi_1(X) \xrightarrow{\cong} \pi_1(Y)$, one has

$$K_2(f, \Lambda) \otimes_{\Lambda} \mathbb{Z} = K_2(f, \mathbb{Z}) = \text{Ker}(H_2(X, \mathbb{Z}) \rightarrow H_2(Y, \mathbb{Z})),$$

and that $H_2(X, \mathbb{Z})$ is finitely generated. By Proposition 2.4 we can successively construct degree 1-maps

$$X \xrightarrow{f} Q, \quad Q_1 \xrightarrow{f_1} Q_2, \quad \dots$$

if we find nondegenerate stably free nontrivial submodules in $H_2(Q_k, \Lambda)$, and one has

$$K_2(f_k \circ \dots \circ f_1 \circ f, \Lambda) \cong K_2(f_k, \Lambda) \oplus \dots \oplus K_2(f_1, \Lambda) \oplus K_2(f, \Lambda) \subset H_2(X, \Lambda).$$

Now

$$K_2(f_k \circ \dots \circ f_1 \circ f, \mathbb{Z}) \cong K_2(f_k, \mathbb{Z}) \oplus \dots \oplus K_2(f_1, \mathbb{Z}) \oplus K_2(f, \mathbb{Z}) \subset H_2(X, \mathbb{Z})$$

is finitely generated. Hence, after certain k , we have

$$K_2(f_{k+1}, \Lambda) \otimes_{\Lambda} \mathbb{Z} = K_2(f_{k+1}, \mathbb{Z}) = \{0\}.$$

Kaplansky’s lemma (see remark below) implies $K_2(f_{k+1}, \Lambda) \cong 0$. Therefore, $g = f_k \circ \dots \circ f_1 \circ f : X \rightarrow Q_k$ is of degree 1, and Q_k is minimal. This completes the proof of Corollary 1.2.

Remark In [10, p.122], the following result is stated:

Lemma *Let \mathbb{F} be a field of characteristic zero, and π an arbitrary group. Let $A = \mathbb{F}[\pi]$ be the group algebra, and let $u, v \in M_n(A)$ be 2 ($n \times n$) matrices such that the product vu is the identity matrix I_n . Then $uv = I_n$.*

It has the following consequence (referred to above as “Kaplansky’s lemma”):

Corollary *If $K_2(f, \Lambda) \otimes_{\Lambda} \mathbb{Q} \cong 0$, then $K_2(f, \Lambda) \cong 0$.*

Proof We know that $K_2 = K_2(f, \Lambda)$ is stably free, i.e. $K_2 \oplus \Lambda^a \cong \Lambda^b$, where a and b are positive integers. Tensoring with \mathbb{Q} implies that $a = b$. Let $h : K_2 \oplus \Lambda^a \rightarrow \Lambda^b$ be an isomorphism, and consider

$$u = h \circ i : \Lambda^a \xrightarrow{c} K_2 \oplus \Lambda^a \xrightarrow{h} \Lambda^a$$

and

$$v = \text{pr} \circ h^{-1} : \Lambda^a \xrightarrow{c} K_2 \oplus \Lambda^a \xrightarrow{\text{pr}} \Lambda^a.$$

Obviously $v \circ u = \text{Id}$, and hence $u \circ v = \text{Id}$. This implies that $K_2 \subset \text{Ker}(u \circ v) \cong 0$. □

Note also that $K_2 \otimes_{\Lambda} \mathbb{Q} \cong 0$ is equivalent to $K_2 \otimes_{\Lambda} \mathbb{Z} \cong 0$.

Of course, starting with X one cannot in general assume that there is only one minimal P and degree 1-map $f : X \rightarrow P$ with $f_* : \pi_1(X) \xrightarrow{\cong} \pi_1(P)$.

Problem 2.5 *Construct examples of X that admit several minimal Poincaré 4-complexes P_i and degree 1-maps $f_i : X \rightarrow P_i$ satisfying $f_{i*} : \pi_1(X) \xrightarrow{\cong} \pi_1(P_i)$.*

The next proposition completes the description of the correspondence between stably free Λ -modules with nondegenerate Hermitian forms and degree 1-maps of Poincaré 4-complexes. However, we have to assume that $\pi_1(X)$ does not contain elements of order 2.

Proposition 2.6 *Let X be a Poincaré 4-complex and G a stably free Λ -module with nondegenerate Hermitian form. Then there are a Poincaré 4-complex Y and a degree 1-map $f : Y \rightarrow X$ such that $K_2(f, \Lambda) \cong G$, λ_Y restricted to $K_2(f, \Lambda)$ coincides with λ on G under the isomorphism. Moreover, $f_* : \pi_1(Y) \rightarrow \pi_1(X)$ is an isomorphism.*

Proof Let first G be free of rank m . The proof proceeds as in [7]. Here we begin with $Y' = X\#(\#_1^m \mathbb{C}P^2)$ and the Hermitian form λ , and continue as in Section 3 of [7] to construct $f : Y \rightarrow X$. If G is stably free, that is, $G \oplus H \cong \Lambda^m$, where $H = \Lambda^t$, we begin with $Y' = X\#(\#_1^m \mathbb{C}P^2)$ and the Hermitian form $\lambda' = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ on $G \oplus H$, and construct a degree 1-map $f'' : Y'' \rightarrow X$ with $K_2(f'', \Lambda) = G \oplus H$, and $\lambda_{Y''}$ restricted to $K_2(f'', \Lambda)$ is equal to λ' . Now $H \subset H_2(Y'', \Lambda)$ is Λ -free, and $\lambda_{Y''}$ restricted to H is non-singular. As in the proof of Proposition 2.4 we can construct a degree 1-map $f : Y \rightarrow X$ with $K_2(f, \Lambda) \cong G$. \square

3. A general result on the uniqueness of strongly minimal models

Let π be a finitely presented group. Suppose we are given strongly minimal Poincaré 4-complexes P and P' with $\pi_1(P) \cong \pi \cong \pi_1(P')$. For simplicity, we denote $\pi'_1 = \pi_1(P')$, $\pi_1 = \pi_1(P)$, $\Lambda' = \mathbb{Z}[\pi'_1]$, and $\Lambda = \mathbb{Z}[\pi_1]$. Then we have

$$\begin{aligned} p^* : H^2(B\pi_1, \Lambda) &\xrightarrow{\cong} H^2(P, \Lambda) \\ p'^* : H^2(B\pi'_1, \Lambda') &\xrightarrow{\cong} H^2(P', \Lambda') \end{aligned}$$

where $p : P \rightarrow B\pi_1$ and $p' : P' \rightarrow B\pi'_1$ are the classifying maps. We denote by $\chi : D \rightarrow B\pi_1$ and $\chi' : D' \rightarrow B\pi'_1$ the 2-stage Postnikov fibrations with fibers $K(\pi_2(P), 2)$ and $K(\pi_2(P'), 2)$, respectively. Spaces D and D' are obtained from P and P' , respectively, by adding cells of dimension ≥ 4 so that $\pi_q(D) \cong 0 \cong \pi_q(D')$ for every $q \geq 3$, and the inclusions $f : P \rightarrow D$ and $f' : P' \rightarrow D'$ induce isomorphisms $f_* : \pi_i(P) \rightarrow \pi_i(D)$ and $f'_* : \pi_i(P') \rightarrow \pi_i(D')$, for every $i < 3$. We shall often write it as diagrams

$$\begin{array}{ccc} P & \xrightarrow{f} & D \\ \parallel & & \downarrow \chi \\ P & \xrightarrow{p} & B\pi_1 \end{array} \qquad \begin{array}{ccc} P' & \xrightarrow{f'} & D' \\ \parallel & & \downarrow \chi' \\ P' & \xrightarrow{p'} & B\pi'_1. \end{array}$$

We choose an isomorphism $\alpha : \pi_1 \rightarrow \pi'_1$. It determines an isomorphism $\Lambda \rightarrow \Lambda'$ of rings. For the sake of simplicity we shall identify Λ' with Λ via this isomorphism when we use it as coefficients in (co)homology groups. We define

$$\beta : H_2(P, \Lambda) \rightarrow H_2(P', \Lambda)$$

by the following diagram

$$\begin{array}{ccccc} H^2(B\pi_1, \Lambda) & \xrightarrow[\cong]{p^*} & H^2(P, \Lambda) & \xrightarrow[\cong]{\cap[P]} & H_2(P, \Lambda) \\ (B\alpha)^* \uparrow & & & & \downarrow \beta \\ H^2(B\pi'_1, \Lambda) & \xrightarrow[\cong]{p'^*} & H^2(P', \Lambda) & \xrightarrow[\cong]{\cap[P']} & H_2(P', \Lambda). \end{array} \tag{3.1}$$

