Turkish Journal of Mathematics

Volume 38 | Number 3 Article 13

1-1-2014

Counting pseudo-Anosov mapping classes on the 3-punctured projective plane

BLAZEJ SZEPIETOWSKI

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 524 – 534

doi:10.3906/mat-1211-7

Counting pseudo-Anosov mapping classes on the 3-punctured projective plane

Błażej SZEPIETOWSKI*
Institute of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland

Received: 04.11.2012 • **Accepted:** 19.10.2013 • **Published Online:** 14.03.2014 • **Printed:** 11.04.2014

Abstract: We prove that in the pure mapping class group of the 3-punctured projective plane equipped with the word metric induced by certain generating set, the ratio of the number of pseudo-Anosov elements to the number of all elements in a ball centered at the identity tends to one, as the radius of the ball tends to infinity. We also compute growth functions of the sets of reducible and pseudo-Anosov elements.

Key words: Mapping class group, nonorientable surface, growth functions

1. Introduction

Let G be a group with a finite generating set A. For $x \in G$ the *length* of x with respect to A is defined to be the minimum number of factors needed to express x as a product of elements of A and their inverses. We denote it by $||x||_A$. The *word metric* on G with respect to A is defined as $d_A(x,y) = ||xy^{-1}||_A$ for $x, y \in G$. For a subset $X \subset G$, the *growth function* of X with respect to A is the function $f(z)$ defined by the power series $\sum_{n=0}^{\infty} C_n z^n$, where the coefficient C_n is equal to the number of elements of length n in X. The *density* $d(X)$ of X with respect to A is defined as

$$d(X) = \lim_{n\to\infty} \frac{\#(\mathcal{B}(n) \cap X)}{\#\mathcal{B}(n)},$$

where $\mathcal{B}(n)$ is the set of elements of G of length at most n (it is the ball of radius n, centered at the identity, with respect to the word metric induced by A), and $\#$ denotes the cardinality.

Let S be a compact surface with a finite set P of distinguished points in the interior of S called *punctures*. We denote as $\mathrm{Homeo}(S,P)$ the topological group of all, orientation preserving if S is orientable, homeomorphisms of S that preserve P and fix the boundary of S pointwise. The *mapping class group* of (S,P) is $\mathcal{M}(S,P) = \pi_0\mathrm{Homeo}(S,P)$. Elements of $\mathcal{M}(S,P)$ are isotopy classes of homeomorphisms in $\mathrm{Homeo}(S,P)$. By the *pure mapping class group* of (S,P) we understand in this paper the subgroup $\mathcal{PM}(S,P)$ of $\mathcal{M}(S,P)$ consisting of the isotopy classes of homeomorphisms fixing every puncture and also preserving local orientation at every puncture. Since the groups $\mathcal{M}(S,P)$ and $\mathcal{PM}(S,P)$ are finitely generated, it makes sense to study growth functions and densities of their subsets, with respect to various finite generating sets.

*Correspondence: blaszep@mat.ug.edu.pl
2010 *AMS Mathematics Subject Classification:* Primary 57N05, Secondary 20F38.

Suppose that $\partial S = \emptyset$ and the Euler characteristic of $S \backslash P$ is negative. Let $\mathcal{C}(S,P)$ denote the set of isotopy classes of simple closed curves on $S\backslash P$ not bounding a disc with less than 2 punctures. The group $\mathcal{M}(S,P)$ acts on $\mathcal{C}(S,P)$. An element of $\mathcal{M}(S,P)$ is called *reducible* if it fixes a nonempty finite collection of pairwise disjoint elements of $\mathcal{C}(S,P)$. An element of $\mathcal{M}(S,P)$ that has infinite order and is not reducible is called *pseudo-Anosov*. By the Nielsen–Thurston classification of surface homeomorphisms (see [4, Chapter 13]), a pseudo-Anosov mapping class can be represented by a pseudo-Anosov homeomorphism h, such that there is a pair F^s, F^u of transverse measured foliations on S, such that $h(F^s) = \lambda^{-1}F^s$ and $h(F^u) = \lambda F^u$ for some $\lambda > 1$.

In this paper we consider the case when (S,P) is the projective plane with 3 punctures. The pure mapping class group $\mathcal{PM}(S,P)$ is free of rank 3. We fix free generators of $\mathcal{PM}(S,P)$ and consider the induced word metric. We prove the following results.

Theorem 1.1 *The growth functions of the sets of reducible and pseudo-Anosov elements in* $\mathcal{PM}(S,P)$ *are rational.*

We compute these growth functions explicitly.

Theorem 1.2 *Let* $\mathcal{P}$ *be the set of pseudo-Anosov elements in* $\mathcal{PM}(S,P)$. *Then* $d(\mathcal{P}) = 1$.

Analogous results were proved in [10] in the case when S is the torus, and in [1] for the 4-holed sphere. Our results, as well as those in [1, 10], give a partial answer to Question 3.14 and confirm Conjecture 3.15 in [3] in a special case. Similar results on "genericity" of pseudo-Anosovs, in the sense of random walks and not the word metric, were proved in the papers [6, 7, 8]. This paper seems to be the first in which problems of this type are considered for a nonorientable surface.

This paper is organised as follows: In Section 2 we give an algebraic characterisation of reducible elements in the pure mapping class group of the 3-punctured projective plane. In Section 3 we count for each $n \geq 1$ the numbers of reducible elements of length n and also determine growth functions of certain sets of reducible elements. The main results are proved in Section 4.

