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Characteristic classes on Grassmannians

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Abstract: In this paper, we study the geometry and topology on the oriented Grassmann manifolds. In particular, we use characteristic classes and the Poincaré duality to study the homology groups of Grassmann manifolds. We show that for $k = 2$ or $n \leq 8$, the cohomology groups $H^*(G(k,n),\mathbb{R})$ are generated by the first Pontrjagin class, the Euler classes of the canonical vector bundles. In these cases, the Poincaré duality: $H^q(G(k,n),\mathbb{R}) \to H_{k(n-k)-q}(G(k,n),\mathbb{R})$ can be expressed explicitly.

Key words: Grassmann manifold, fibre bundle, characteristic class, homology group, Poincaré duality

1. Introduction

Let $G(k,n)$ be the Grassmann manifold formed by all oriented k-dimensional subspaces of Euclidean space $\mathbb{R}^n$. For any $\pi \in G(k,n)$, there are orthonormal vectors $e_1, \cdots, e_k$ such that π can be represented by $e_1 \wedge \cdots \wedge e_k$. Thus $G(k,n)$ becomes a submanifold of the space $\bigwedge^k(\mathbb{R}^n)$; then we can use moving frame to study the Grassmann manifolds.

There are 2 canonical vector bundles $E = E(k,n)$ and $F = F(k,n)$ over $G(k,n)$ with fibres generated by vectors of the subspaces and the vectors orthogonal to the subspaces, respectively. Then we have Pontrjagin classes $p_i(E)$ and $p_j(F)$ with the relationship

$$(1 + p_1(E) + \cdots)(1 + p_1(F) + \cdots) = 1.$$

If k or $n - k$ is an even number, we have Euler class $e(E)$ or $e(F)$.

The oriented Grassmann manifolds are classifying spaces for oriented vector bundles. For any oriented vector bundle $\tau\colon\ \xi \to M$ with fibre type $\mathbb{R}^k$, there is a map $g\colon\ M \to G(k,n)$ such that ξ is isomorphic to the induced bundle g^*E. If the maps $g_1, g_2\colon\ M \to G(k,n)$ are homotopic, the induced bundles g_1^*E and g_2^*E are isomorphic. Then the characteristic classes of the vector bundle ξ are the pullback of the characteristic classes of the vector bundle E.

In this paper, we study the geometry and topology on the oriented Grassmann manifolds. In particular, we use characteristic classes and the Poincaré duality to study the homology groups of oriented Grassmann manifolds. The characteristic classes of the canonical vector bundles can be represented by curvature and

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are the harmonic forms, see [5, 7, 8, 15, 20]. For $k = 2$ or $n \leq 8$, we show that the cohomology groups $H^*(G(k,n),\mathbb{R})$ are generated by the first Pontrjagin class $p_1(E)$ and the Euler classes $e(E)$, $e(F)$ if k or $n-k$ is even. In these cases, the Poincaré duality: $H^q(G(k,n),\mathbb{R}) \to H_{k(n-k)-q}(G(k,n),\mathbb{R})$ can be given explicitly.

In §2, we compute volumes of some homogeneous spaces that are needed in the later discussion. In §3, we study the Poincaré duality on oriented compact Riemannian manifolds. The results are Theorem 3.1.

The Poincaré polynomials of Grassmann manifolds $G(k,n)$ for $k = 2$ or $n \leq 8$ are listed at the end of §3, which give the real homology groups of Grassmann manifolds. From [12], we know that the tangent space of Grassmann manifolds is isomorphic to tensor products of the canonical vector bundles. In §4 we use the splitting principle of the characteristic class to study the relationship among these vector bundles, and show that the characteristic classes of the tangent bundle on Grassmann manifolds can be represented by that of canonical vector bundles.

In §5, we study $G(2,N)$; the main results are Theorem 5.5. In §6, we study the Grassmann manifold $G(3,6)$; the main results are Theorem 6.1.

In §7, §8 we study the Grassmann manifold $G(3,7)$ and $G(3,8)$; the main results are Theorem 7.5 and 8.4. In §9, we study $G(4,8)$; the main results are Theorem 9.4, 9.5.

As an application, in §5 and §9, we consider the Gauss maps of submanifolds in Euclidean spaces. The results generalize the work by Chern and Spanier [4]. For example, if $g\colon M \to G(4,8)$ is the Gauss map of an immersion $f\colon M \to \mathbb{R}^8$ of a compact oriented 4-dimensional manifold, we have

$$g_*[M] = \frac{1}{2}\chi(M)[G(4,5)] + \lambda[G(1,5)] + \frac{3}{2}\tau(M)[G(2,4)],$$

where $\lambda = \frac{1}{2}\int_M e(F(4,8))$ and $\tau(M)$ is the signature of M. $\lambda = 0$ if f is an imbedding.

In §10 we use Gysin sequence to compute the cohomology of the homogeneous space $ASSOC = G_2/SO(4)$, which was studied by Borel and Hirzebruch [6].

The cohomology groups of infinite Grassmann manifold $G(k,\mathbb{R}^\infty)$ are simple; they are generated by Pontrjagin classes and the Euler class (if k is even) of the canonical vector bundle freely; see [13], p.179.

The computations on specific Grassmann manifolds like $G(3,7)$ or $G(4,8)$ have important implications on the theory of calibrated submanifolds like associative, coassociative, or Cayley submanifolds of Riemannian 7-8-manifolds of G_2 or $Spin_7$ holonomy. This work has many applications like [1, 11] among potential others. In [1, 11], there are applications to associative, coassociative submanifolds of G_2 manifolds.

2. The volumes of homogeneous spaces

For any $\pi \in G(k,n)$, there are orthonormal vectors $e_1, \cdots, e_k$ such that π can be represented by $e_1 \wedge \cdots \wedge e_k$. These give an imbedding of $G(k,n)$ in Euclidean space $\bigwedge^k(\mathbb{R}^n)$; see [2, 20]. Let $e_1, e_2, \cdots, e_n$ be orthonormal frame fields on $\mathbb{R}^n$ such that $G(k,n)$ is generated by $e_1 \wedge \cdots \wedge e_k$ locally. The vectors $e_1, e_2, \cdots, e_n$ can be viewed as functions on Grassmann manifolds. Let $\mathrm{d}e_A = \sum\limits_{B=1}^{n} \omega_A^B e_B$, $\omega_A^B = \langle \mathrm{d}e_A, e_B\rangle$ be 1 forms on $G(k,n)$. From $\mathrm{d}^2 e_A = 0$, we have $\mathrm{d}\omega_A^B = \sum\limits_{C=1}^{n} \omega_A^C \wedge \omega_C^B$. By

$$\mathrm{d}(e_1 \wedge \cdots \wedge e_k) = \sum_{i=1}^{k}\sum_{\alpha=k+1}^{n} \omega_i^\alpha E_{i\alpha},$$

$$E_{i\alpha} = e_1 \cdots e_{i-1} e_\alpha e_{i+1} \cdots e_k,\ i = 1, \cdots, k,\ \alpha = k+1, \cdots, n,$$

we know $E_{i\alpha}$ forms a basis of $T_{e_1 \cdots e_k} G(k,n)$ and ω_i^α is their dual basis.

$$\mathrm{d}s^2 = \langle \mathrm{d}(e_1 \wedge \cdots \wedge e_k), \mathrm{d}(e_1 \wedge \cdots \wedge e_k) \rangle = \sum_{i,\alpha} (\omega_i^\alpha)^2$$

is the induced metric on $G(k,n)$. Differential $E_{i\alpha} = e_1 \cdots e_{i-1} e_\alpha e_{i+1} \cdots e_k$, by Gauss equation, we get the Riemannian connection ∇ on $G(k,n)$,

$$\nabla E_{i\alpha} = \sum_{j=1}^{k} \omega_i^j E_{j\alpha} + \sum_{\beta=k+1}^{n} \omega_\alpha^\beta E_{i\beta}.$$

Grassmann manifold $G(k,n)$ is oriented; the orientation is given by the volume form

$$\omega_1^{k+1} \wedge \omega_2^{k+1} \wedge \cdots \wedge \omega_k^{k+1} \wedge \cdots \wedge \omega_1^n \wedge \omega_2^n \wedge \cdots \wedge \omega_k^n.$$

For later use we compute the volumes for some homogeneous spaces. We first compute the volume of special orthogonal group $SO(n)$.