The next diagram explains the compatibility of the k -invariants $k_P^3 \in H^3(B\pi_1, \pi_2(P))$ and $k_{P'}^3 \in H^3(B\pi'_1, \pi_2(P'))$:

$$\begin{array}{ccc}
 \text{Hom}_\Lambda(H_2(P, \Lambda), H_2(P, \Lambda)) & \longrightarrow & H^3(B\pi_1, H_2(P, \Lambda)) \\
 \downarrow \beta_\# & & \downarrow \beta_\# \\
 \text{Hom}_\Lambda(H_2(P, \Lambda), H_2(P', \Lambda)) & \longrightarrow & H^3(B\pi_1, H_2(P', \Lambda)) \\
 \uparrow \beta^\# & & \uparrow (B\alpha)^* \\
 \text{Hom}_\Lambda(H_2(P', \Lambda), H_2(P', \Lambda)) & \longrightarrow & H^3(B\pi'_1, H_2(P', \Lambda))
 \end{array} \tag{3.2}$$

where the top (resp. bottom) horizontal map sends Id into k_P^3 (resp. $k_{P'}^3$), and on the left (resp. right) vertical side we have $\beta_\#(\text{Id}) = \beta = \beta^\#(\text{Id})$ (resp. $\beta_\#(k_P^3) = (B\alpha)^*(k_{P'}^3)$). Therefore, there is a homotopy equivalence $h : D \rightarrow D'$ such that the diagram

$$\begin{array}{ccc}
 D & \xrightarrow{h} & D' \\
 \downarrow \chi & & \downarrow \\
 B\pi_1 & \xrightarrow{B\alpha} & B\pi'_1
 \end{array}$$

commutes (up to homotopy). Furthermore, Diagram (3.1) can be completed to the following diagram

$$\begin{array}{ccccccccc}
 H^2(D, \Lambda) & \xleftarrow[\cong]{\chi^*} & H^2(B\pi_1, \Lambda) & \xrightarrow{p^*} & H^2(P, \Lambda) & \xrightarrow{\cap[P]} & H_2(P, \Lambda) & \xrightarrow{f_*} & H_2(D, \Lambda) \\
 h^* \uparrow & & \uparrow (B\alpha)^* & & & & \beta \downarrow & & \downarrow h_* \\
 H^2(D', \Lambda) & \xleftarrow[\cong]{\chi'^*} & H^2(B\pi'_1, \Lambda) & \xrightarrow{p'^*} & H^2(P', \Lambda) & \xrightarrow{\cap[P']} & H_2(P', \Lambda) & \xrightarrow{f'_*} & H_2(D', \Lambda)
 \end{array} \tag{3.3}$$

where

$$\begin{array}{ccc}
 H^2(D, \Lambda) & \xrightarrow{f_*} & H^2(P, \Lambda) & & H^2(D', \Lambda) & \xrightarrow{f'_*} & H^2(P', \Lambda) \\
 \parallel & & \uparrow p^* & & \parallel & & \uparrow p'^* \\
 H^2(D, \Lambda) & \xleftarrow[\cong]{\chi^*} & H^2(B\pi_1, \Lambda) & & H^2(D', \Lambda) & \xleftarrow[\cong]{\chi'^*} & H^2(B\pi'_1, \Lambda)
 \end{array}$$

Note that all the maps are Λ -isomorphisms.

At this point it is convenient to introduce the map

$$G : H_4(D, \mathbb{Z}) \rightarrow \text{Hom}_\Lambda(H^2(D, \Lambda), \overline{H}_2(D, \Lambda))$$

using the equivariant cap-product construction, and similarly G' for D' . From Diagram (3.1) we summarize as follows:

Corollary 3.1 *Diagram (3.1) commutes, and the composed horizontal homomorphisms (from left to right) are $G(f_*[P])$ and $G'(f'_*[P'])$.*

We again invoke Wall's theorem (Lemma 2.3) and identify

$$P = K \cup_\varphi D^4 \qquad P' = K' \cup_{\varphi'} D'^4$$

where K and K' are 3-complexes, and $\varphi : \mathbb{S}^3 \rightarrow K$ and $\varphi' : \mathbb{S}^3 \rightarrow K'$ are the attaching maps of the 4-cells D^4 and D'^4 , respectively. Hence, (D, K) and (D', K') are relative CW-complexes with cells in dimensions $k \geq 4$, that is, $D^{(3)} = K$ and $D'^{(3)} = K'$. Approximate $h : D \rightarrow D'$ by a cellular map (again denoted by h). Then

$$h^{(3)} = h|_K : K \rightarrow K'$$

and

$$\begin{array}{ccc} D & \xrightarrow{h} & D' \\ i \uparrow & & \uparrow i' \\ K & \xrightarrow{h^{(3)}} & K' \end{array}$$

commutes, where $i : K \subset D$ and $i' : K' \subset D'$ are the inclusion maps.

Proposition 3.2 (a) $h^{(3)} : K \rightarrow K'$ extends to $\phi : P \rightarrow P'$ if $h_* f_* [P] = \ell f'_* [P'] \in H_4(D', \mathbb{Z})$ for some $\ell \in \mathbb{Z}$; and

(b) If $f'_* : H_4(P', \mathbb{Z}) \rightarrow H_4(D', \mathbb{Z})$ is injective and $\ell = \pm 1$, then ϕ is of degree ± 1 ; hence, it is a homotopy equivalence.

Proof (a) The obstruction to extending $h^{(3)}$ belongs to

$$\begin{aligned} H^4(P, \pi_3(P')) &\cong H_0(P, \pi_3(P')) \cong \mathbb{Z} \otimes_{\Lambda} \pi_3(P') \cong \mathbb{Z} \otimes_{\Lambda} \pi_4(D', P') \\ &\cong \mathbb{Z} \otimes_{\Lambda} H_4(D', P', \Lambda) = H_4(D', P', \mathbb{Z}) \end{aligned}$$

(one applies among others: $\pi_3(D') = \pi_3(D) = 0$ and the Hurewicz theorem). The obstruction in $\mathbb{Z} \otimes_{\Lambda} \pi_3(P')$ is given by the image of $[h^{(3)} \circ \varphi] \in \pi_3(K')$ under the composite map

$$\pi_3(K') \longrightarrow \pi_3(P') \longrightarrow \pi_3(P') \otimes_{\Lambda} \mathbb{Z}.$$

The obstruction in $H_4(D', P', \mathbb{Z})$ is given by the induced map of the composition

$$(D^4, \mathbb{S}^3) \xrightarrow{\varphi} (P, K) \subset (D, K) \xrightarrow{h} (D', K') \subset (D', P')$$

and hence it is the image of $[P] \in H_4(P, \mathbb{Z})$ under the composition on the bottom horizontal row in the following diagram:

$$\begin{array}{ccccccc} H_4(P, K, \mathbb{Z}) & \longrightarrow & H_4(D, K, \mathbb{Z}) & \xrightarrow{h_*} & H_4(D', K', \mathbb{Z}) & \xlongequal{\quad} & H_4(D', K', \mathbb{Z}) \\ \cong \uparrow & & \uparrow & & \uparrow & & \downarrow \\ H_4(P, \mathbb{Z}) & \xrightarrow{f_*} & H_4(D, \mathbb{Z}) & \xrightarrow{h_*} & H_4(D', \mathbb{Z}) & \longrightarrow & H_4(D', P', \mathbb{Z}). \end{array} \tag{3.4}$$

Hence, the obstruction vanishes if and only if $h_* f_* [P] = \ell f'_* [P']$ for some $\ell \in \mathbb{Z}$.