2. Pure mapping class group of the 3-punctured projective plane

For the rest of this paper let S be the projective plane obtained from the standard unit disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$ by identifying antipodal points on ∂D. Let z_1, z_2, z_3 denote the images in S of the points $-\frac{3}{4}i, \frac{3}{4}e^{\frac{\pi i}{6}}, \frac{3}{4}e^{\frac{5\pi i}{6}}$ respectively. We fix $P = \{z_1, z_2, z_3\}$ and denote $\mathcal{PM}(S,P)$ simply as $\mathcal{PM}(S)$. We also fix the local orientation at each puncture z_i induced by the standard orientation of D.

A simple closed curve γ on S is called *nonseparating* if $S\backslash\gamma$ is connected, and *separating* otherwise. Every nonseparating curve on S is *one-sided*, which means that its regular neighbourhood is a Möbius strip. Let μ_0 be the image of ∂D in S and let μ_1, μ_2, μ_3 be the images in S of the line segments respectively $t, te^{\frac{2\pi i}{3}}, te^{\frac{\pi i}{3}}$ for $t \in [-1,1]$. Note that these are one-sided curves. For $i = 0,1,2,3$ let D_i be the disc obtained by cutting S along μ_i ($D_0 = D$) and fix the orientation of D_i induced by the local orientation at z_1. For $j = 1,2,3$ let α_j and β_j be the separating curves in the Figure. We fix Dehn twists T_{α_j}, T_{β_j}, such that T_{α_j} are right with respect to the orientation of D_0, T_{β_2} and T_{β_3} are right with respect to the orientation of D_1, and T_{β_1} is right

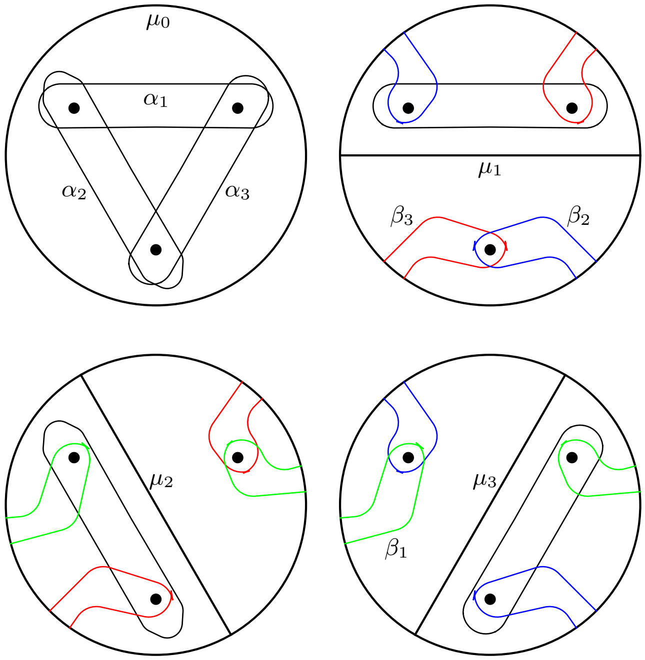

Figure. Curves on the 3-punctured projective plane S.

with respect to the orientation of D_2. Then in $\mathcal{PM}(S)$ we have the following relations:

$$\text{(L1)}\quad T_{\alpha_1}T_{\alpha_2}T_{\alpha_3}=1,\qquad \text{(L2)}\quad T_{\alpha_1}^{-1}T_{\beta_2}T_{\beta_3}=1,$$

$$\text{(L3)}\quad T_{\alpha_2}T_{\beta_1}T_{\beta_3}=1,\qquad \text{(L4)}\quad T_{\alpha_3}T_{\beta_2}T_{\beta_1}^{-1}=1.$$

They all follow from the well-known lantern relation between Dehn twists supported on a 4-holed sphere (see [4, Proposition 5.1]). In the lantern relation one has a product of 3 twists on one side of the equality and a product of 4 twists about the boundary components of the sphere on the other side. In our situation, however, the 4 twists are trivial, because they are about curves bounding once-punctured discs and a Möbius band.

Theorem 2.1 ([9, Theorem 7.5]) *The group $\mathcal{PM}(S)$ is freely generated by T_{α_1}, T_{α_2}, T_{β_1}.*

Since a free group is torsion free, every element of $\mathcal{PM}(S)$ is either reducible or pseudo-Anosov.

Lemma 2.2 *Let M be the Möbius strip with one puncture $p\in M$. Then $\mathcal{PM}(M,\{p\})$ is generated by a Dehn twist about the boundary of M.*

Proof Let F be the projective plane obtained from M by gluing a disc with a puncture q along ∂M. Since every $h\in \mathrm{Homeo}(M,\{p\})$ may be extended by the identity on the disc to $h'\in\mathrm{Homeo}(F,\{p,q\})$, we have a homomorphism $\mathcal{PM}(M,\{p\})\to\mathcal{PM}(F,\{p,q\})$, which fits in the following short exact sequence (see [9, Section 7])

$$1\to\mathbb{Z}\to\mathcal{PM}(M,\{p\})\to\mathcal{PM}(F,\{p,q\})\to 1,$$

where $\mathbb{Z}$ is generated by a Dehn twist $T_{\partial M}$. By [5, Corollary 4.6], $\mathcal{M}(F,\{p,q\})$ is isomorphic to the dihedral group of order 8, and since $\mathcal{PM}(F,\{p,q\})$ is a subgroup of index 8, thus it is trivial (note that in [5] a slightly different definition of the pure mapping class group of a nonorientable surface is used; its elements are allowed to reverse local orientation at the punctures). □