Let $gl(n,\mathbb{R})$ be the set of all $n \times n$ real matrices with the inner product

$$\langle X, Y \rangle = \mathrm{tr}\,(XY^t) = \sum_{A,B} X_{AB} Y_{AB}, \quad X = (X_{AB}), Y = (Y_{CD}) \in gl(n,\mathbb{R}).$$

Then $gl(n,\mathbb{R})$ is a Euclidean space and $SO(n)$ is a Riemannian submanifold of $gl(n,\mathbb{R})$. Represent the elements of $SO(n)$ by $G = (e_1, \cdots, e_n)^t$, where e_A is the A-th row of G. The vectors $e_1, \cdots, e_n$ can be viewed as functions of $SO(n)$; then $\omega_A^B = \langle \mathrm{d}e_A, e_B \rangle = \mathrm{d}e_A \cdot e_B^t$ are 1 forms on $SO(n)$, $\omega_A^B + \omega_B^A = 0$. Let E_{BC} be the matrix with 1 in the B-th row, C-th column, the others being zero. We have

$$\mathrm{d}GG^{-1} = (\omega_A^B) = \sum_{A,B} \omega_A^B E_{AB}, \quad \mathrm{d}G = \sum_{A<B} \omega_A^B (E_{AB} - E_{BA}) G.$$

Then $\{(E_{AB} - E_{BA})G\}$ is a basis of $T_G SO(n)$ and

$$\mathrm{d}s^2 = \langle \mathrm{d}G, \mathrm{d}G \rangle = 2 \sum_{A<B} \omega_A^B \otimes \omega_A^B$$

is a Riemannian metric on $SO(n)$.

Proposition 2.1 *The volume of $SO(n)$ is*

$$V(SO(n)) = 2^{\frac{1}{2}(n-1)} V(S^{n-1}) V(SO(n-1)) = 2^{\frac{1}{4}n(n-1)} V(S^{n-1}) \cdots V(S^1).$$

Proof Let $\bar{e}_n = (0, \cdots, 0, 1)$ be a fixed vector. The map $\tau(G) = \bar{e}_n G = e_n$ defines a fibre bundle $\tau\colon\ SO(n) \to S^{n-1}$ with fibres $SO(n-1)$. By $\mathrm{d}e_n = \sum \omega_n^A e_A$,

$$\mathrm{d}V_{S^{n-1}} = \omega_1^n \cdots \omega_{n-1}^n$$

is the volume element of S^{n-1}. The volume element of $SO(n)$ can be represented by

$$\mathrm{d}V_{SO(n)} = (\sqrt{2})^{\frac{1}{2}n(n-1)} \prod_{A<B} \omega_A^B = 2^{\frac{1}{2}(n-1)}(\sqrt{2})^{\frac{1}{2}(n-1)(n-2)} \prod_{A<B<n} \omega_A^B \cdot \tau^* \mathrm{d}V_{S^{n-1}},$$

restricting $(\sqrt{2})^{\frac{1}{2}(n-1)(n-2)} \prod_{A<B<n} \omega_A^B$ on the fibres of τ are the volume elements of the fibres. Integration $\mathrm{d}V_{SO(n)}$ along the fibre of τ first, then on S^{n-1}, shows

$$V(SO(n)) = 2^{\frac{1}{2}(n-1)} V(S^{n-1}) V(SO(n-1)).$$

□

As we know $V(S^m) = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma(\frac{m+1}{2})}$,

$$V(S^{2n-1}) = \frac{2\pi^n}{(n-1)!}, \quad V(S^{2n}) = \frac{2^{2n+1} n! \pi^n}{(2n)!}.$$

To compute the volume of $G(k,n)$, we use principle bundle $SO(n) \to G(k,n)$ with the Lie group $SO(k) \times SO(n-k)$ as fibres.

Proposition 2.2 *The volume of Grassmann manifold $G(k,n)$ is*

$$V(G(k,n)) = \frac{V(SO(n))}{2^{\frac{1}{2}k(n-k)} V(SO(k)) V(SO(n-k))} = \frac{V(S^{n-1}) \cdots V(S^{n-k})}{V(S^{k-1}) \cdots V(S^1)}.$$

The proof is similar to that of Proposition 2.1. By simple computation, we have

$$V(G(2,n+2)) = \frac{2(2\pi)^n}{n!}, \quad V(G(3,6)) = \frac{2}{3}\pi^5,$$

$$V(G(3,7)) = \frac{16}{45}\pi^6, \quad V(G(3,8)) = \frac{2}{45}\pi^8, \quad V(G(4,8)) = \frac{8}{135}\pi^8.$$

Now we compute the volume of complex Grassmann manifold $G_{\mathbb{C}}(k,n)$. Let J be the natural complex structure on $\mathbb{C}^n = \mathbb{R}^{2n}$ and $s_1, \cdots, s_k$ be Hermitian orthonormal basis of $\pi \in G_{\mathbb{C}}(k,n)$. Let e_{2i-1}, $e_{2i} = Je_{2i-1} \in \mathbb{R}^{2n}$ be the realization vectors of s_i, $\sqrt{-1}\, s_i$ respectively. Then $e_1 e_2 \cdots e_{2k-1} e_{2k} \in G(2k,2n)$, and $G_{\mathbb{C}}(k,n)$ becomes a submanifold of $G(2k,2n)$.

Let $U(n) = \{G \in gl(n,\mathbb{C}) \mid G \cdot \overline{G}^t = I\}$ be the unitary group and the Hermitian inner product of $X = (X_{AB}), Y = (Y_{CD}) \in gl(n,\mathbb{C})$ be

$$\langle X, Y \rangle = \mathrm{tr}\,(X\overline{Y}^t) = \sum_{A,B} X_{AB} \overline{Y}_{AB}.$$

Let $G = (s_1, \cdots, s_n)^t \in U(n)$ represented by the rows of G, $\omega_A^B = \langle \mathrm{d}s_A, s_B \rangle = \mathrm{d}s_A \cdot \bar{s}_B^t$ be 1 forms on $U(n)$. Let $\omega_A^B = \varphi_A^B + \sqrt{-1}\,\psi_A^B$. From $\omega_A^B + \overline{\omega}_B^A = 0$ we have $\varphi_A^B + \varphi_B^A = 0$, $\psi_A^B - \psi_B^A = 0$. Then

$$\begin{aligned} \mathrm{d}G &= \sum_{A,B} \omega_A^B E_{AB} G \\ &= \sum_{A<B} \varphi_A^B (E_{AB} - E_{BA}) G + \sqrt{-1}\,\{ \sum_{A<B} \psi_A^B (E_{AB} + E_{BA}) G + \sum_A \psi_A^A E_{AA} G \}, \end{aligned}$$

and

$$\mathrm{d}s^2 = \langle \mathrm{d}G, \mathrm{d}G \rangle = 2\sum_{A<B} (\varphi_A^B \otimes \varphi_A^B + \psi_A^B \otimes \psi_A^B) + \sum_A \psi_A^A \otimes \psi_A^A$$

is a Riemannian metric on $U(n)$. The volume element is

$$\mathrm{d}V_{U(n)} = 2^{\frac{1}{2}n(n-1)} \psi_1^1 \cdots \psi_n^n \prod_{A<B} \varphi_A^B \psi_A^B.$$