(b) If $\phi : P \rightarrow P'$ exists, then it is such that the diagram

$$\begin{array}{ccccc}
 H_4(P, \mathbb{Z}) & \xrightarrow{\phi_*} & H_4(P', \mathbb{Z}) & \xrightarrow{f'_*} & H_4(D', \mathbb{Z}) \\
 \downarrow \cong & & \downarrow \cong & & \parallel \\
 H_4(P, K, \mathbb{Z}) & \xrightarrow{\phi_*} & H_4(P', K', \mathbb{Z}) & & H_4(D', \mathbb{Z}) \\
 \downarrow f_* & & \downarrow f'_* & & \parallel \\
 H_4(D, K, \mathbb{Z}) & \xrightarrow{h_*} & H_4(D', K', \mathbb{Z}) & \longleftarrow & H_4(D', \mathbb{Z})
 \end{array}$$

commutes. Hence, $f'_* \phi_* [P] = h_* f_* [P] = \pm f'_* [P']$ implies $\phi_* [P] = \pm [P']$ since f'_* is injective. Using the Poincaré duality one obtains

$$\phi_* : H_*(P, \Lambda) \xrightarrow{\cong} H_*(P', \Lambda).$$

Because $\phi_* : \pi_1(P) \rightarrow \pi_1(P')$ is an isomorphism, the map $\phi : P \rightarrow P'$ is a homotopy equivalence by the Hurewicz–Whitehead theorem. □

Proof of Theorem 1.3 We have a commutative diagram (up to homotopy)

$$\begin{array}{ccccccc}
 D & \xlongequal{\quad} & D & \xrightarrow{h} & D' & \xlongequal{\quad} & D' \\
 f \uparrow & & \downarrow & & \downarrow & & \uparrow f' \\
 P & \xrightarrow{p} & B\pi_1 & \xrightarrow{B\alpha} & B\pi'_1 & \xleftarrow{p'} & P'
 \end{array}$$

where $h : D \rightarrow D'$ is a homotopy equivalence. Consider the diagram

$$\begin{array}{ccc}
 H_4(D, \mathbb{Z}) & \xrightarrow{G} & \text{Hom}_\Lambda(H^2(D, \Lambda), \overline{H}_2(D, \Lambda)) \\
 h_* \downarrow & & \downarrow T \\
 H_4(D', \mathbb{Z}) & \xrightarrow{G'} & \text{Hom}_\Lambda(H^2(D', \Lambda), \overline{H}_2(D', \Lambda))
 \end{array} \tag{3.5}$$

where $\cap z$ is the cap product with $z \in H_4(D, \mathbb{Z})$. Similarly, \cap' . The map T is defined by $T(\xi) = h_* \circ \xi \circ h^*$. Note that T is an isomorphism. □

Lemma 3.3 *Diagram (3.5) commutes.*

Proof Given $x \in H_4(D, \mathbb{Z})$ and $u' \in H^2(D', \mathbb{Z})$, then we have

$$TG(x)(u') = h_*(h^*(u') \cap x) = u' \cap h_*(x) = G'h_*(x)$$

as required. □

Now consider the diagram

$$\begin{array}{ccccc}
 H_4(P, \mathbb{Z}) & \xrightarrow{f_*} & H_4(D, \mathbb{Z}) & \xrightarrow{G} & \text{Hom}_\Lambda(H^2(D, \Lambda), \overline{H}_2(D, \Lambda)) \\
 & & \downarrow h_* & & \downarrow T \\
 H_4(P', \mathbb{Z}) & \xrightarrow{f'_*} & H_4(D', \mathbb{Z}) & \xrightarrow{G'} & \text{Hom}_\Lambda(H^2(D', \Lambda), \overline{H}_2(D', \Lambda)).
 \end{array}$$

It follows from Corollary 3.1 that

$$TGf_*[P] = G'f'_*[P'],$$

and from $TG = G'h_*$ we get $G'h_*f_*[P] = G'f'_*[P']$; hence, $h_*f_*[P] = f'_*[P']$. So Proposition 3.2 (a) holds with $\ell = 1$.

A similar diagram as (5) holds for the space P' :

$$\begin{array}{ccc} H_4(P', \mathbb{Z}) & \xrightarrow{G''} & \text{Hom}_\Lambda(H^2(P', \Lambda), \overline{H}_2(P', \Lambda)) \cong \text{Hom}_\Lambda(\overline{H}_2(P', \Lambda), \overline{H}_2(P', \Lambda)) \\ f'_* \downarrow & & \downarrow T \\ H_4(D', \mathbb{Z}) & \xrightarrow{G'} & \text{Hom}_\Lambda(H^2(D', \Lambda), \overline{H}_2(D', \Lambda)) \end{array}$$

with $T(\xi) = f_* \circ \xi \circ f^*$. Since T is an isomorphism, f'_* is injective if and only if the map G'' is injective. Now observe that under the maps the generator $[P']$ goes to Id. The upper right isomorphism is induced by Poincaré duality. Hence G'' is injective if and only if Id is not of finite order. Now $H_2(P', \Lambda) \cong H^2(B\pi'_1, \Lambda) \cong H^2(B\pi_1, \Lambda)$. The claim now follows from Proposition 3.2(b).

4. Construction of strongly minimal models

The principal examples of fundamental groups π admitting a strongly minimal model P are discussed in [5]. These are groups of geometric dimension equal to 2, i.e. $B\pi$ is a 2-dimensional aspherical complex. It is easy to see that the boundary of a regular neighborhood N of an embedding $B\pi \subset \mathbb{R}^5$ is a strongly minimal model for π (see [5]). Here we show that the map G is not injective, and hence we cannot expect uniqueness up to homotopy equivalence. In fact, we are going to classify all strongly minimal models fixing π by elements of the kernel of G . Note that all k -invariants vanish since $B\pi$ is a 2-complex. We assume $H_4(P, \Lambda) \cong 0$, i.e. that π is infinite (which holds for the known examples).

4.1. Computation of Ker G

We fix π as above, and for convenience also one strongly minimal model P , say $P = \partial N$. We have the following 2-stage Postnikov system.

$$\begin{array}{ccc} D & \xrightarrow{x} & B\pi \\ f \uparrow & & \uparrow p \\ P & \xlongequal{\quad} & P \end{array}$$

Lemma 4.1 *There is an exact sequence*

$$0 \longrightarrow \Gamma(\pi_2) \otimes_\Lambda \mathbb{Z} \longrightarrow H_4(D, \mathbb{Z}) \longrightarrow H_2(B\pi, H_2(D, \Lambda)) \longrightarrow 0$$

where $\pi_2 = \pi_2(P) \cong \pi_2(D)$.

Proof This follows from the spectral sequence

$$E_{pq}^2 = H_p(B\pi, H_q(D, \Lambda)) \xrightarrow{p+q=n} H_n(D, \mathbb{Z}).$$

Taking $n = 4$, we have $E_{pq}^2 = E_{pq}^\infty = [F_p H_4(D, \mathbb{Z})]/[F_{p-1} H_4(D, \mathbb{Z})]$ with filtration

$$0 \cong F_{-1} H_4 \subset F_0 H_4 \subset F_1 H_4 \subset F_2 H_4 \subset F_3 H_4 \subset F_4 H_4(D, \mathbb{Z}) = H_4(D, \mathbb{Z}).$$

The result follows since $E_{22}^2 = H_2(B\pi, H_2(D, \Lambda))$, $E_{04}^2 = H_0(B\pi, H_4(D, \Lambda)) = H_4(D, \Lambda) \otimes_\Lambda \mathbb{Z}$, and $E_{pq}^2 \cong 0$ else for $p + q = 4$. □

Remark Similarly one gets the exact sequence

$$0 \longrightarrow H_1(P, H_3(P, \Lambda)) \longrightarrow H_4(P, \mathbb{Z}) \longrightarrow H_2(B\pi, H_2(P, \Lambda)) \longrightarrow 0.$$

In particular, $H_2(B\pi, H_2(D, \Lambda))$ is a quotient of \mathbb{Z} because $H_2(D, \Lambda) \cong H_2(P, \Lambda)$ and $H_4(P, \mathbb{Z}) \cong \mathbb{Z}$.

Lemma 4.2 *The kernel of*

$$G : H_4(D, \mathbb{Z}) \rightarrow \text{Hom}_{\Lambda-\Lambda}(H^2(D, \Lambda), \overline{H}_2(D, \Lambda))$$

is $\Gamma(\pi_2) \otimes_\Lambda \mathbb{Z}$.

Proof The map $\chi^* : H^2(B\pi, \Lambda) \rightarrow H^2(D, \Lambda)$ is an isomorphism, and $H^2(B\pi, \Lambda) \cong [\text{Hom}_\Lambda(C_2(\widetilde{B}\pi), \Lambda)]/[\text{Im } \delta^1]$, where

$$\delta^1 : \text{Hom}_\Lambda(C_1(\widetilde{B}\pi), \Lambda) \rightarrow \text{Hom}_\Lambda(C_2(\widetilde{B}\pi), \Lambda)$$

is the co-boundary map. The composition

$$\begin{array}{ccc} \text{Hom}_{\Lambda-\Lambda}(H^2(B\pi, \Lambda), \overline{H}_2(D, \Lambda)) & \longrightarrow & \text{Hom}_\Lambda(\text{Hom}_\Lambda(C_2(\widetilde{B}\pi), \Lambda), \overline{H}_2(D, \Lambda)) \\ \cong \uparrow & & \\ \text{Hom}_\Lambda(H^2(D, \Lambda), H_2(D, \Lambda)) & & \end{array}$$

is obviously injective. Because $C_2(\widetilde{B}\pi)$ is Λ -free, there is a canonical isomorphism

$$\text{Hom}_{\Lambda-\Lambda}(\text{Hom}_\Lambda(C_2(\widetilde{B}\pi), \Lambda), \overline{H}_2(D, \Lambda)) \cong C_2(\widetilde{B}\pi) \otimes_\Lambda H_2(D, \Lambda).$$