Proposition 2.3 *An element of $\mathcal{PM}(S)$ is reducible if and only if it fixes an isotopy class of one-sided curves.*

Proof Let h be a reducible homeomorphism of S. By definition, there is a set C of disjoint nonisotopic simple closed curves such that $h(C) = C$. If C contains a one-sided curve, then since any 2 one-sided curves on S intersect, C contains only one such curve, and this curve is fixed by h. If C does not contain a one-sided curve, then it consists of a single separating curve γ. Let E and M be the connected components of the surface obtained by cutting S along γ, where E is a punctured disc and M is a Möbius strip with at most one puncture. Clearly h preserves M and E, and since it preserves local orientation at the punctures, it also preserves orientation of E. It follows that h preserves orientation of γ and changing h by an isotopy we may assume that it is equal to the identity on γ. Let $h' = h|_M$. If there is no puncture in M then h' is isotopic to the identity on M by an isotopy fixing ∂M (see [2, Theorem 3.4]), while if there is a puncture in M, then h' is isotopic to some power of a Dehn twist about ∂M, by Lemma 2.2. In particular h is isotopic to a homeomorphism fixing a one-sided curve on M. □

We say that 2 simple closed curves γ_1 and γ_2 are $\mathcal{PM}(S)$*-equivalent* if $\gamma_1 = h(\gamma_2)$ for some $h \in \mathrm{Homeo}(S,P)$ fixing every puncture and preserving local orientation at every puncture.

Lemma 2.4 *Every one-sided simple closed curve on S is $\mathcal{PM}(S)$-equivalent to μ_i for some $i \in \{0,1,2,3\}$.*

Proof Let γ be a one-sided simple closed curve and let E be the disc obtained by cutting S along γ. Fix the orientation of E induced by the local orientation at z_1. Let us compare the local orientations at z_2 and z_3 to the orientation of E. There are 4 cases.

Case 1. The local orientations at z_2 and z_3 agree with the orientation of E. Then there is an orientation preserving homeomorphism $f\colon D_0 \to E$, preserving the punctures, which commutes with the gluings giving back S. Thus f induces $h \in \mathrm{Homeo}(S,P)$ such that $h(\mu_0) = \gamma$.

Case 2. The local orientations at z_2 and z_3 are opposite to the orientation of E. Then there is an orientation preserving homeomorphism $f\colon D_1 \to E$ inducing $h \in \mathrm{Homeo}(S,P)$ such that $h(\mu_1) = \gamma$.

Case 3. The local orientation at z_3 agrees with the orientation of E, whereas that at z_2 is opposite. Then there is an orientation preserving homeomorphism $f\colon D_2 \to E$ inducing $h \in \mathrm{Homeo}(S,P)$ such that $h(\mu_2) = \gamma$.

Case 4. The local orientation at z_2 agrees with the orientation of E, whereas that at z_3 is opposite. Then there is $h \in \mathrm{Homeo}(S,P)$ such that $h(\mu_3) = \gamma$. □

The following corollary follows immediately from Proposition 2.3 and Lemma 2.4.

Corollary 2.5 *An element of $\mathcal{PM}(S)$ is reducible if and only if it is conjugate to an element fixing the isotopy class of μ_i for some $i \in \{0,1,2,3\}$.*

For a group G and elements $x_1,\dots,x_k \in G$ we denote by $\langle x_1,\dots,x_k\rangle$ the subgroup of G generated by $x_1,\dots,x_k$.

Proposition 2.6 *For $i = 0,1,2,3$ let $\mathcal{S}_i$ denote the stabiliser in $\mathcal{PM}(S)$ of the isotopy class of μ_i. Then* $\mathcal{S}_0 = \langle T_{\alpha_1}, T_{\alpha_2}\rangle$, $\mathcal{S}_1 = \langle T_{\alpha_1}, T_{\alpha_2}T_{\beta_1}\rangle$, $\mathcal{S}_2 = \langle T_{\alpha_2}, T_{\beta_1}\rangle$, $\mathcal{S}_3 = \langle T_{\alpha_1}T_{\alpha_2}, T_{\beta_1}\rangle$.

Proof Fix $i \in \{0,1,2,3\}$ and consider the group $\mathcal{PM}(D_i,P)$. Since every homeomorphism of D_i equal to the identity on ∂D_i induces a homeomorphism of S, we have a homomorphism $\varphi_i\colon \mathcal{PM}(D_i,P)\to\mathcal{PM}(S,P)$. The image of φ_i is equal to $\mathcal{S}_i$, because every homeomorphism of S that fixes μ_i and preserves local orientation at the punctures must also preserve orientation of μ_i, and thus it is isotopic to a homeomorphism equal to the identity on μ_i. The group $\mathcal{PM}(D_i,P)$ is well known to be isomorphic to the pure braid group on 3 strands, and it is generated by Dehn twists about 3 curves, each curve surrounding 2 punctures, and each 2 curves intersecting each other twice (see [4, Chapter 9]). It follows that $\mathcal{S}_0=\langle T_{\alpha_1},T_{\alpha_2},T_{\alpha_3}\rangle$, $\mathcal{S}_1=\langle T_{\alpha_1},T_{\beta_2},T_{\beta_3}\rangle$, $\mathcal{S}_2=\langle T_{\beta_1},T_{\alpha_2},T_{\beta_3}\rangle$, $\mathcal{S}_3=\langle T_{\beta_1},T_{\beta_2},T_{\alpha_3}\rangle$. By the lantern relations (L1–L4) only 2 twists are needed to generate $\mathcal{S}_i$, and since $T_{\alpha_1}T_{\alpha_2}=T_{\alpha_3}^{-1}$ by (L1) and $T_{\alpha_2}T_{\beta_1}=T_{\beta_3}^{-1}$ by (L2), the proposition follows. □