Proposition 2.3 *(1) The volume of $U(n)$ is*

$$V(U(n)) = 2^{n-1} V(S^{2n-1}) V(U(n-1)) = 2^{\frac{1}{2}n(n-1)} V(S^{2n-1}) V(S^{2n-3}) \cdots V(S^1);$$

(2) As Riemannian submanifold of $G(2k,2n)$, the volume of $G_{\mathbb{C}}(k,n)$ is

$$V(G_{\mathbb{C}}(k,n)) = \frac{V(U(n))}{V(U(k))V(U(n-k))};$$

(3) The volume of $\mathbb{C}P^n = G_{\mathbb{C}}(1,n+1)$ is

$$V(\mathbb{C}P^n) = \frac{(2\pi)^n}{n!}.$$

Proof Let $\bar{e}_n = (0,\cdots,0,1)$ be a fixed vector. The map $\tau(G) = \bar{e}_n G = s_n$ defines a fibre bundle $\tau\colon\ U(n) \to S^{2n-1}$ with fibre type $U(n-1)$. From $\mathrm{d}s_n = \sum\ \omega_n^A s_A$ and $\omega_n^A = \varphi_n^A + \sqrt{-1}\,\psi_n^A,\ \varphi_n^n = 0$, we have the volume element of S^{2n-1},

$$\mathrm{d}V_{S^{2n-1}} = \varphi_1^n \psi_1^n \cdots \varphi_{n-1}^n \psi_{n-1}^n \psi_n^n.$$

Then the volume element of $U(n)$ can be represented by

$$\mathrm{d}V_{U(n)} = 2^{n-1} \tau^* \mathrm{d}V_{S^{2n-1}} \cdot \mathrm{d}V_{U(n-1)}.$$

These prove (1).

As noted above, the map $[s_1 \cdots s_k] \mapsto e_1 e_2 \cdots e_{2k-1} e_{2k}$ gives an imbedding of $G_{\mathbb{C}}(k,n)$ in $G(2k,2n)$. From $\mathrm{d}s_i = \sum\ \omega_i^j s_j + \sum\ \omega_i^\alpha s_\alpha,\ \omega_i^j = \varphi_i^j + \sqrt{-1}\,\psi_i^j,\ \ \omega_i^\alpha = \varphi_i^\alpha + \sqrt{-1}\,\psi_i^\alpha$, we have

$$\mathrm{d}e_{2i-1} = \sum\ (\varphi_i^j e_{2j-1} + \psi_i^j e_{2j}) + \sum\ (\varphi_i^\alpha e_{2\alpha-1} + \psi_i^\alpha e_{2\alpha}),$$

$$\mathrm{d}e_{2i} = \sum\ (\varphi_i^j e_{2j} - \psi_i^j e_{2j-1}) + \sum\ (\varphi_i^\alpha e_{2\alpha} - \psi_i^\alpha e_{2\alpha-1}).$$

Then

$$\mathrm{d}(e_1 e_2 \cdots e_{2k-1} e_{2k}) = \sum_{i,\alpha}\ \varphi_i^\alpha (E_{2i-1\,2\alpha-1} + E_{2i\,2\alpha}) + \sum_{i,\alpha}\ \psi_i^\alpha (E_{2i-1\,2\alpha} - E_{2i\,2\alpha-1}),$$

$$\mathrm{d}V_{G_{\mathbb{C}}(k,n)} = 2^{k(n-k)} \varphi_1^{k+1} \psi_1^{k+1} \cdots \varphi_k^n \psi_k^n.$$

The rest is similar to that of Proposition 2.1. □

The symmetric space $SLAG = SU(n)/SO(n)$ can be imbedded in $G(n,2n)$ as follows. Let $\bar{e}_{2i-1}, \bar{e}_{2i} = J\bar{e}_{2i-1},\ i = 1,\cdots,n$, be a fixed orthonormal basis of $\mathbb{C}^n = \mathbb{R}^{2n}$; the subspace $\{G(\bar{e}_1\bar{e}_3\cdots\bar{e}_{2n-1}) \mid G \in SU(n) \subset SO(2n)\}$ is diffeomorphic to $SLAG = SU(n)/SO(n)$.

Proposition 2.4 *(1) The volume of special unitary group $SU(n)$ is*

$$V(SU(n)) = 2^{n-1}\sqrt{\frac{n}{n-1}}V(S^{2n-1})V(SU(n-1));$$

(2) The volume of $SLAG$ is

$$V(SLAG) = \frac{V(SU(n))}{V(SO(n))}.$$

Proof The proof is similar to that of Proposition 2.1. Let $G = (s_1, \cdots, s_n)^t \in SU(n)$, $\omega_A^B = \mathrm{d}s_A \cdot \bar{s}_B^t$. From $\det G = 1$ we have $\sum\limits_{A=1}^{n} \omega_A^A = 0$; then $\psi_n^n = -\sum\limits_{B\neq n} \psi_B^B$. The Riemannian metric on $SU(n)$ is

$$\begin{aligned}
\mathrm{d}s^2 &= 2\sum_{A<B}(\varphi_A^B \otimes \varphi_A^B + \psi_A^B \otimes \psi_A^B) + \sum_{B\neq n}\psi_B^B \otimes \psi_B^B + \psi_n^n \otimes \psi_n^n \\
&= 2\sum_{A<B}(\varphi_A^B \otimes \varphi_A^B + \psi_A^B \otimes \psi_A^B) \\
&\quad +(\psi_1^1, \cdots, \psi_{n-1}^{n-1}) \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix} \begin{pmatrix} \psi_1^1 \\ \vdots \\ \psi_{n-1}^{n-1} \end{pmatrix}.
\end{aligned}$$

Then

$$\mathrm{d}V_{SU(n)} = 2^{\frac{1}{2}n(n-1)}\sqrt{n}\psi_1^1 \cdots \psi_{n-1}^{n-1} \prod_{A<B} \varphi_A^B \psi_A^B.$$

The volume of special unitary group $SU(n)$ is

$$V(SU(n)) = 2^{n-1}\sqrt{\frac{n}{n-1}}V(S^{2n-1})V(SU(n-1)).$$

Let e_{2A-1}, $e_{2A} = Je_{2A-1}$ be the realization vectors of s_A, $\sqrt{-1}\,s_A$ respectively. $SLAG$ is generated by $G(\bar{e}_1\bar{e}_3 \cdots \bar{e}_{2n-1}) = e_1e_3 \cdots e_{2n-1}$,

$$\mathrm{d}(e_1e_3 \cdots e_{2n-1}) = \sum \psi_B^B(E_{2B-1\,2B} - E_{2n-1\,2n}) + \sum_{A<B} \psi_A^B(E_{2A-1\,2B} + E_{2B-1\,2A}),$$

$$\mathrm{d}s^2 = 2\sum_{A<B}\psi_A^B \otimes \psi_A^B + 2\sum_{B\neq n}\psi_B^B \otimes \psi_B^B + \sum_{B\neq C<n}\psi_B^B \otimes \psi_C^C.$$