Composing all these maps gives an injective map

$$\text{Hom}_{\Lambda-\Lambda}(H^2(D, \Lambda), \overline{H}_2(D, \Lambda)) \rightarrow C_2(\widetilde{B}\pi) \otimes_\Lambda H_2(D, \Lambda).$$

The composition with G gives a map $H_4(D, \mathbb{Z}) \rightarrow C_2(\widetilde{B}\pi) \otimes_\Lambda H_2(D, \Lambda)$ with image of the 2-cycle subgroup of the complex $C_*(\widetilde{B}\pi) \otimes_\Lambda H_2(D, \Lambda)$, i.e. $H_2(B\pi, H_2(D, \Lambda))$. This is the map $H_4(D, \mathbb{Z}) \rightarrow H_2(B\pi, H_2(D, \Lambda))$ of Lemma 4.1. In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{\Lambda-\Lambda}(H^2(D, \Lambda), \overline{H}_2(D, \Lambda)) & \longrightarrow & H_2(B\pi, H_2(D, \Lambda)) \\ \uparrow G & & \uparrow \\ H_4(D, \mathbb{Z}) & \xlongequal{\quad} & H_4(D, \mathbb{Z}) \end{array}$$

where the horizontal map is injective. The result now follows from Lemma 4.1. □

Supplement to Lemma 4.1. If P and P' are 2 strongly minimal models for π , let

$$\begin{array}{ccc} P & \xrightarrow{f} & D \\ \parallel & & \downarrow \chi \\ P & \xrightarrow[p]{} & B\pi \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{f'} & D' \\ \parallel & & \downarrow \chi' \\ P' & \xrightarrow[p']{} & B\pi \end{array}$$

be the 2 associated 2-stage Postnikov systems. Let $h : D \rightarrow D'$ be the homotopy equivalence constructed in Section 3. Then the diagram

$$\begin{array}{ccc} H_4(D, \mathbb{Z}) & \longrightarrow & H_2(B\pi, H_2(D, \Lambda)) \\ h_* \downarrow & & \downarrow \\ H_4(D', \mathbb{Z}) & \longrightarrow & H_2(B\pi, H_2(D', \Lambda)) \end{array}$$

commutes. The right vertical map is induced by $h_* : H_2(D, \Lambda) \rightarrow H_2(D', \Lambda)$.

4.2. Construction of strongly minimal models

We choose a strongly minimal model P for π . By Wall's theorem [13], P is homotopy equivalent to $K \cup_{\varphi_1} D^4$, where K is a 3-complex, and $\varphi_1 : \mathbb{S}^3 \rightarrow K$ is the attaching map of the only 4-cell. This representation is unique, i.e. given a homotopy equivalence

$$K_1 \cup_{\varphi_1} D^4 \xrightarrow{h} K_2 \cup_{\varphi_2} D^4$$

there is a homotopy equivalence of pairs $(K_1, \varphi_1(\mathbb{S}^3)) \rightarrow (K_2, \varphi_2(\mathbb{S}^3))$ (see [13, p.222]). We simply write $P = K \cup_{\varphi_1} D^4$ and change the attaching map $[\varphi_1] \in \pi_3(K)$ by an element $[\varphi] \in \Gamma(\pi_2)$, i.e. $[\varphi] \in \Gamma(\pi_2) = \text{Im}(\pi_3(K^{(2)}) \rightarrow \pi_3(K))$, and we consider $X = K \cup_{\varphi_2} D^4$, where $\varphi_2 = \varphi_1 + \varphi$ and $\varphi : \mathbb{S}^3 \rightarrow K^{(2)}$. Let $q : X \rightarrow B\pi$ be the classifying map. It follows that $q^* : H^2(B\pi, \Lambda) \rightarrow H^2(X, \Lambda)$ is an isomorphism. If X is a Poincaré 4-complex, then X is a strongly minimal model for π .

4.3. Proof of the Poincaré duality

(I) We have an isomorphism $\pi_4(X, K) \rightarrow H_4(X, K, \Lambda) \cong \Lambda$. Let us consider the diagram of Whitehead's sequences:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \pi_4(X, K) & \xrightarrow{\cong} & H_4(X, K, \Lambda) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(\pi_2) & \longrightarrow & \pi_3(K) & \longrightarrow & H_3(K, \Lambda) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ & & \Gamma(\pi_2) & \longrightarrow & \pi_3(X) & \longrightarrow & H_3(X, \Lambda) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

One has a similar diagram if we replace X by P . Under the Hurewicz map, $[\varphi_1]$ and $[\varphi_2]$ go to the same element in $H_3(K, \Lambda)$, which coincides with the images of the generators of $H_4(P, K, \Lambda)$ resp. $H_4(X, K, \Lambda)$ under the connecting homomorphism, and hence $H_3(X, \Lambda) \cong H_3(P, \Lambda)$. Moreover, this gives us the following:

Lemma 4.3 $H_4(X, \mathbb{Z}) \cong \mathbb{Z}$

Proof Tensoring with $\otimes_{\Lambda} \mathbb{Z}$ the upper part of the above diagram gives

$$\begin{array}{ccccc} \pi_4(X, K) \otimes_{\Lambda} \mathbb{Z} & \xrightarrow{\cong} & H_4(X, K, \Lambda) \otimes_{\Lambda} \mathbb{Z} & \xrightarrow{\cong} & H_4(X, K, \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(\pi_2) \otimes_{\Lambda} \mathbb{Z} & \longrightarrow & \pi_3(K) \otimes_{\Lambda} \mathbb{Z} & \longrightarrow & H_3(K, \mathbb{Z}) \end{array}$$

and similarly for X replaced by P (we do not claim the exactness of the lower row). Now $H_4(P, K, \mathbb{Z}) \rightarrow H_3(K, \mathbb{Z})$ is the zero map. By the argument above, $[\varphi_1] \otimes_{\Lambda} 1$ and $[\varphi_2] \otimes_{\Lambda} 1$ map to the same element in $H_3(K, \Lambda) \otimes_{\Lambda} \mathbb{Z}$, and hence the generators of $H_4(X, K, \mathbb{Z})$ resp. $H_4(P, K, \mathbb{Z})$ map to the same element in $H_3(K, \mathbb{Z})$ under the connecting homomorphisms. Thus, $H_4(X, K, \mathbb{Z}) \rightarrow H_3(K, \mathbb{Z})$ is the zero map. Therefore, there is an isomorphism $H_4(X, \mathbb{Z}) \rightarrow H_4(X, K, \mathbb{Z}) \cong \mathbb{Z}$. \square

Let $[X] \in H_4(X, \mathbb{Z})$ be a generator. We have to study

$$\cap[X] : H^p(X, \Lambda) \rightarrow H_{4-p}(X, \Lambda).$$

To examine the cases $p = 1$ and $p = 3$, we introduce an auxiliary space $Y = K \cup_{\varphi_1, \varphi} \{D^4, D^4\}$, obtained from K by attaching two 4-cells with attaching maps φ_1 and φ . Note that $Y = P \cup_{\varphi} D^4$.

(II) Case $p = 1$

Let $i : P \rightarrow Y$ be the inclusion, and $j : X \rightarrow Y$ be the map induced by $K \subset Y$ and

$$\varphi_2 = \varphi_1 + \varphi : \mathbb{S}^3 \longrightarrow \mathbb{S}^3 \vee \mathbb{S}^3 \xrightarrow{\varphi_1 \vee \varphi} K.$$

We have the following maps of pairs:

$$\begin{array}{ccc} (D^4, \mathbb{S}^3) & \xrightarrow{\bar{i} \circ \bar{\varphi}_1} & (Y, K) \\ \bar{\varphi}_1 \downarrow & & \uparrow \bar{i} \\ (P, K) & \xlongequal{\quad} & (P, K) \end{array} \quad \begin{array}{ccc} (D^4, \mathbb{S}^3) & \xrightarrow{\bar{j} \circ \bar{\varphi}_2} & (Y, K) \\ \bar{\varphi}_2 \downarrow & & \uparrow \bar{j} \\ (X, K) & \xlongequal{\quad} & (X, K) \end{array}$$

and $\bar{\varphi} : (D^4, \mathbb{S}^3) \rightarrow (Y, K)$. Obviously, $\bar{\varphi}_2 = \bar{\varphi}_1 + \bar{\varphi} : (D^4, \mathbb{S}^3) \rightarrow (Y, K)$ is the 4-cell $[\varphi_1]$ “slided” over $[\varphi]$. Since $[\bar{\varphi}] \in \Gamma(\pi_2)$, $\bar{\varphi}$ factors as follows:

$$\begin{array}{ccc} (D^4, \mathbb{S}^3) & \xrightarrow{\bar{k} \circ \bar{\varphi}} & (Y, K) \\ \bar{\varphi} \downarrow & & \uparrow \bar{k} \\ (K^{(2)} \cup_{\varphi} D^4, K^{(2)}) & \xlongequal{\quad} & (K^{(2)} \cup_{\varphi} D^4, K^{(2)}) \end{array}$$