3. Counting some words in the free group of rank 3

Let $\mathcal{F}=\mathcal{F}(a,b,c)$ be the free group on generators a,b,c. The elements of $\mathcal{F}$ are reduced words in the letters $a,a^{-1},b,b^{-1},c,c^{-1}$. By a word in $\mathcal{F}$ we always mean a reduced word. A word is *cyclically reduced* if its first letter is different from the inverse of its last letter. The number of letters in a word $w\in\mathcal{F}$ is *the length of* w denoted as $|w|$.

The following well-known theorem is the solution to the conjugacy problem in a free group.

Theorem 3.1 *Every element of a free group is conjugate to a cyclically reduced word. Two cyclically reduced words are conjugate if and only if one is a cyclic permutation of the other.*

By Theorem 2.1, there is an isomorphism $\rho\colon\mathcal{F}\to\mathcal{PM}(S)$ given by $\rho(a)=T_{\alpha_1}$, $\rho(b)=T_{\alpha_2}$, $\rho(c)=T_{\beta_1}$, which is an isometry with respect to the word metrics induced by the generating sets $\{a,b,c\}$ of $\mathcal{F}$ and $\{T_{\alpha_1},T_{\alpha_2},T_{\beta_1}\}$ of $\mathcal{PM}(S)$. Via this isomorphism we identify $\mathcal{F}$ with $\mathcal{PM}(S)$.

For $w_1,\ldots,w_k\in\mathcal{F}$ we denote by $\mathcal{C}(w_1,\ldots,w_k)$ the set of elements of $\mathcal{F}$ that are conjugate to elements of $\langle w_1,\ldots,w_k\rangle$, and by $\mathcal{C}(w_1,\ldots,w_k;n)$ the subset of $\mathcal{C}(w_1,\ldots,w_k)$ consisting of elements of length n.

We also introduce the following notation:

$$\begin{aligned}A_n&=\#\mathcal{C}(b;n),\\ B_n&=\#\mathcal{C}(a,b;n),\\ C_n&=\#\mathcal{C}(abc;n),\\ D_n&=\#(\mathcal{C}(a,bc;n)\backslash(\mathcal{C}(a;n)\cup\mathcal{C}(bc;n))).\end{aligned}$$

Lemma 3.2 *Let R_n be the number of reducible elements of length n in $\mathcal{F}$. Then, for $n\geq 1$*

$$R_n=2B_n+2D_n-A_n-C_n.$$

Proof From Corollary 2.5 and Proposition 2.6 we have

$$R_n=\#(\mathcal{C}(a,b;n)\cup\mathcal{C}(b,c;n)\cup\mathcal{C}(a,bc;n)\cup\mathcal{C}(ab,c;n)).$$

It follows from Theorem 3.1 that

$$\mathcal{C}(a,b)\cap\mathcal{C}(b,c)=\mathcal{C}(b),\quad \mathcal{C}(a,b)\cap\mathcal{C}(a,bc)=\mathcal{C}(a)$$
$$\mathcal{C}(a,b)\cap\mathcal{C}(ab,c)=\mathcal{C}(ab),\quad \mathcal{C}(b,c)\cap\mathcal{C}(a,bc)=\mathcal{C}(bc)$$
$$\mathcal{C}(b,c)\cap\mathcal{C}(ab,c)=\mathcal{C}(c),\quad \mathcal{C}(a,bc)\cap\mathcal{C}(ab,c)=\mathcal{C}(abc).$$

We prove the last equality; the first 5 are easily verified. Let $w\in\mathcal{C}(a,bc)\cap\mathcal{C}(ab,c)$ be nontrivial. Then w is conjugate to a word

$$w_1=a^{x_1}(bc)^{x_2}\cdots a^{x_{2k-1}}(bc)^{x_{2k}},$$

where x_i are integers, and we may assume that w_1 is cyclically reduced. Analogously, w is conjugate to a cyclically reduced word of the form

$$w_2=(ab)^{y_1}c^{y_2}\cdots(ab)^{y_{2l-1}}c^{y_{2l}}.$$

By Theorem 3.1, w_1 is a cyclic permutation of w_2. It follows that w_1 is neither a power of a nor a power of bc. Therefore we can assume $x_i\neq 0$ for $1\leq i\leq 2k$ and $k\geq 1$. By replacing w by w^{-1} if necessary, we may assume $x_1>0$. Note that none of the words aa, ac^{-1}, cb, ca^{-1} can appear as a sub-word of a cyclic permutation of w_2. It follows that $x_i=1$ for $1\leq i\leq 2k$; hence $w_1=(abc)^k$ and $w\in\mathcal{C}(abc)$. We have shown that $\mathcal{C}(a,bc)\cap\mathcal{C}(ab,c)\subseteq\mathcal{C}(abc)$, and the opposite inclusion is obvious.