Then

$$\mathrm{d}V_{SLAG} = 2^{\frac{1}{4}n(n-1)}\sqrt{n}\psi_1^1 \cdots \psi_{n-1}^{n-1} \prod_{A<B} \psi_A^B.$$

Let τ: $SU(n) \to SLAG$ be the projection with fibres $SO(n)$. Restricting $\mathrm{d}s_i = \sum \omega_i^j s_j + \sum \omega_i^\alpha s_\alpha$ on the fibre of τ, we have $\omega_i^\alpha = 0$ and $\psi_i^j = 0$; then $\mathrm{d}V_{SO(n)} = 2^{\frac{1}{4}n(n-1)} \prod_{A<B} \varphi_A^B$ is the volume element of the fibres. This completes the proof. □

Let $Sp(n) = \{G \in gl(n, \mathbb{H}) \mid G \cdot \overline{G}^t = I\}$ be the symplectic group, and $G_{\mathbb{H}}(k,n) = \frac{Sp(n)}{Sp(k)\times Sp(n-k)}$ be the quaternion Grassmann manifold which can also be imbedded in $G(4k, 4n)$. The following proposition can be proved as Proposition 2.3.

Proposition 2.5 *(1) The volume of $Sp(n)$ is*

$$V(Sp(n)) = 4^{n-1}V(S^{4n-1})V(Sp(n-1)) = 2^{n(n-1)}V(S^{4n-1})V(S^{4n-5})\cdots V(S^3);$$

(2) As Riemannian submanifold of $G(4k, 4n)$, the volume of $G_{\mathbb{H}}(k,n)$ is

$$V(G_{\mathbb{H}}(k,n)) = \frac{2^{2k(n-k)}V(Sp(n))}{V(Sp(k))V(Sp(n-k))}.$$

As $\mathbb{H}P^n = G_{\mathbb{H}}(1, n+1)$, we have

$$V(\mathbb{H}P^n) = \frac{(4\pi)^{2n}}{(2n+1)!}.$$

3. The Poincaré duality

Let M be a compact oriented Riemannian manifold and $H_q(M) = H_q(M, \mathbb{R})$ its q-th singular homology group, and $H^q(M) = H^q(M, \mathbb{R})$ be the q-th de Rham cohomology group. For any $[\xi] \in H^q(M)$ and $[z] = [\sum \lambda_i \sigma_i] \in H_q(M)$, we can define

$$[\xi]([z]) = \int_z \xi = \sum \lambda_i \int_{\sigma_i} \xi = \sum \lambda_i \int_{\triangle^q} \sigma_i^* \xi,$$

where every singular simplex $\sigma_i \colon \triangle^q \to M$ is differentiable. If $[\xi] \in H^q(M, \mathbb{Z})$ and $[z] \in H_q(M, \mathbb{Z})$, we have $[\xi]([z]) \in \mathbb{Z}$. By universal coefficients theorem, we have

$$H^q(M, \mathbb{R}) \cong \mathrm{Hom}\,(H_q(M, \mathbb{R}), \mathbb{R}),$$

and

$$H^q(M, \mathbb{Z}) \cong \mathrm{Hom}\,(H_q(M, \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}\,(H_{q-1}(M, \mathbb{Z}), \mathbb{Z}).$$

On the other hand, we have Poincaré duality

$$D\colon\ H^q(M, \mathbb{R}) \to H_{n-q}(M, \mathbb{R}), \quad n = \dim M.$$

For any $[\xi] \in H^q(M)$, $D[\xi] \in H_{n-q}(M)$, we have

$$[\eta](D[\xi]) = \int_{D[\xi]} \eta = \int_M \xi \wedge \eta$$

for any $[\eta] \in H^{n-q}(M)$.

In the following, we use harmonic forms to represent the Poincaré duality. Let $\varphi_1, \cdots, \varphi_k$ be the basis of $H^q(M)$ and $[T_i] = D(\varphi_i)$ be their Poincaré duals. By Hodge Theorem, we can assume that $\varphi_1, \cdots, \varphi_k$ are all the harmonic forms on M. Then $*\varphi_1, \cdots, *\varphi_k$ are also the harmonic forms and form a basis of $H^{n-q}(M)$. Let

$$a_{ij} = (\varphi_i, \varphi_j) = \int_M \langle \varphi_i, \varphi_j \rangle \, \mathrm{d}V_M = \int_M \varphi_i \wedge *\varphi_j$$

be the inner product of differential forms φ_i, φ_j. Let $\psi_1, \cdots, \psi_k$ be the dual basis of $[T_1], \cdots, [T_k]$, also represented by harmonic forms. Assuming $\psi_j = \sum *\varphi_i b_{ij}$, by Poincaré duality,

$$\delta_{ij} = \int_{T_i} \psi_j = \int_M \varphi_i \wedge \psi_j = \int_M \sum \varphi_i \wedge *\varphi_l b_{lj} = \sum a_{il} b_{lj}.$$

This shows $(b_{ij}) = (a_{ij})^{-1}$, and we have

$$(\psi_1, \cdots, \psi_k) = (*\varphi_1, \cdots, *\varphi_k)(a_{ij})^{-1}.$$

Theorem 3.1 *Let $\varphi_1, \cdots, \varphi_k$ be a basis of the cohomology group $H^q(M)$ represented by harmonic forms. Let $[T_1], \cdots, [T_k] \in H_{n-q}(M)$ be the dual of $(\psi_1, \cdots, \psi_k) = (*\varphi_1, \cdots, *\varphi_k)(a_{ij})^{-1}$, where $a_{ij} = (\varphi_i, \varphi_j)$. The Poincaré duality $D\colon H^q(M) \to H_{n-q}(M)$ is given by*

$$D(\varphi_i) = [T_i].$$

Furthermore, if $[S_1], \cdots, [S_k]$ are the dual basis of $\varphi_1, \cdots, \varphi_k$, then

$$D(\psi_i) = (-1)^{q(n-q)}[S_i].$$

Proof The equations $D(\psi_i) = (-1)^{q(n-q)}[S_i]$ follow from $**\varphi_i = (-1)^{q(n-q)}\varphi_i$ and $(\varphi_i, \varphi_j) = (*\varphi_i, *\varphi_j)$. □

Theorem 3.1 can be applied to the Poincaré duality $D\colon H^q(M, \mathbb{Z}) \to H_{n-q}(M, \mathbb{Z})$ if we ignore the torsion elements of $H^q(M, \mathbb{Z})$.

The q-th Betti number is the common dimension of the real homology and cohomology groups $H_q(G(k,n))$ and $H^q(G(k,n))$ (and is also the rank of $H_q(G(k,n), \mathbb{Z})$ and $H^q(G(k,n), \mathbb{Z})$). The Poincaré polynomials, with the Betti numbers as coefficients, are given by the following Table (see [7, 8, 18]).