From this one sees that $\bar{j}_*[\bar{\varphi}_2] - \bar{i}_*[\bar{\varphi}_1]$ belongs to

$$\text{Im}(H_4(K^{(2)} \cup_{\varphi} D^4, K^{(2)}) \rightarrow H_4(Y, K)).$$

The diagram

$$\begin{array}{ccccc} H_4(X) & \xrightarrow{j_*} & H_4(Y) & \xleftarrow{i_*} & H_4(P) \\ \cong \downarrow & & \downarrow & & \downarrow \cong \\ H_4(X, K) & \xrightarrow{\bar{j}_*} & H_4(Y, K) & \xleftarrow{\bar{i}_*} & H_4(P, K) \end{array}$$

as well as injectivity of $H_4(Y) \rightarrow H_4(Y, K)$ and the isomorphism

$$H_4(K^{(2)} \cup_{\varphi} D^4) \rightarrow H_4(K^{(2)} \cup_{\varphi} D^4, K^{(2)})$$

prove the following:

Lemma 4.4 $j_*[X] - i_*[P]$ belongs to $\text{Im}(H_4(K^{(2)} \cup_{\varphi} D^4) \rightarrow H_4(Y))$.

Corollary 4.5 Taking cap-products with $i_*[P]$ and $j_*[X] : H^1(Y, \Lambda) \rightarrow H_3(Y, \Lambda)$ gives the same map.

Proof Let $\theta \in H_4(K^{(2)} \cup_{\varphi} D^4)$ map to $j_*[X] - i_*[P]$. Then the diagram

$$\begin{array}{ccc} H^1(Y, \Lambda) & \xrightarrow{\cap j_*[X] - \cap i_*[P]} & H_3(Y, \Lambda) \\ \cong \downarrow & & \uparrow \\ H^1(K^{(2)} \cup_{\varphi} D^4, \Lambda) & \xrightarrow{-\cap \theta} & H_3(K^{(2)} \cup_{\varphi} D^4, \Lambda) \cong 0 \end{array}$$

commutes. □

Lemma 4.6 $i_* : H_3(P, \Lambda) \rightarrow H_3(Y, \Lambda)$ is an isomorphism.

Proof Since $Y = P \cup_{\varphi} D^4$, i_* is surjective. Let us consider the diagram

$$\begin{array}{ccccc} H_4(K^{(2)} \cup_{\varphi} D^4, K^{(2)}, \Lambda) & \longrightarrow & H_4(Y, P, \Lambda) & \longrightarrow & H_3(P, \Lambda) \\ \cong \uparrow & & \cong \uparrow & & \uparrow \\ H_4(K^{(2)} \cup_{\varphi} D^4, \Lambda) & \xrightarrow{\cong} & H_4(K^{(2)} \cup_{\varphi} D^4, K^{(2)}, \Lambda) & \longrightarrow & H_3(K^{(2)} \cup_{\varphi} D^4, \Lambda), \end{array}$$

which shows that $H_4(Y, P, \Lambda) \rightarrow H_3(P, \Lambda)$ is the zero map. □

Lemma 4.7 $j_* : H_3(X, \Lambda) \rightarrow H_3(Y, \Lambda)$ is an isomorphism.

Proof The map j_* is surjective because $Y^{(3)} = K = X^{(3)}$. We identify $H_4(Y, K, \Lambda) \cong \Lambda \oplus \Lambda$ according to the diagram

$$\begin{array}{ccccc}
 & 0 & 0 & 0 & \\
 & \downarrow & \downarrow & \uparrow & \\
 H_4(D^4, \mathbb{S}^3, \Lambda) & \xrightarrow[\cong]{\bar{\varphi}_{1*}} & H_4(P, K, \Lambda) & & H_4(Y, X, \Lambda) \\
 & \bar{i}_* \downarrow & \downarrow & \uparrow & \\
 & H_4(Y, K, \Lambda) & \cong & H_4(Y, K, \Lambda) & \cong & H_4(Y, K, \Lambda) & \xleftarrow{\bar{k}_* \bar{\varphi}_*} & H_4(D^4, \mathbb{S}^3, \Lambda) \\
 & \bar{j}_* \uparrow & & & \downarrow & & \downarrow \cong & \\
 H_4(D^4, \mathbb{S}^3, \Lambda) & \xrightarrow[\cong]{\bar{\varphi}_{1*} + \bar{\varphi}_*} & H_4(X, K, \Lambda) & & H_4(Y, P, \Lambda) & \cong & H_4(Y, P, \Lambda) \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

where $\bar{i}_*[\bar{\varphi}_1] = (1, 0) \in \Lambda \oplus \Lambda$ and $\bar{k}_*[\bar{\varphi}] = (0, 1) \in \Lambda \oplus \Lambda$. The map $\bar{k}_* \bar{\varphi}_*$ defines a splitting of $H_4(Y, P, \Lambda) \rightarrow H_4(Y, K, \Lambda)$. Since: $H_4(Y, \Lambda) \rightarrow H_4(Y, P, \Lambda)$ is an isomorphism (here we use our assumption $H_4(P, \Lambda) \cong 0$ and Lemma 4.6), the image of $H_4(Y, \Lambda)$ in $H_4(Y, K, \Lambda) \cong \Lambda \oplus \Lambda$ is generated by $(0, 1)$. Thus, we can write the following diagram.

$$\begin{array}{ccccc}
 & & H_4(Y, \Lambda) & & \\
 & & \downarrow & & \\
 \Lambda & \longrightarrow & \Lambda \oplus \Lambda & \longrightarrow & (\Lambda \oplus \Lambda)/\Lambda(1, 1) \\
 \parallel & & \parallel & & \\
 \Lambda & \longrightarrow & \Lambda \oplus \Lambda & \longrightarrow & \Lambda
 \end{array}$$

The map \bar{j}_* corresponds to $\Lambda \rightarrow \Lambda \oplus \Lambda$ defined by $1 \rightarrow (1, 1)$. Hence, the map $H_4(Y, \Lambda) \rightarrow H_4(Y, X, \Lambda)$ corresponds to the isomorphism $\Lambda \rightarrow (\Lambda \oplus \Lambda)/\Lambda(1, 1)$ defined by $1 \rightarrow [(0, 1)]$, the class of $(0, 1)$ in the quotient. Therefore, we have an isomorphism $H_3(X, \Lambda) \rightarrow H_3(Y, \Lambda)$. \square

Lemma 4.8 *The map $\cap[X] : H^1(X, \Lambda) \rightarrow H_3(X, \Lambda)$ is an isomorphism.*

Proof This follows from the diagram

$$\begin{array}{ccc}
 H^1(X, \Lambda) & \xrightarrow{\cap[X]} & H_3(X, \Lambda) \\
 j^* \uparrow \cong & & \cong \downarrow j_* \\
 H^1(Y, \Lambda) & \xrightarrow{\cap j_*[X]} & H_3(Y, \Lambda) \\
 i^* \downarrow \cong & & \cong \uparrow i_* \\
 H^1(P, \Lambda) & \xrightarrow[\cong]{\cap[P]} & H_3(P, \Lambda)
 \end{array}$$

and $\cap j_*[X] = \cap i_*[P] : H^1(Y, \Lambda) \rightarrow H_3(Y, \Lambda)$. \square

(III) Case $p = 3$

Now we look N at the case $\cap[X] : H^3(X, \Lambda) \rightarrow H_1(X, \Lambda) \cong 0$, i.e. we have to show that $H^3(X, \Lambda) \cong 0$. Note that the sequence

$$0 \longrightarrow H^3(K, \Lambda) \longrightarrow H^4(P, K, \Lambda) \longrightarrow H^4(P, \Lambda) \longrightarrow 0$$

is exact. Since $H^4(P, \Lambda) \cong H_0(P, \Lambda) \cong \mathbb{Z}$, this sequence coincides with

$$0 \longrightarrow I(\Lambda) \longrightarrow \Lambda \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where ϵ is the augmentation, and $I(\Lambda) = \text{Ker } \epsilon$. Let us consider the following diagram.