For $n\geq 1$ we have

$$R_n=\#\mathcal{C}(a,b;n)+\#\mathcal{C}(b,c;n)+\#(\mathcal{C}(a,bc;n)\backslash(\mathcal{C}(a;n)\cup\mathcal{C}(bc;n)))$$
$$+\#(\mathcal{C}(ab,c;n)\backslash(\mathcal{C}(c;n)\cup\mathcal{C}(ab;n)))-\#\mathcal{C}(b;n)-\#\mathcal{C}(abc;n).$$

The lemma follows because $\#(\mathcal{C}(ab,c;n)\backslash(\mathcal{C}(c;n)\cup\mathcal{C}(ab;n)))=D_n$ and $\#\mathcal{C}(b,c;n)=B_n$. □

Lemma 3.3 *For $k\geq 0$ we have $A_{2k+1}=A_{2k+2}=2\cdot 5^k$. The growth function of $\mathcal{C}(b)$ with respect to the generators a,b,c is $f_1(x)=\frac{1+2x-3x^2}{1-5x^2}$.*

Proof Every element of $\mathcal{C}(b)$ can be expressed uniquely in the form $w=ub^iu^{-1}$, where $i\in\mathbb{Z}$ and u is a word whose last letter is not $b^{\pm1}$. Let us fix $k\geq 0$. Observe there is a bijection $\mathcal{C}(b;2k+1)\to\mathcal{C}(b;2k+2)$ defined as $ub^iu^{-1}\mapsto ub^{i+1}u^{-1}$. Thus $A_{2k+1}=A_{2k+2}$. Let us count the words in $\mathcal{C}(b;2k+1)$. Every such word is of the form $w=ub^{\varepsilon(2i+1)}u^{-1}$, where u is a word whose last letter is not $b^{\pm1}$ of length $k-i$ for $0\leq i\leq k$ and $\varepsilon\in\{-1,1\}$. For a fixed i, there are 2 choices for ϵ, and if $i<k$ then there are $4\cdot 5^{k-i-1}$ choices for u. Thus

$$A_{2k+1}=2+\sum_{i=0}^{k-1}2\cdot 4\cdot 5^{k-i-1}=2+8\cdot 5^{k-1}\sum_{i=0}^{k-1}5^{-i}=2\cdot 5^k.$$

Now we can compute the growth function.

$$\begin{aligned}f_1(x)&=1+\sum_{k=0}^{\infty}(A_{2k+1}x^{2k+1}+A_{2k+2}x^{2k+2})=1+(1+x)\sum_{k=0}^{\infty}2\cdot 5^kx^{2k+1}\\&=1+(1+x)2x\sum_{k=0}^{\infty}(5x^2)^k=1+\frac{2x(1+x)}{1-5x^2}=\frac{1+2x-3x^2}{1-5x^2}.\end{aligned}$$

□

Lemma 3.4 *For $k \geq 0$ we have $B_{2k+1} = \frac{1}{3}B_{2k+2} = 6 \cdot 9^k - 2 \cdot 5^k$. The growth function of $\mathcal{C}(a,b)$ with respect to the generators a, b, c is*

$$f_2(x) = 1 + \frac{6x}{1-3x} - \frac{2x(1+3x)}{1-5x^2}.$$

Proof Every element of $\mathcal{C}(a,b)$ is either a word in $\langle a,b\rangle$ or it is of the form $uc^{\varepsilon}wc^{-\varepsilon}u^{-1}$, where $w \in \langle a,b\rangle$, $\varepsilon \in \{-1,1\}$, and u is a word whose last letter is not $c^{-\varepsilon}$. For $i \geq 1$ there are $4 \cdot 3^{i-1}$ words of length i in $\langle a,b\rangle$. It follows that $B_{2k+2} = 3B_{2k+1}$ for $k \geq 0$. Let us count words of the form $uc^{\varepsilon}wc^{-\varepsilon}u^{-1}$ of length $2k+1$. Suppose that $|w| = 2i+1$ for $0 \leq i \leq k-1$. Then $|u| = k-i-1$ and we have $4 \cdot 3^{2i}$ choices for w, 2 choices for ε, and 5^{k-i-1} choices for u. Thus

$$\begin{aligned} B_{2k+1} &= 4 \cdot 3^{2k} + \sum_{i=0}^{k-1} 8 \cdot 3^{2i} \cdot 5^{k-i-1} = 4 \cdot 3^{2k} + 8 \cdot 5^{k-1} \sum_{i=0}^{k-1} \left(\frac{9}{5}\right)^i = \\ &= 6 \cdot 9^k - 2 \cdot 5^k. \end{aligned}$$

$$\begin{aligned} f_2(x) &= 1 + \sum_{k=0}^{\infty} B_{2k+1}x^{2k+1} + 3B_{2k+1}x^{2k+2} \\ &= 1 + (1+3x)x\sum_{k=0}^{\infty}(6 \cdot 9^k - 2 \cdot 5^k)x^{2k} \\ &= 1 + (1+3x)x\left(\frac{6}{1-9x^2} - \frac{2}{1-5x^2}\right) = 1 + \frac{6x}{1-3x} - \frac{2x(1+3x)}{1-5x^2}. \end{aligned}$$

□

Lemma 3.5 *For $k \geq 0$ we have $C_{6k+3} = C_{6(k+1)} = \frac{6}{31}(5^{3k+2}+6)$, $C_{6k+5} = C_{6(k+1)+2} = 5C_{6k+3} - 6$, $C_{6(k+1)+1} = C_{6(k+1)+4} = 5C_{6k+5}$. The growth function of $\mathcal{C}(abc)$ with respect to the generators a, b, c is*

$$f_3(x) = 1 + \frac{6x^3}{31}\left(\frac{25(1+x^3)(1+5x^2+25x^4)}{1-(5x^2)^3} + \frac{6-x^2-5x^4}{1-x^3}\right).$$

Proof Every nontrivial element of $\mathcal{C}(abc)$ can be expressed uniquely in the form uv^iu^{-1}, where $i \geq 1$, $v \in \{(abc)^{\pm1}, (bca)^{\pm1}, (cab)^{\pm1}\}$ and u is a word whose last letter is neither equal to the last letter of v nor to the inverse of the first letter of v.