Grassmannian	Poincaré polynomial
$G(1, n+1)$	$1 + t^n$
$G(2, 2n+1)$	$1 + t^2 + t^4 + \cdots + t^{4n-2}$
$G(2, 2n+2)$	$(1 + t^{2n})(1 + t^2 + \cdots + t^{2n})$
$G(3, 6)$	$(1 + t^4)(1 + t^5)$
$G(3, 7)$	$(1 + t^4 + t^8)(1 + t^4)$
$G(3, 8)$	$(1 + t^4 + t^8)(1 + t^7)$
$G(4, 8)$	$(1 + t^4 + t^8)(1 + t^4)^2$

4. The vector bundles on $G(k,n)$

Let $\tau_1\colon E(k,n) \to G(k,n)$ be the canonical vector bundle on Grassmann manifold $G(k,n)$, and the fibre over $\pi \in G(k,n)$ be the vectors of π. $E = E(k,n)$ is a Riemannian vector bundle with the induced metric. Let $e_1, \cdots, e_k, e_{k+1}, \cdots, e_n$ be orthonormal frame fields on $\mathbb{R}^n$, $G(k,n)$ is locally generated by

$e_1 \cdots e_k = e_1 \wedge \cdots \wedge e_k$. Then $e_1, \cdots, e_k$ are local orthonormal sections of the vector bundle τ_1. From $\mathrm{d}e_i = \sum_{j=1}^{k} \omega_i^j e_j + \sum_{\alpha=k+1}^{n} \omega_i^\alpha e_\alpha$, we know that $\nabla e_i = \sum \omega_i^j e_j$ defines a Riemannian connection on τ_1. From $\nabla^2 e_i = \sum (\mathrm{d}\omega_i^j - \sum \omega_i^l \wedge \omega_l^j) e_j$, we have curvature forms

$$\Omega_i^j = \mathrm{d}\omega_i^j - \sum \omega_i^l \wedge \omega_l^j = \sum \omega_i^\alpha \wedge \omega_\alpha^j.$$

The total Pontrjagin classes of the vector bundle $\tau_1\colon E \to G(k,n)$ are

$$p(E) = 1 + p_1(E) + p_2(E) + \cdots = \det(I + \frac{1}{2\pi}(\Omega_i^j)).$$

If k is even, we have Euler class

$$e(E) = \frac{(-1)^{\frac{k}{2}}}{(4\pi)^{\frac{k}{2}}(\frac{k}{2})!} \sum_{i_1,\cdots,i_k} \varepsilon(i_1 i_2 \cdots i_k) \Omega_{i_1 i_2} \Omega_{i_3 i_4} \cdots \Omega_{i_{k-1} i_k}.$$

Similarly, we can define vector bundle $\tau_2\colon F = F(k,n) \to G(k,n)$ on Grassmann manifold $G(k,n)$; the fibre over $e_1 \cdots e_k \in G(k,n)$ is the vectors orthogonal to $e_1, \cdots, e_k$. Then $e_{k+1}, \cdots, e_n$ are local orthonormal sections of F. From $\mathrm{d}e_\alpha = \sum \omega_\alpha^\beta e_\beta + \sum \omega_\alpha^i e_i$, we have Riemannian connection $\nabla e_\alpha = \sum \omega_\alpha^\beta e_\beta$. The curvature forms are given by

$$\nabla^2 e_\alpha = \sum \Omega_\alpha^\beta e_\beta, \quad \Omega_\alpha^\beta = \sum \omega_\alpha^i \wedge \omega_i^\beta.$$

The total Pontrjagin classes of the vector bundle $\tau_2\colon F \to G(k,n)$ are

$$p(F) = 1 + p_1(F) + p_2(F) + \cdots = \det(I + \frac{1}{2\pi}(\Omega_\alpha^\beta)).$$

The direct sum $E(k,n) \oplus F(k,n) = G(k,n) \times \mathbb{R}^n$ is trivial, and we have

$$(1 + p_1(E) + p_2(E) + \cdots) \cdot (1 + p_1(F) + p_2(F) + \cdots) = 1.$$

Then Pontrjagin classes of F are determined by that of E. For example, we have

$$p_1(F) = -p_1(E), \quad p_2(F) = p_1^2(E) - p_2(E), \quad p_3(F) = -p_1^3(E) + 2p_1(E)p_2(E) - p_3(E).$$

Let $*\colon \bigwedge^k(\mathbb{R}^n) \to \bigwedge^{n-k}(\mathbb{R}^n)$ be the star operator, $*G(k,n) = G(n-k,n)$, and the canonical vector bundles $E(k,n), F(k,n)$ are interchanged under the map $*$.

Proposition 4.1 *The tangent space $TG(k,n)$ of a Grassmann manifold is isomorphic to tensor product $E(k,n) \otimes F(k,n)$. If $k(n-k)$ is even, we have*

$$e(G(k,n)) = e(E(k,n) \otimes F(k,n)).$$

Proof Let $e_1, e_2, \cdots, e_n$ be an oriented orthonormal basis of $\mathbb{R}^n$, the fibre of $E(k,n)$ over $x = e_1 \wedge \cdots \wedge e_k \in G(k,n)$ is generated by $e_1, \cdots, e_k$ and the fibre of $F(k,n)$ over x is generated by $e_{k+1}, \cdots, e_n$. On the other hand, the tangent space $T_xG(k,n)$ is generated by $E_{i\,\alpha} = e_1 \wedge \cdots \wedge e_{i-1} \wedge e_\alpha \wedge e_{i+1} \cdots \wedge e_k$. It is easy to see that the map $E_{i\,\alpha} \mapsto e_i \otimes e_\alpha$ gives an isomorphism from tangent bundle $TG(k,n)$ to tensor product $E \otimes F$. See also [12].

The isomorphism $TG(k,n) \to E(k,n) \otimes F(k,n)$ preserves the connections on $TG(k,n)$ and $E \otimes F$, respectively, where the connection on $E \otimes F$ is

$$\nabla (e_i \otimes e_\alpha) = \sum \omega_i^j e_j \otimes e_\alpha + \sum \omega_\alpha^\beta e_i \otimes e_\beta.$$

□

In the following we use the splitting principle of the characteristic class to study the relationship among these vector bundles. We study the oriented Grassmann manifold $G(2k,2n)$; the other cases can be discussed similarly. Let $s_1, \cdots, s_{2k}$ be the orthonormal sections of vector bundle $E(2k,2n)$ such that the curvature of Riemannian connection has the form

$$\frac{1}{2\pi}\nabla^2 \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{2k-1} \\ s_{2k} \end{pmatrix} = \begin{pmatrix} 0 & -x_1 & & & \\ x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -x_k \\ & & & x_k & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{2k-1} \\ s_{2k} \end{pmatrix}.$$

The total Pontrjagin classes and the Euler class of $E = E(2k,2n)$ are

$$p(E) = \prod_{i=1}^{k} (1 + x_i^2), \quad e(E) = x_1 \cdots x_k.$$

Similarly, assuming $t_{2k+1}, t_{2k+2}, \cdots, t_{2n}$ are the orthonormal sections of vector bundle $F(2k,2n)$, the curvature of the Riemannian connection has the form

$$\frac{1}{2\pi}\nabla^2 \begin{pmatrix} t_{2k+1} \\ t_{2k+2} \\ \vdots \\ t_{2n-1} \\ t_{2n} \end{pmatrix} = \begin{pmatrix} 0 & -y_{k+1} & & & \\ y_{k+1} & 0 & & & \\ & & \ddots & & \\ & & & 0 & -y_n \\ & & & y_n & 0 \end{pmatrix} \begin{pmatrix} t_{2k+1} \\ t_{2k+2} \\ \vdots \\ t_{2n-1} \\ t_{2n} \end{pmatrix}.$$

The total Pontrjagin classes and the Euler class of $F = F(2k,2n)$ are

$$p(F) = \prod_{\alpha=k+1}^{n} (1 + y_\alpha^2), \quad e(F) = y_{k+1} \cdots y_n.$$