$$\begin{array}{ccccc} H^3(K, \Lambda) & \longrightarrow & H^4(Y, P, \Lambda) & \xrightarrow[\cong]{\bar{\varphi}^*} & H^4(D^4, \mathbb{S}^3, \Lambda) \\ \parallel & & \uparrow & & \\ H^3(K, \Lambda) & \longrightarrow & H^4(Y, K, \Lambda) & \xrightarrow{\bar{j}^*} & H^4(X, K, \Lambda) \\ \parallel & & \downarrow \bar{i}_* & & \\ H^3(K, \Lambda) & \longrightarrow & H^4(P, K, \Lambda) & \xrightarrow[\cong]{\bar{\varphi}_1^*} & H^4(D^4, \mathbb{S}^3, \Lambda) \end{array}$$

The 2 vertical maps split $H^4(Y, K, \Lambda) \cong \Lambda \oplus \Lambda$ so that

$$\bar{i}^* : H^4(Y, K, \Lambda) \cong \Lambda \oplus \Lambda \rightarrow \Lambda \cong H^4(P, K, \Lambda)$$

projects onto the first component and $H^4(Y, K, \Lambda) \rightarrow H^4(Y, P, \Lambda) \cong \Lambda$ projects onto the second component. Since the composition $H^3(K, \Lambda) \rightarrow H^4(Y, P, \Lambda)$ is the zero map, we can identify the image of $H^3(K, \Lambda) \rightarrow H^4(Y, K, \Lambda)$ with $(I(\Lambda), 0) \subset \Lambda \oplus \Lambda$. The map \bar{j}^* is the sum $\Lambda \oplus \Lambda \rightarrow \Lambda$ since the generator of $H^4(X, K, \Lambda) \cong \Lambda$ maps under

$$\bar{\varphi}_1^* + \bar{\varphi}^* : H^4(X, K, \Lambda) \rightarrow H^4(D^4, \mathbb{S}^3, \Lambda) \cong \Lambda$$

to a generator. Hence, the image of

$$H^3(K, \Lambda) \longrightarrow H^4(Y, K, \Lambda) \xrightarrow{\bar{j}^*} H^4(X, K, \Lambda)$$

is $I(\Lambda) \subset \Lambda$, i.e. $H^3(K, \Lambda) \rightarrow H^4(X, K, \Lambda)$ is injective. The long exact sequence of the pair (X, K) implies $H^3(X, \Lambda) \cong 0$.

(IV) Case $p = 4$

Remark The last argument also implies $H^4(X, \Lambda) \cong \Lambda/I(\Lambda) \cong \mathbb{Z}$. We have proven the first part of the following:

Lemma 4.9 $H^3(X, \Lambda) \cong 0$, $H^4(X, \Lambda) \cong \mathbb{Z}$, and $\cap[X] : H^4(X, \Lambda) \rightarrow H_0(X, \Lambda)$ is an isomorphism.

Proof The second part follows from the well-known property of cap-products indicated in the following diagram:

$$\begin{array}{ccc} \mathbb{Z} \cong H^4(X, \Lambda) & \xrightarrow{\cap[X]} & H_0(X, \Lambda) \cong \mathbb{Z} \\ \epsilon \uparrow & & \uparrow \epsilon \\ \Lambda \cong H^4(X, K, \Lambda) = \text{Hom}_\Lambda(C_4(\tilde{X}, \tilde{K}), \Lambda) & \xrightarrow[\cong]{A} & C_0(\tilde{X}) \cong \Lambda \end{array}$$

Here $A(\alpha) = \alpha(1)$, $1 \in C_4(\tilde{X}, \tilde{K})$ being the generator. Observe that $H_0(X, \Lambda) = C_0(\tilde{X})/\partial_1 C_1(\tilde{X})$, so ϵ corresponds to the canonical map $C_0(\tilde{X}) \rightarrow C_0(\tilde{X})/\partial_1 C_1(\tilde{X})$ (we may assume that X has one 0-cell). \square

(V) Case $p = 2$

Recall the 2-stage Postnikov system for P :

$$\begin{array}{ccc} P & \xrightarrow{f} & D \\ \parallel & & \downarrow x \\ P & \xrightarrow{p} & B\pi. \end{array}$$

Let $f_0 = f|_K$. Given any $\psi : \mathbb{S}^3 \rightarrow K$, a canonical map $g : K \cup_\psi D^4 \rightarrow D$ can be constructed as follows: Let $H : \mathbb{S}^3 \times I \rightarrow D$ be the zero homotopy of the composition $f_0 \circ \psi : \mathbb{S}^3 \rightarrow D$. It factors over

$$D^4 = (\mathbb{S}^3 \times I)/\mathbb{S}^3 \times \{1\} \xrightarrow{\hat{H}} D.$$

Then $g = f_0 \cup \hat{H} : K \cup_\psi D^4 \rightarrow D$. Since $\pi_q(D) \cong 0$ for $q \geq 3$, g is unique up to homotopy. In our case, we have $\psi = \varphi_2 = \varphi_1 + \varphi$ with $[\varphi] \in \Gamma(\pi_2)$, where $\varphi : \mathbb{S}^3 \rightarrow K^{(2)}$, i.e. we need the zero homotopy of the composition

$$\mathbb{S}^3 \longrightarrow \mathbb{S}^3 \vee \mathbb{S}^3 \xrightarrow{\varphi_1 \vee \varphi} K \vee K^{(2)} \xrightarrow{f_0 \vee f_0} D \vee D \longrightarrow D.$$

We take the wedge of the zero homotopies $H : \mathbb{S}^3 \times I \rightarrow D$ for $f_0 \circ \varphi_1$ and $H_0 : \mathbb{S}^3 \times I \rightarrow D$ for $f_0 \circ \varphi$. This gives us the following:

Lemma 4.10 *Let $g_0 = f_0 \cup \hat{H}_0 : K^{(2)} \cup_\varphi D^4 \rightarrow D$ denote the canonical extension and $\theta \in H_4(K^{(2)} \cup_\varphi D^4, \mathbb{Z})$ the canonical generator. Then we have*

$$g_*[X] = f_*[P] + (g_0)_*(\theta).$$

Corollary 4.11 $(g_0)_*(\theta) \in \text{Ker } G \subset H_4(D, \mathbb{Z})$. In particular,

$$\cap f_*[P] = \cap g_*[X] : H^2(D, \Lambda) \rightarrow H_2(D, \Lambda);$$

that is, the map $\cap[X] : H^2(X, \Lambda) \rightarrow H_2(X, \Lambda)$ is an isomorphism.

Proof The above spectral sequence applied to $K^{(2)} \cup_\varphi D^4$ gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \otimes_\Lambda H_4(K^{(2)} \cup_\varphi D^4, \Lambda) & \longrightarrow & H_4(K^{(2)} \cup_\varphi D^4, \mathbb{Z}) & & \\ & & & & \longrightarrow & H_2(B\pi, H_2(K^{(2)} \cup_\varphi D^4, \Lambda)) & \longrightarrow 0. \end{array}$$

The first map is an isomorphism, so $H_2(B\pi, H_2(K^{(2)} \cup_{\varphi} D^4, \Lambda)) \cong 0$. Comparison with the exact sequence for D :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z} \otimes_{\Lambda} H_4(K^{(2)} \cup_{\varphi} D^4, \Lambda) & \longrightarrow & H_4(K^{(2)} \cup_{\varphi} D^4, \mathbb{Z}) & \longrightarrow & 0 \\
 & & & & \downarrow (g_0)_* & & \downarrow \\
 & & & & G : H_4(D, \mathbb{Z}) & \longrightarrow & \text{Hom}_{\Lambda}(H^2(D, \Lambda), H_2(D, \Lambda))
 \end{array}$$

gives the result. □

Theorem 4.12 *Suppose $B\pi$ is homotopy equivalent to a 2-dimensional complex. Let $\pi_2 = H^2(B\pi, \Lambda)$. Then, if we fix one model P , we obtain all models by the above construction.*

Proof Fixing P , we constructed for any $[\varphi] \in \pi_2$ a strongly minimal model. Conversely, let $X = K \cup_{\psi} D^4$ be a minimal model, where $\psi : \mathbb{S}^3 \rightarrow K$ is the attaching map. The map $f : X \rightarrow D$ into the 2-stage Postnikov space D is given by the zero homotopy of

$$\mathbb{S}^3 \xrightarrow{\psi} K \xrightarrow{f_0} D;$$

that is,

$$\begin{array}{ccc}
 \mathbb{S}^3 \times I & \xrightarrow{H} & D \\
 \downarrow & & \uparrow \hat{H} \\
 D^4 = (\mathbb{S}^3 \times I) / \mathbb{S}^3 \times \{1\} & \xlongequal{\quad} & D^4
 \end{array}$$

with $f = f_0 \cup \hat{H}$. Let us consider $\hat{H} : (D^4, \mathbb{S}^3) \rightarrow (D, K)$ and let

$$\bar{\psi} : (D^4, \mathbb{S}^3) \rightarrow (X, K)$$

be the top cell. The diagram

$$\begin{array}{ccccc}
 H_4(X, \mathbb{Z}) & \xrightarrow{\cong} & H_4(X, K, \mathbb{Z}) & \xleftarrow[\cong]{\bar{\psi}} & H_4(D^4, \mathbb{S}^3, \mathbb{Z}) \\
 f_* \downarrow & & & & \parallel \\
 H_4(D, \mathbb{Z}) & \longrightarrow & H_4(D, K, \mathbb{Z}) & \xleftarrow{\hat{H}_*} & H_4(D^4, \mathbb{S}^3, \mathbb{Z})
 \end{array}$$

shows that $f_*[X]$ depends only on $\psi \otimes_{\Lambda} 1 \in \pi_3(K) \otimes_{\Lambda} \mathbb{Z}$. Note that $H_4(D, \mathbb{Z}) \rightarrow H_4(D, K, \mathbb{Z})$ is injective. This also demonstrates that the above construction only depends on ξ , not on the choice of $[\varphi] \in \Gamma(\pi_2)$ with $[\varphi] \otimes_{\Lambda} 1 = \xi$.