Let us count the elements of $\mathcal{C}(abc; 6k+3)$. Every such element is of the form $uv^{2i+1}u^{-1}$, where u, v are as above, $0 \leq i \leq k$, and $|u| = 3(k-i)$. There are 6 choices for v and if $i < k$ then there are $4 \cdot 5^{3(k-i)-1}$ choices for u. Thus

$$C_{6k+3} = 6 + 24\sum_{i=0}^{k-1} 5^{3(k-i)-1} = 6 + 24 \cdot 5^{3k-1}\sum_{i=0}^{k-1} 5^{-3i} = \frac{6}{31}(5^{3k+2}+6).$$

Every element of $\mathcal{C}(abc; 6k+5)$ is of the form $\alpha w\alpha^{-1}$ for $w \in \mathcal{C}(abc; 6k+3)$, where α is a single letter. For each w there are 4 choices for α if w is cyclically reduced, and 5 choices otherwise. There are 6 cyclically reduced words in $\mathcal{C}(abc;6k+3)$, namely v^{2k+1} for $v \in \{(abc)^{\pm 1},(bca)^{\pm 1},(cab)^{\pm 1}\}$; hence $C_{6k+5}=5C_{6k+3}-6=\frac{6}{31}(5^{3k+3}-1)$.

Similarly, every element of $\mathcal{C}(abc;6k+7)$ is of the form $\alpha w \alpha^{-1}$ for $w \in \mathcal{C}(abc;6k+5)$, where α is a single letter. Since the words in $\mathcal{C}(abc;6k+5)$ are not cyclically reduced, hence $C_{6k+7}=5C_{6k+5}$.

Observe that the mapping $uv^iu^{-1} \mapsto uv^{i+1}u^{-1}$ defines bijections $\mathcal{C}(abc;6k+3) \to \mathcal{C}(abc;6k+6)$, $\mathcal{C}(abc;6k+5) \to \mathcal{C}(abc;6k+8)$ and $\mathcal{C}(abc;6k+7) \to \mathcal{C}(abc;6k+10)$. Thus $C_{6k+3}=C_{6k+6}$, $C_{6k+5}=C_{6k+8}$ and $C_{6k+7}=C_{6k+10}$.

Since $C_1=C_2=C_4=0$, thus

$$\begin{aligned} f_3(x) = 1 &+ \sum_{k=0}^{\infty} C_{6k+3}(x^{6k+3}+x^{6k+6}) \\ &+ \sum_{k=0}^{\infty} C_{6k+5}(x^{6k+5}+x^{6k+8}+5x^{6k+7}+5x^{6k+10}) \\ &= 1+x^3(1+x^3)\sum_{k=0}^{\infty} C_{6k+3}x^{6k} + x^5(1+x^3)(1+5x^2)\sum_{k=0}^{\infty} C_{6k+5}x^{6k}. \end{aligned}$$

We have

$$\sum_{k=0}^{\infty} C_{6k+3}x^{6k} = \frac{6}{31}\sum_{k=0}^{\infty}(5^{3k+2}+6)x^{6k} = \frac{6}{31}\left(\frac{25}{1-(5x^2)^3}+\frac{6}{1-x^6}\right)$$

$$\sum_{k=0}^{\infty} C_{6k+5}x^{6k} = \frac{6}{31}\sum_{k=0}^{\infty}(5^{3k+3}-1)x^{6k} = \frac{6}{31}\left(\frac{125}{1-(5x^2)^3}-\frac{1}{1-x^6}\right)$$

It follows that $f_3(x)$ can be expressed by the formula given in the lemma. □

Lemma 3.6 *Let E_n denote the number of cyclically reduced words in $\mathcal{C}(a,bc;n)\backslash(\mathcal{C}(a;n)\cup\mathcal{C}(bc;n))$. Then for $n \geq 0$ we have*

$$E_{n+3}=E_{n+2}+E_{n+1}+3E_n+8+(-1)^n 4. \tag{3.1}$$

Proof Let us define some subsets of $\mathcal{C}(a,bc;n)$:

$\mathcal{E}_n$ – the set of cyclically reduced words in $\mathcal{C}(a,bc;n)\backslash(\mathcal{C}(a;n)\cup\mathcal{C}(bc;n))$,

$\mathcal{X}_n$ – the set of words of length n, of the form $a^{\varepsilon_1}u(bc)^{\varepsilon_2}$,

$\overline{\mathcal{X}_n}$ – the set of words of length n, of the form $(bc)^{\varepsilon_1}ua^{\varepsilon_2}$,

$\mathcal{Y}_n$ – the set of words of length n, of the form $a^{\varepsilon_1}ua^{\varepsilon_2}$,

where $\varepsilon_i \in \{-1,1\}$ for $i=1,2$ and $u \in \langle a,bc\rangle$. Note that $\mathcal{X}_n$ and $\overline{\mathcal{X}_n}$ are subsets of $\mathcal{E}_n$, but $\mathcal{Y}_n$ is not, as it contains words that are not cyclically reduced, and powers of a. The mapping $w \mapsto w^{-1}$ defines a bijection $\mathcal{X}_n \to \overline{\mathcal{X}_n}$. We define $X_n=\#\mathcal{X}_n=\#\overline{\mathcal{X}_n}$, $Y_n=\#\mathcal{Y}_n$.