$s_{2i-1} \otimes t_{2\alpha-1}, s_{2i} \otimes t_{2\alpha-1}, s_{2i-1} \otimes t_{2\alpha}, s_{2i} \otimes t_{2\alpha}$ are the local orthonormal sections of vector bundle $E \otimes F \cong TG(2k,2n)$. The curvature of Riemannian connection on $E \otimes F$ is given by

$$\frac{1}{2\pi}\nabla^2 \begin{pmatrix} s_{2i-1} \otimes t_{2\alpha-1} \\ s_{2i} \otimes t_{2\alpha-1} \\ s_{2i-1} \otimes t_{2\alpha} \\ s_{2i} \otimes t_{2\alpha} \end{pmatrix} = \begin{pmatrix} 0 & -x_i & -y_\alpha & 0 \\ x_i & 0 & 0 & -y_\alpha \\ y_\alpha & 0 & 0 & -x_i \\ 0 & y_\alpha & x_i & 0 \end{pmatrix} \begin{pmatrix} s_{2i-1} \otimes t_{2\alpha-1} \\ s_{2i} \otimes t_{2\alpha-1} \\ s_{2i-1} \otimes t_{2\alpha} \\ s_{2i} \otimes t_{2\alpha} \end{pmatrix}.$$

Then we have

Lemma 4.2 *(1)* $e(TG(2k,2n)) = e(E\otimes F) = \prod_{i,\alpha}\,(x_i^2 - y_\alpha^2)$;

(2) $p(TG(2k,2n)) = p(E\otimes F) = \prod_{i,\alpha}\,(1 + 2(x_i^2 + y_\alpha^2) + (x_i^2 - y_\alpha^2)^2)$.

By simple computation, we have

$$p_1(TG(2k,2n)) = (2n-2k)p_1(E) + 2kp_1(F) = 2(n-2k)p_1(E).$$

In particular, $p_1(TG(2k,4k)) = 0$.

In the next section, we shall show

$$e(TG(2,2n+2)) = (n+1)e^{2n}(E(2,2n+2)),$$

$$e(TG(2,2n+3)) = (n+1)e^{2n+1}(E(2,2n+3)).$$

We can also show

$$e(TG(3,7)) = 3e^3(F(3,7)), \quad e(TG(4,8)) = 6e^4(E(4,8)) = 6e^4(F(4,8)).$$

5. The cases of $G(2,N)$

In this section, we study the real homology of Grassmann manifold $G(2,N)$.

As is well known, the oriented Grassmann manifold $G(2,N)$ is a Kähler manifold and can be imbedded in a complex projective space. Here we give a new proof. Let e_1, e_2 be the oriented orthonormal basis of $\pi\in G(2,N)$, $e_1\mapsto e_2,\ e_2\mapsto -e_1$ defines an almost complex structure

$$J\colon\ T_\pi G(2,N)\to T_\pi G(2,N),$$

$$E_{1\,\alpha} = e_\alpha\wedge e_2 \mapsto -e_\alpha\wedge e_1 = E_{2\,\alpha},\ E_{2\,\alpha} = e_1\wedge e_\alpha\mapsto e_2\wedge e_\alpha = -E_{1\,\alpha}.$$

It is easy to see that J is well defined and preserves the metric on $G(2,N)$.

Proposition 5.1 $G(2,N)$ *is a Kähler manifold with complex structure* J.

Proof Let ∇ be the Riemannian connection on $TG(2,N)$ defined above. We have

$$(\nabla J)E_{i\,\alpha} = \nabla(JE_{i\,\alpha}) - J(\nabla E_{i\,\alpha}) = 0,\quad i=1,2.$$

Hence, $\nabla J = 0$, J is a complex structure and $G(2,N)$ is a Kähler manifold. □

The Euler classes of canonical vector bundles $E = E(2,2n+2)$ and $F = F(2,2n+2)$ can be represented by

$$e(E) = \frac{1}{2\pi}\sum_{\alpha=3}^{2n+2}\,\omega_1^\alpha\wedge\omega_2^\alpha,$$

$$e(F) = \frac{(-1)^n}{(4\pi)^n n!}\sum\,\varepsilon(\alpha_1\alpha_2\cdots\alpha_{2n})\Omega_{\alpha_1\alpha_2}\wedge\cdots\wedge\Omega_{\alpha_{2n-1}\alpha_{2n}}.$$

For $k<2n$, $G(2,k+2)$ is a submanifold of $G(2,2n+2)$ whose elements are contained in a fixed $k+2$-dimensional subspace of $\mathbb{R}^{2n+2}$, $i\colon\ G(2,k+2)\to G(2,2n+2)$ the inclusion. Then, $E(2,k+2) = i^*E(2,2n+2)$ and $e(E(2,k+2)) = i^*e(E(2,2n+2))$. Let $G(1,2n+1)$ be a submanifold of $G(2,2n+2)$ with elements $e_1\wedge\bar{e}_2,\ \bar{e}_2 = (0,\cdots,0,1)$, $j\colon\ G(1,2n+1)\to G(2,2n+2)$ be the inclusion.

Theorem 5.2 *For Grassmann manifold $G(2,2n+2)$, we have*

(1) $p_q(F) = (-1)^q p_1^q(E) = (-1)^q e^{2q}(E), \quad q = 1, \cdots, n;$

(2) The Pontrjagin classes and Euler class of tangent bundle $TG(2,2n+2)$ can be represented by the Euler class of E,

$$p_1(G(2,2n+2)) = 2(n-1)e^2(E), \quad p_2(G(2,2n+2)) = (2n^2-5n+9)e^4(E), \ \cdots,$$

$$e(G(2,2n+2)) = (n+1)e^{2n}(E);$$

(3) $\int_{G(1,2n+1)} e(F) = \int_{G(2,k+2)} e^k(E) = 2, \quad k = 1, \cdots, 2n; \ \int_{G(1,2n+1)} e^n(E) = \int_{G(2,n+2)} e(F) = 0.$

Proof For Grassmann manifold $G(2,2n+2)$, we have $p_1(E) = e^2(E)$, $p_n(F) = e^2(F)$. From $(1+p_1(E)) \cdot (1+p_1(F)+p_2(F)+\cdots+p_n(F)) = 1$, we have

$$1 + p_1(F) + p_2(F) + \cdots + p_n(F) = \frac{1}{1+p_1(E)} = 1 + \sum_{q=1}^{n} (-1)^q p_1^q(E).$$

Hence, $p_q(F) = (-1)^q p_1^q(E) = (-1)^q e^{2q}(E)$, $p_n(F) = e^2(F) = (-1)^n e^{2n}(E)$. This proves (1).

By Lemma 4.2, note that $x_1 = e(E)$, the Euler class of $G(2,2n+2)$ is

$$\begin{aligned} e(TG(2,2n+2)) &= (x_1^2 - y_2^2)(x_1^2 - y_3^2)\cdots(x_1^2 - y_{n+1}^2) \\ &= x_1^{2n} - x_1^{2n-2}p_1(F) + x_1^{2n-4}p_2(F) - \cdots + (-1)^n p_n(F) \\ &= (n+1)e^{2n}(E). \end{aligned}$$

By Gauss–Bonnet formula, we have

$$\chi(G(2,2n+2)) = \int_{G(2,2n+2)} e(G(2,2n+2)) = 2n+2.$$

From (1) and

$$p(G(2,2n+2)) = \prod_{\alpha=2}^{n+1} (1 + 2(x_1^2 + y_\alpha^2) + (x_1^2 - y_\alpha^2)^2),$$

we can prove (2).