It remains to be shown that any minimal model X' is homotopy equivalent to some model X obtained by the above construction. Write

$$X' = K' \cup_{\psi} D^4 \xrightarrow{f'} D',$$

where D' is the 2-stage Postnikov space, K' is a 3-dimensional complex, and $\psi : \mathbb{S}^3 \rightarrow K'$ is the attaching map. Recall our standard model:

$$P = K \cup_{\varphi_1} D^4 \xrightarrow{f} D.$$

In Section 3 we constructed a homotopy equivalence $h : D' \rightarrow D$ sending $K' \rightarrow K$. Lemma 3.3 implies

$$h_* f'_*[X'] - f_*[P] \in \text{Ker } G = \Gamma(\pi_2) \otimes_{\Lambda} \mathbb{Z}.$$

By Lemma 4.1 of Section 4 choose $[\varphi] \in \Gamma(\pi_2)$ so that $[\varphi] \otimes_{\Lambda} 1 = h_* f'_*[X'] - f_*[P]$, and $\varphi : \mathbb{S}^3 \rightarrow K^{(2)} \subset K$. As in Part V of Section 4, we build $X = K \cup_{\varphi_2} D^4$, with $\varphi_2 = \varphi_1 + \varphi$, and $g : X \rightarrow D$. Let $g_0 : K^{(2)} \cup_{\varphi} D^4 \rightarrow D$ be the canonically defined map from the zero homotopy of $\mathbb{S}^3 \rightarrow K^{(2)} \rightarrow D$. Then we have (use Lemma 4.10) $g_*[X] = f_*[P] + (g_0)_*(\theta)$, where $\theta \in H_4(K^{(2)} \cup_{\varphi} D^4, \mathbb{Z})$ is a generator. But $(g_0)_*(\theta) = h_* f'_*[X'] - f_*[P]$, as can be seen from the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_4(K^{(2)} \cup_{\varphi} D^4, \Lambda) \otimes_{\Lambda} \mathbb{Z} & \xrightarrow{\cong} & H_4(K^{(2)} \cup_{\varphi} D^4, \mathbb{Z}) & & \\ & & \downarrow (g_0)_* \otimes_{\Lambda} 1 & & \downarrow (g_0)_* & & \\ 0 & \longrightarrow & \Gamma(\pi_2) \otimes_{\Lambda} \mathbb{Z} = H_4(D, \Lambda) \otimes_{\Lambda} \mathbb{Z} & \longrightarrow & H_4(D, \mathbb{Z}) & \longrightarrow & H_2(B\pi, H_2(D, \Lambda)) \longrightarrow 0. \end{array}$$

Therefore, $g_*[X] = h_* f'_*[X']$. By Proposition 3.2 and the proof of Theorem 1.3 (where we have to use that π_2 is not a torsion group) we obtain a homotopy equivalence $X' \rightarrow X$. □

5. Non-uniqueness of strongly minimal models: examples

In Section 4 we constructed minimal models for all elements of $\Gamma(\pi_2)$. In this section we address the question of uniqueness up to homotopy equivalence. Recall that for 2 models X and X' we have a homotopy equivalence between the 2-stage Postnikov systems (assuming that the first k -invariants are compatible). It is deduced from Diagram (3.2) in Section 3, i.e. we have the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & D & \xrightarrow{h} & D' & \xleftarrow{f'} & X' \\ & & \downarrow & & \downarrow & & \\ & & B\pi_1 & \longrightarrow & B\pi_1 & & \end{array}$$

If $X = K \cup_{\varphi} D^4$ and $X' = K' \cup_{\psi} D^4$, then D and D' are constructed from the 3-complexes K and K' , respectively, by adjoining cells of dimension greater or equal to 4. Proposition 3.2 defines an obstruction to extending the restriction $h^{(3)} : K \rightarrow K'$ to a homotopy equivalence $X \rightarrow X'$. Also, if this obstruction does not vanish, it could be that X is homotopy equivalent to X' . We use h to identify $D \rightarrow B\pi_1$ with $D' \rightarrow B\pi_1$. All this makes sense if $B\pi_1$ is an aspherical 2-complex. From now on we shall consider only Baumslag–Solitar groups $B(k)$, $k \neq 0$, and aspherical surface fundamental groups. For any such model X we obtain $H_3(X, \Lambda) \cong H^1(X, \Lambda) \cong H^1(B\pi, \Lambda) \cong 0$ by Lemma 6.2 of [5] (here $\pi = \pi_1$, as usual). Since $H_4(X, \Lambda) \cong 0$, we get an isomorphism from $H_4(X, K, \Lambda)$ onto $H_3(K, \Lambda)$, i.e. $H_3(K, \Lambda) \cong \Lambda$. Furthermore, the canonical generator of $H_4(X, K, \Lambda)$, given by the attaching map φ , defines a generator of $H_3(K, \Lambda)$ and a splitting $s_X : H_3(K, \Lambda) \rightarrow \pi_3(K)$ of the Whitehead sequence given by the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\pi_2) & \xrightarrow{i_*} & \pi_3(K) & \xrightarrow{H} & H_3(K, \Lambda) \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \cong \\ & & & & \pi_4(X, K) & \xrightarrow{\cong} & H_4(X, K, \Lambda). \end{array}$$

Then s_X defines a splitting $t_X : \pi_3(K) \rightarrow \Gamma(\pi_2)$. From the Whitehead sequence of X , we have an isomorphism from $\Gamma(\pi_2)$ onto $\pi_3(X)$, and t_X can also be defined by the following diagram:

$$\begin{array}{ccc} \Gamma(\pi_2) & \xleftarrow{t_X} & \pi_3(K) \\ \cong \downarrow & & \downarrow j_* \\ \pi_3(X) & \xlongequal{\quad} & \pi_3(X). \end{array}$$

Conversely, t_X defines s_X by the well-known procedure using the projection operator $i_* \circ t_X$. If $X = K \cup_\varphi D^4$ and $X' = K \cup_\psi D^4$ are homotopy equivalent models, there is a homotopy equivalence of pairs (see [13], Theorem 2.4)

$$g : (K, \varphi(\mathbb{S}^3)) \rightarrow (K, \psi(\mathbb{S}^3))$$

inducing the diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\pi_2) & \xrightarrow{i_*} & \pi_3(K) & \xrightarrow{j_*} & \pi_3(X) \\ & & g_* \downarrow & & g_* \downarrow & & g_* \downarrow \\ 0 & \longrightarrow & \Gamma(\pi_2) & \xrightarrow{i'_*} & \pi_3(K) & \xrightarrow{j'_*} & \pi_3(X') \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\pi_2) & \xrightarrow{i_*} & \pi_3(K) & \xrightarrow{H} & H_3(K, \Lambda) \longrightarrow 0 \\ & & g_* \downarrow & & g_* \downarrow & & g_* \downarrow \\ 0 & \longrightarrow & \Gamma(\pi_2) & \xrightarrow{i'_*} & \pi_3(K) & \xrightarrow{H} & H_3(K, \Lambda) \longrightarrow 0. \end{array}$$

Hence, all splittings $t_X, t_{X'}, s_X$, and $s_{X'}$ commute with the induced homomorphisms g_* . In the following we fix one model $X = K \cup_\varphi D^4$. We are going to construct models $X' = K \cup_\psi D^4$ that are not homotopy equivalent to X . Let us denote by $1 \in H_3(K, \Lambda)$ the generator defined by X , i.e. $s_X(1) = [\varphi]$. Let $\theta : \Gamma(\pi_2) \rightarrow \Gamma(\pi_2)$ be an isomorphism. Then $\theta \circ t_X = t : \pi_3(K) \rightarrow \Gamma(\pi_2)$ is a splitting. It defines a splitting $s : H_3(K, \Lambda) \rightarrow \pi_3(K)$. Then $s(1) = s_X(1) + i_*(a)$ for some $a \in \Gamma(\pi_2)$. As in Section 4, we construct the model $X' = K \cup_\psi D^4$ with $[\psi] = s(1)$.