Every element of $\mathcal{X}_{n+2}$ is of the form $w(bc)^{\varepsilon}$ for $w \in \mathcal{X}_n \cup \mathcal{Y}_n$. Conversely, if $n>0$, then for $w \in \mathcal{X}_n$ there is 1 element of the form $w(bc)^{\varepsilon}$ in $\mathcal{X}_{n+2}$, while for $w \in \mathcal{Y}_n$ there are 2 such elements. Thus $X_{n+2}=X_n+2Y_n$. Similarly we have $Y_{n+1}=Y_n+2X_n$. Now we can obtain a recursive equation for X_n as follows: $X_{n+3}-X_{n+1}=2Y_{n+1}=2Y_n+4X_n=X_{n+2}-X_n+4X_n$. Thus for $n \geq 1$ we have

$$X_{n+3}=X_{n+2}+X_{n+1}+3X_n. \tag{3.2}$$

For $n \geq 1$ we define a mapping $\iota\colon \mathcal{E}_n \to \mathcal{E}_{n+2}$. Let $w \in \mathcal{E}_n$. By the definition of $\mathcal{E}_n$ and Theorem 3.1, w is a word of length n in $\langle a, bc\rangle$, possibly cyclically permuted, that is neither a power of a nor a power of bc. We set

$$\iota(w) = \begin{cases} a^{\varepsilon}ua^{2\varepsilon} & \text{if } w = a^{\varepsilon}u \\ (bc)^{\varepsilon}u(bc)^{\varepsilon} & \text{if } w = (bc)^{\varepsilon}u \\ cubcb & \text{if } w = cub \\ b^{-1}u(bc)^{-1}c^{-1} & \text{if } w = b^{-1}uc^{-1}, \end{cases}$$

where $\varepsilon \in \{-1, 1\}$. Note that ι is injective and

$$\mathcal{E}_{n+2} = \iota(\mathcal{E}_n) \cup \mathcal{X}_{n+2} \cup \overline{\mathcal{X}_{n+2}} \cup \mathcal{Z} \cup \mathcal{U},$$

where $\mathcal{Z}$ is the set of words of the form $a^{\varepsilon_1}u(bc)^{\varepsilon_2}a^{\varepsilon_1}$, and $\mathcal{U}$ is the set of words of the form $cua^{\varepsilon_1}b$ or $b^{-1}ua^{\varepsilon_1}c^{-1}$, where $\varepsilon_i \in \{-1, 1\}$ for $i = 1, 2$ and $u \in \langle a, bc\rangle$. There are bijections $\mathcal{X}_{n+1} \to \mathcal{Z}$ given by $a^{\varepsilon_1}u(bc)^{\varepsilon_2} \mapsto a^{\varepsilon_1}u(bc)^{\varepsilon_2}a^{\varepsilon_1}$, and $\overline{\mathcal{X}_{n+2}} \to \mathcal{U}$ given by $bcua^{\varepsilon} \mapsto cua^{\varepsilon}b$, $(bc)^{-1}ua^{\varepsilon} \mapsto b^{-1}ua^{\varepsilon}c^{-1}$. Thus $\#\mathcal{Z} = X_{n+1}$, $\#\mathcal{U} = X_{n+2}$ and

$$E_{n+2} = 3X_{n+2} + X_{n+1} + E_n. \tag{3.3}$$

We have $E_n = X_n = 0$ for $n \leq 2$, $\mathcal{X}_3 = \{a^{\varepsilon_1}(bc)^{\varepsilon_2} \mid \varepsilon_1, \varepsilon_2 \in \{-1, 1\}\}$, $\mathcal{X}_4 = \{a^{2\varepsilon_1}(bc)^{\varepsilon_2} \mid \varepsilon_1, \varepsilon_2 \in \{-1, 1\}\}$; thus $X_3 = X_4 = 4$, $E_3 = 12$ and $E_4 = 16$. Thus (3.1) holds for $n = 0$ and $n = 1$. It is now routine to prove that (3.1) holds for all $n \geq 0$ by induction, using (3.3) and (3.2). □

Lemma 3.7 *For $n \geq 0$ we have*

$$D_{n+3} = D_{n+2} + D_{n+1} + 3D_n + \varphi(n), \tag{3.4}$$

where $\varphi(2k+1) = 4 \cdot 5^k$, $\varphi(2k) = 12 \cdot 5^k$ for $k \geq 0$. The growth function of $\mathcal{C}(a, bc)\backslash(\mathcal{C}(a) \cup \mathcal{C}(bc))$ with respect to the generators a, b, c is

$$f_4(x) = \frac{4x^3(3 + x)}{(1 - 5x^2)(1 - x - x^2 - 3x^3)}.$$

Proof Let $\mathcal{D}_n = \mathcal{C}(a, bc; n)\backslash(\mathcal{C}(a; n) \cup \mathcal{C}(bc; n))$. Every element of $\mathcal{D}_{n+2}$ that is not cyclically reduced is of the form $\alpha u \alpha^{-1}$, where α is a letter and $u \in \mathcal{D}_n$. Conversely, if $n \geq 1$, then for every $u \in \mathcal{D}_n$ there are 5 elements of the form $\alpha u \alpha^{-1}$ in $\mathcal{D}_{n+2}$ if u is not cyclically reduced, or 4 such words if u is cyclically reduced. Thus $D_{n+2} - E_{n+2} = 5(D_n - E_n) + 4E_n$, which gives, for $n \geq 0$,

$$D_{n+2} = E_{n+2} - E_n + 5D_n. \tag{3.5}$$

We have $D_n = E_n = 0$ for $n \leq 2$, $D_3 = E_3 = 12$ and $D_4 = E_4 = 16$. Thus (3.4) holds for $n = 0$ and $n = 1$. It is now routine to prove that (3.4) holds for all $n \geq 0$ by induction, using (3.5) and (3.1) from Lemma 3.6.