Restricting the Euler class $e(E)$ on $G(2,k+2)$, we have

$$i^* e(E) = \frac{1}{2\pi} \sum_{\alpha=3}^{k+2} \omega_1^\alpha \wedge \omega_2^\alpha.$$

Then

$$i^* e^k(E) = \frac{k!}{(2\pi)^k} \omega_1^3 \wedge \omega_2^3 \wedge \cdots \wedge \omega_1^{k+2} \wedge \omega_2^{k+2} = \frac{k!}{(2\pi)^k} \mathrm{d}\, V_{G(2,k+2)},$$

where $\mathrm{d}\, V_{G(2,k+2)}$ is the volume element of $G(2,k+2)$. Then

$$\int_{G(2,k+2)} e^k(E) = 2, \quad k = 1, \cdots, 2n.$$

Restricting on $G(1,2n+1)$, $\omega_2^\alpha = 0$, we have

$$e(F(1,2n+1)) = j^* e(F(2,2n+2)) = \frac{(2n)!}{2^{2n} n! \pi^n} \omega_1^3 \wedge \omega_1^4 \wedge \cdots \wedge \omega_1^{2n+2}.$$

It is easy to see that $e(F(1,2n+1))$ is the Euler class of the tangent bundle of $S^{2n} = G(1,2n+1)$ and $\omega_1^3 \wedge \omega_1^4 \wedge \cdots \wedge \omega_1^{2n+2}$ is the volume element. Hence by Gauss–Bonnet formula or by direct computation, we have

$$\int_{G(1,2n+1)} j^* e(F(2,2n+2)) = 2.$$

Furthermore, from $\omega_2^\alpha|_{G(1,2n+1)} = 0$ and $\Omega_{\alpha\beta}|_{G(2,n+2)} = 0$ for $\alpha, \beta > n+2$, we have

$$\int_{G(1,2n+1)} j^* e^n(E) = 0, \quad \int_{G(2,n+2)} i^* e(F) = 0.$$

By $p_1^k(E) = e^{2k}(E)$, we have

$$\int_{G(2,2k+2)} p_1^k(E) = 2.$$

□

The Poincaré polynomial of $G(2,2n+2)$ is

$$p_t(G(2,2n+2)) = 1 + t^2 + \cdots + t^{2n-2} + 2t^{2n} + t^{2n+2} + \cdots + t^{4n}.$$

By Theorem 5.2, we have

(1) For $k \neq n$, $e^k(E) \in H^{2k}(G(2,2n+2))$, $G(2,k+2) \in H_{2k}(G(2,2n+2))$ are the generators respectively;

(2) $e^n(E), e(F) \in H^{2n}(G(2,2n+2))$ and $G(2,n+2), G(1,2n+1) \in H_{2n}(G(2,2n+2))$ are the generators.

The characteristic classes $e^k(E), e(F)$ and the submanifolds $G(2,k+2), G(1,2n+1)$ are integral cohomology and homology classes, respectively. However, they need not be the generators of the integral cohomology and homology groups. For example, when $k \neq n$, from $\int_{G(2,k+2)} e^k(E) = 2$ we know that $[G(2,k+2)] \in H_{2k}(G(2,2n+2),\mathbb{Z})$, $e^k(E) \in H^{2k}(G(2,2n+2),\mathbb{Z})$ cannot be generators simultaneously. Now we compute $\int_{\mathbb{C}P^k} e^k(E)$ and $\int_{\mathbb{C}P^n} e(F)$.

Let J be a complex structure on $\mathbb{R}^{2k+2} \subset \mathbb{R}^{2n+2}$ and $\mathbb{C}P^k = \{e_1 J e_1 \mid e_1 \in S^{2k+1}\}$. Let $e_1, e_2 = Je_1, e_{2\alpha-1}, e_{2\alpha} = Je_{2\alpha-1}$, $\alpha = 2,3,\cdots,k+1$, be local orthonormal frame fields on $\mathbb{R}^{2k+2}$. By $\mathrm{d}e_2 = J\mathrm{d}e_1$ we have $\omega_1^{2\alpha-1} = \omega_2^{2\alpha}$, $\omega_1^{2\alpha} = -\omega_2^{2\alpha-1}$; then

$$\mathrm{d}(e_1 \wedge e_2) = \sum_{\alpha=2}^{k+1} \omega_1^{2\alpha-1}(E_{1\,2\alpha-1} + E_{2\,2\alpha}) + \sum_{\alpha=2}^{k+1} \omega_1^{2\alpha}(E_{1\,2\alpha} - E_{2\,2\alpha-1}).$$

The oriented volume element of $\mathbb{C}P^k$ is $\mathrm{d}V = 2^k \omega_1^3 \wedge \omega_1^4 \wedge \cdots \wedge \omega_1^{2k+2}$.

Let $i\colon \mathbb{C}P^k \to G(2,2n+2)$ be inclusion, we have

$$i^* e^k(E) = (-1)^k \frac{k!}{\pi^k} \omega_1^3 \wedge \omega_1^4 \wedge \cdots \wedge \omega_1^{2k+2}.$$

By Proposition 2.3 (3),

$$\int_{\mathbb{C}P^k} i^* e^k(E) = (-1)^k.$$

By $p_k(E) = e^{2k}(E)$, we have $\int_{\mathbb{C}P^{2k}} i^* p_k(E) = 1$.

For $n = k$, J induces a complex structure on the induced bundle $i^*F \to \mathbb{C}P^n$. Let $F_{\mathbb{C}}$ be the complex vector bundle formed by the $(1,0)$-vectors of $i^*F \otimes \mathbb{C}$. By $i^*e(F) = c_n(F_{\mathbb{C}})$ (see [19]), we can show

$$\int_{\mathbb{C}P^n} i^* e(F) = \int_{\mathbb{C}P^n} c_n(F_{\mathbb{C}}) = 1.$$

See also Chern [3].

Let $\bar{J}$ be a complex structure on $\mathbb{R}^{2k+2}$, and the orientation given by $\bar{J}$ is opposite to that of J. Let $\overline{\mathbb{C}P}^k = \{v \wedge \bar{J}v \mid v \in S^{2k+1}\}$ be the complex projective space. The orientation on the vector bundle $E(2, 2n+2)|_{\overline{\mathbb{C}P}^k}$ is given by $v, \bar{J}v$, and we have

$$\int_{\overline{\mathbb{C}P}^k} i^* e^k(E) = (-1)^k.$$

Let $\widetilde{F}_{\mathbb{C}}$ be the complex vector bundle formed by the $(1,0)$-vectors of $F \otimes \mathbb{C}|_{\overline{\mathbb{C}P}^n}$. The orientation on realization vector bundle of $\widetilde{F}_{\mathbb{C}}$ given by $\bar{J}$ is opposite to that of $F|_{\overline{\mathbb{C}P}^n}$. Hence $e(F|_{\overline{\mathbb{C}P}^n}) = -c_n(\widetilde{F}_{\mathbb{C}})$ and we have

$$\int_{\overline{\mathbb{C}P}^n} e(F) = -\int_{\overline{\mathbb{C}P}^n} c_n(\widetilde{F}_{\mathbb{C}}) = -1.$$