Proposition 5.1 *If θ is not induced by an isomorphism $\pi_2 \rightarrow \pi_2$, then X' is not homotopy equivalent to X .*

Proof Any homotopy equivalence $g : X \rightarrow X'$ induces

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\pi_2) & \longrightarrow & \pi_3(K) & \longrightarrow & \pi_3(X) \\ & & g_* \downarrow & & g_* \downarrow & & g_* \downarrow \\ 0 & \longrightarrow & \Gamma(\pi_2) & \longrightarrow & \pi_3(K) & \longrightarrow & \pi_3(X'). \end{array}$$

However, $g_* : \Gamma(\pi_2) \rightarrow \Gamma(\pi_2)$ is never θ . □

Examples Let $X = F \times \mathbb{S}^2$, where F is a closed oriented aspherical surface. Then $\pi_2(X) \cong \mathbb{Z}$, $\Gamma(\pi_2) \cong \mathbb{Z}$ and $-\text{Id} : \Gamma(\pi_2) \rightarrow \Gamma(\pi_2)$ is not induced by an isomorphism $\pi_2 \rightarrow \pi_2$. This easily follows from the Γ -functor property. There are inclusions $\pi_2 \rightarrow \Gamma(\pi_2)$ and $\Gamma(\pi_2) \rightarrow \pi_2 \otimes \pi_2$ (because π_2 is free abelian) such that the

composition $\pi_2 \rightarrow \Gamma(\pi_2) \rightarrow \pi_2 \otimes \pi_2$ sends x to $x \otimes x$. In the case when $\pi = B(k)$, π_2 is free abelian (see [5], Lemma 6.2 V), one obtains such θ in this case, too. On the other hand, if θ is induced by an isomorphism $\beta : \pi_2 \rightarrow \pi_2$, one needs more to construct a homotopy equivalence. By [15], Theorem 3, one gets a map $g : K \rightarrow K$, but the induced maps g_* do not necessarily commute with the splittings s_X and $s_{X'}$.

Supplement to the aspherical surface case. In the example $F \times \mathbb{S}^2$ there are 2 models, namely $F \times \mathbb{S}^2$ and the non-trivial \mathbb{S}^2 -bundle $E \rightarrow F$ with the second Stiefel–Whitney class $\neq 0$ (see, for example, [3], Appendix). Here it is also convenient to consider the map

$$F_{\mathbb{Z}} : H_4(D, \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(H^2(D, \mathbb{Z}) \otimes H^2(D, \mathbb{Z}), \mathbb{Z})$$

given by

$$F_{\mathbb{Z}}(x)(u \otimes v) := x \cap (u \cup v),$$

where $D = F \times \mathbb{C}P^\infty$. Then $F_{\mathbb{Z}}$ is injective. If $f_0 : F \times \mathbb{S}^2 \rightarrow D$ and $f_1 : E \rightarrow D$ are Postnikov maps, then $F_{\mathbb{Z}}(f_{0*}[F \times \mathbb{S}^2])$ and $F_{\mathbb{Z}}(f_{1*}[E])$ are the integral intersection forms of $F \times \mathbb{S}^2$ and E , respectively. Moreover, these forms are respectively given by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

(see [3]). It was shown in [9], Section 5, that $F \times \mathbb{S}^2$ and E are the only models up to homotopy equivalence.

6. Final remarks

The following map was defined in [2]:

$$F : H_4(D, \mathbb{Z}) \rightarrow \text{Hom}_{\Lambda-\Lambda}(H^2(D, \Lambda) \otimes_{\mathbb{Z}} \overline{H}^2(D, \Lambda), \Lambda),$$

to classify Poincaré 4-complexes X , where $D \rightarrow B\pi$ is a 2-stage Postnikov system for X . Here $H^2(D, \Lambda) \otimes_{\mathbb{Z}} \overline{H}^2(D, \Lambda)$ carries the obvious Λ -bimodule structure. It was proven therein that F is injective for free non-abelian groups π . The maps F and G are related by the following diagram:

$$\begin{array}{ccc} H_4(D, \mathbb{Z}) & \xrightarrow{G} & \text{Hom}_{\Lambda-\Lambda}(H^2(D, \Lambda), \overline{H}_2(D, \Lambda)) \\ \parallel & & \downarrow H \\ H_4(D, \mathbb{Z}) & \xrightarrow{F} & \text{Hom}_{\Lambda-\Lambda}(H^2(D, \Lambda) \otimes_{\mathbb{Z}} \overline{H}^2(D, \Lambda), \Lambda), \end{array}$$

where $H(\varphi)(u \otimes v) = \overline{\hat{u}(\varphi(v))}$, and \hat{u} is the image of u under

$$H^2(D, \Lambda) \rightarrow \text{Hom}_{\Lambda}(H_2(D, \Lambda), \Lambda).$$

Obviously, G is injective if F is injective. If $f : X \rightarrow D$ is a map such that $f_* : \pi_q(X) \rightarrow \pi_q(D)$ is an isomorphism for $q = 1, 2$, then $F(f_*[X]) \circ (f^* \otimes f^*)$ is the equivariant intersection form on X , and $f_*G(f_*[X])f^* : H^2(X, \Lambda) \rightarrow \overline{H}_2(X, \Lambda)$ is the Poincaré duality isomorphism. It is convenient to denote $F(f_*[X])$ as the “intersection type” and $G(f_*[X])$ as the “Poincaré duality type” of X . The Poincaré duality type determines the intersection type. In this sense it is a stronger “invariant”. For \mathbb{S}^2 -bundles over aspherical 2-surfaces all intersection types vanish, whereas the Poincaré types are non-trivial.

Acknowledgments

This work was performed under the auspices of the GNSAGA of the CNR (National Research Council) of Italy and partially supported by the MIUR (Ministero per la Ricerca Scientifica e Tecnologica) of Italy, and by the Slovenian Research Agency grants P1-0292-0101 and J1-4144-0101. We thank the referee for comments and suggestions.

References

- [1] Baues, H.J.: *Combinatorial Homotopy and 4-Dimensional Complexes*. Berlin. Walter de Gruyter 1991.
- [2] Cavicchioli, A., Hegenbarth, F.: On 4-manifolds with free fundamental groups. *Forum Math.* 6, 415–429 (1994).
- [3] Cavicchioli, A., Hegenbarth, F., Repovš, D.: Four-manifolds with surface fundamental groups. *Trans. Amer. Math. Soc.* 349, 4007–4019 (1997).
- [4] Hambleton, I., Kreck, M.: On the classification of topological 4-manifolds with finite fundamental group. *Math. Ann.* 280, 85–104 (1988).
- [5] Hambleton, I., Kreck, M., Teichner, P.: Topological 4-manifolds with geometrically 2-dimensional fundamental group. *J. Topology Anal.* 1, 123–151 (2009).
- [6] Hegenbarth, F., Piccarreta, S.: On Poincaré 4-complexes with free fundamental groups. *Hiroshima Math. J.* 32, 145–154 (2002).
- [7] Hegenbarth, F., Repovš, D., Spaggiari, F.: Connected sums of 4-manifolds. *Topology Appl.* 146–147, 209–225 (2005).
- [8] Hillman, J.A.: PD_4 -complexes with strongly minimal models. *Topology Appl.* 153, 2413–2424 (2006).
- [9] Hillman, J.A.: Strongly minimal PD_4 -complexes. *Topology Appl.* 156, 1565–1577 (2009).
- [10] Kaplansky, I.R.: *Fields and Rings*. Chicago. University of Chicago Press 1969
- [11] MacLane, S., Whitehead, J.H.C.: On the 3-type of a complex. *Proc. Nat. Acad. Sci.* 36, 41–48 (1950)
- [12] Pamuk, M.: Homotopy self-equivalences of 4-manifolds with PD_2 -fundamental group. *Topology Appl.* 156, 1445–1458 (2009).
- [13] Wall, C.T.C.: Poincaré complexes. *Ann. of Math.* 86, 213–245 (1967).
- [14] Wall, C.T.C.: *Surgery on Compact Manifolds*. London. Academic Press 1970
- [15] Whitehead, J.H.C.: On a certain exact sequence. *Ann. of Math.* 52 (2), 51–110 (1950).