Now we can compute the growth function.

$$\begin{aligned}
f_4(x) &= \sum_{n=0}^{\infty} D_n x^n = x^3 \sum_{n=0}^{\infty} D_{n+3} x^n \\
&= x^3 \sum_{n=0}^{\infty} (D_{n+2} + D_{n+1} + 3D_n + \varphi(n)) x^n \\
&= x f_4(x) + x^2 f_4(x) + 3x^3 f_4(x) + x^3 \sum_{k=0}^{\infty} 5^k (12x^{2k} + 4x^{2k+1}) \\
&= (x + x^2 + 3x^3) f_4(x) + \frac{4x^3(3+x)}{1-5x^2},
\end{aligned}$$

and the lemma is proved. □

4. Growth functions and density of reducible and pseudo-Anosov elements

In this section we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $f(x)$ and $g(x)$ denote the growth functions of the sets of reducible and pseudo-Anosov elements respectively. Since $f(x)+g(x)$ is the growth function of $\mathcal{PM}(S)$, we have

$$f(x) + g(x) = 1 + 6\sum_{n=1}^{\infty} 5^{n-1} x^n = \frac{1+x}{1-5x}.$$

Let $f_1(x), f_2(x), f_3(x), f_4(x)$ be the growth functions computed in Lemmas 3.3, 3.4, 3.5, 3.7. By Lemma 3.2 we have

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} R_n x^n = 1 + \sum_{n=1}^{\infty} (2B_n + 2D_n - A_n - C_n) x^n \\
&= 1 + 2f_2(x) + 2f_4(x) - f_1(x) - f_3(x),
\end{aligned}$$

which is a rational function. Since $f(x)$ and $f(x)+g(x)$ are rational, so is $g(x)$. □

Let $f(n)$ and $g(n)$ be 2 sequences of nonnegative numbers. We write $f(n) = \Theta(g(n))$ if there exist 2 positive numbers c_1, c_2 such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for all but finitely many n.

Lemma 4.1 *Let $\mathcal{R}$ be the set of reducible elements in $\mathcal{PM}(S)$. Then $\#(\mathcal{B}(n) \cap \mathcal{R}) = \Theta(3^n)$.*

Proof Since we have the isometry $\rho\colon \mathcal{F} \to \mathcal{PM}(S)$,

$$\#(\mathcal{B}(n) \cap \mathcal{R}) = \sum_{k=0}^{n} R_k.$$

Clearly it suffices to show that $R_n = \Theta(3^n)$. We have $R_n > B_n$ and, by Lemma 3.2, $R_n < 2(B_n + D_n)$. Since $B_n = \Theta(3^n)$ by Lemma 3.4, it suffices to show that $D_n < 3^n$. That is easily proved by induction, using (3.4) from Lemma 3.7 and the inequality $\varphi(n) \leq 12 \cdot 3^n$. □

Proof of Theorem 1.2. By Lemma 4.1 we have $\#(\mathcal{B}(n) \cap \mathcal{R}) = \Theta(3^n)$, and since

$$\#\mathcal{B}(n) = 1 + 6\sum_{k=0}^{n-1} 5^k = \frac{3 \cdot 5^n - 1}{2},$$

thus $d(\mathcal{R}) = 0$. The result follows, because $d(\mathcal{P}) = 1 - d(\mathcal{R})$. □

Acknowledgements

This paper was partially written during my visit to Institut de Mathématiques de Bourgogne in Dijon in the period 01.10.2011 – 30.09.2012 supported by the MNiSW "Mobility Plus" Program 639/MOB/2011/0. I wish to thank the Institut for their hospitality. This work was also partially supported by the MNiSW grant N N201 366436.

References

[1] Atalan, F., Korkmaz, M.: The number of pseudo-Anosov elements in the mapping class group of a four-holed sphere. Turk. J. Math. 34 (2010), 585–592.

[2] Epstein, D. B. A.: Curves on 2-manifolds and isotopies. Acta Math. 115 (1966), 83–107.

[3] Farb, B.: Some problems on mapping class group and moduli space. In "Problems on Mapping Class Group and Related Topics", ed. by B. Farb, Proc. Symp. Pure and Applied Math., Vol. 74 (2006), 11–55.

[4] Farb, B., Margalit, D.: A Primer on Mapping Class Groups. PMS-49. Princeton University Press, 2012.

[5] Korkmaz, M.: Mapping class groups of nonorientable surfaces. Geom. Dedicata 89 (2002), 109–133.

[6] Maher, J.: Random walks on the mapping class group. Duke Math. J. 156 (2011), 429–468.

[7] Maher, J.: Exponential decay in the mapping class group. J. London Math. Soc. 86 (2012), 366–386.

[8] Rivin, I.: Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms. Duke Math. J. 142 (2008), 353–379.

[9] Szepietowski, B.: A presentation for the mapping class group of a non-orientable surface from the action on the complex of curves. Osaka J. Math. 45 (2008), 283–326.

[10] Takasawa, M.: Enumeration of Mapping Classes for the Torus. Geom. Dedicata 85 (2001), 11–19.