These prove

Proposition 5.3 *(1) When $k < n$, we have*

$$[G(2, k+2)] = 2(-1)^k [\mathbb{C}P^k] \in H_{2k}(G(2, 2n+2));$$

(2) In the homology group $H_{2n}(G(2, 2n+2))$, we have

$$[G(2, n+2)] = (-1)^n([\mathbb{C}P^n] + [\overline{\mathbb{C}P}^n]),$$

$$[G(1, 2n+1)] = [\mathbb{C}P^n] - [\overline{\mathbb{C}P}^n].$$

For Grassmann manifold $G(2, 2n+3)$, by the splitting principle of the characteristic classes, we can assume that there are oriented orthonormal sections s_1, s_2 and $t_3, t_4, \cdots, t_{2n+2}, t_{2n+3}$ of vector bundle $E = E(2, 2n+3)$ and $F = F(2, 2n+3)$ respectively, such that

$$\frac{1}{2\pi}\nabla^2 \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

$$\frac{1}{2\pi}\nabla^2 \begin{pmatrix} t_3 \\ t_4 \\ \vdots \\ t_{2n+1} \\ t_{2n+2} \\ t_{2n+3} \end{pmatrix} = \begin{pmatrix} 0 & -y_2 & & & & \\ y_2 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & -y_{n+1} & \\ & & & y_{n+1} & 0 & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} t_3 \\ t_4 \\ \vdots \\ t_{2n+1} \\ t_{2n+2} \\ t_{2n+3} \end{pmatrix}.$$

The total Pontrjagin classes of F are $p(F) = \prod_{\alpha=2}^{n+1}(1+y_\alpha^2)$.

$s_1\otimes t_{2\alpha-1}, s_2\otimes t_{2\alpha-1}, s_1\otimes t_{2\alpha}, s_2\otimes t_{2\alpha}$ and $s_1\otimes t_{2n+3}, s_2\otimes t_{2n+3}$ are orthonormal sections of $E\otimes F\cong TG(2,2n+3)$; they also give an orientation on $E\otimes F$. The curvature of $E\otimes F$ is

$$\frac{1}{2\pi}\nabla^2\begin{pmatrix} s_1\otimes t_{2\alpha-1}\\ s_2\otimes t_{2\alpha-1}\\ s_1\otimes t_{2\alpha}\\ s_2\otimes t_{2\alpha}\end{pmatrix} = \begin{pmatrix} 0 & -x & -y_\alpha & 0\\ x & 0 & 0 & -y_\alpha\\ y_\alpha & 0 & 0 & -x\\ 0 & y_\alpha & x & 0\end{pmatrix}\begin{pmatrix} s_1\otimes t_{2\alpha-1}\\ s_2\otimes t_{2\alpha-1}\\ s_1\otimes t_{2\alpha}\\ s_2\otimes t_{2\alpha}\end{pmatrix},$$

$$\frac{1}{2\pi}\nabla^2\begin{pmatrix} s_1\otimes t_{2n+3}\\ s_2\otimes t_{2n+3}\end{pmatrix} = \begin{pmatrix} 0 & -x\\ x & 0\end{pmatrix}\begin{pmatrix} s_1\otimes t_{2n+3}\\ s_2\otimes t_{2n+3}\end{pmatrix}.$$

Hence the Euler class of $G(2,2n+3)$ is

$$e(TG(2,2n+3)) = e(E\otimes F) = x\prod_{\alpha=2}^{n+1}(x^2-y_\alpha^2) = (n+1)e^{2n+1}(E).$$

The odd dimensional homology groups of $G(2,2n+3)$ are trivial, and the even dimensional homology groups are one dimensional. The Euler-Poincaré number is $\chi(G(2,2n+3)) = 2n+2$.

Similar to the case of $G(2,2n+2)$, we have

Theorem 5.4 *(1) The Pontrjagin classes of $F(2,2n+3)$ and $TG(2,2n+3)$ can all be represented by the Euler class $e(E(2,2n+3))$;*

(2) $e(TG(2,2n+3)) = (n+1)e^{2n+1}(E(2,2n+3))$;

(3) $\int_{G(2,k+2)} e^k(E(2,2n+3)) = 2, \quad k=1,\cdots,2n+1$;

(4) $\int_{\mathbb{C}P^k} e^k(E(2,2n+3)) = (-1)^k, \quad k=1,\cdots,n+1$.

As is well known, the Chern, Pontrjagin, and Euler classes are all integral cocycles. Let $D\colon\ H^k(G(2,N),\mathbb{Z}) \to H_{2N-4-k}(G(2,N),\mathbb{Z})$ be the Poincaré duality. The following theorem gives the structure of the integral homology and cohomology of $G(2,N)$.

Theorem 5.5 *(1) When $2k+2<N$, $[\mathbb{C}P^k]$ and $e^k(E(2,N))$ are the generators of $H_{2k}(G(2,N),\mathbb{Z})$ and $H^{2k}(G(2,N),\mathbb{Z})$, respectively;*

(2) When $2k+2>N$, $[G(2,k+2)]$ and $\frac{1}{2}e^k(E(2,N))$ are the generators of $H_{2k}(G(2,N),\mathbb{Z})$ and $H^{2k}(G(2,N),\mathbb{Z})$, respectively;

(3) When $2k+2<N$, $D(e^k(E(2,N))) = [G(2,N-k)]$; when $2k+2>N$, $D(\frac{1}{2}e^k(E(2,N))) = (-1)^{n-k}[\mathbb{C}P^{n-k-2}]$;

(4) $[\mathbb{C}P^n], [\overline{\mathbb{C}P}^n]$ and $\frac{1}{2}(-1)^n e^n(E(2,2n+2)) \pm \frac{1}{2}e(F(2,2n+2))$ are generators of $H_{2n}(G(2,2n+2),\mathbb{Z})$ and $H^{2n}(G(2,2n+2),\mathbb{Z})$, respectively. Furthermore,

$$D(\frac{1}{2}(-1)^n e^n(E(2,2n+2)) + \frac{1}{2}e(F(2,2n+2))) = [\mathbb{C}P^n],$$

$$D(\frac{1}{2}(-1)^n e^n(E(2,2n+2)) - \frac{1}{2}e(F(2,2n+2))) = [\overline{\mathbb{C}P}^n].$$

Proof As is well known, the Euler classes and Pontrjagin classes are harmonic forms and are integral cocycles, and their products are also harmonic forms; see [7, 15]. When $2k+2 < N$, from $\int_{\mathbb{C}P^k} e^k(E(2,N)) = (-1)^k$ we know $\mathbb{C}P^k \in H_{2k}(G(2,N),\mathbb{Z})$ and $e^k(E(2,N)) \in H^{2k}(G(2,N),\mathbb{Z})$ are generators, respectively.

By simple computation, we have

$$e^k(E(2,N)) = \frac{k!}{(2\pi)^k} \sum_{\alpha_1<\cdots<\alpha_k} \omega_1^{\alpha_1}\omega_2^{\alpha_1}\cdots\omega_1^{\alpha_k}\omega_2^{\alpha_k},$$

$$a = (e^k(E(2,N)), e^k(E(2,N))) = \frac{(k!)^2}{(2\pi)^{2k}} C_{N-2}^k V(G(2,N)),$$

$$\begin{aligned} \frac{1}{a} * e^k(E(2,N)) &= \frac{(N-k-2)!}{2(2\pi)^{N-k-2}} \sum_{\beta_1<\cdots<\beta_{N-k-2}} \omega_1^{\beta_1}\omega_2^{\beta_1}\cdots\omega_1^{\beta_{N-k-2}}\omega_2^{\beta_{N-k-2}} \\ &= \frac{1}{2} e^{N-k-2}(E(2,N)). \end{aligned}$$

By Theorem 3.1, $\frac{1}{2}e^{N-k-2}(E(2,N))$ is a generator of $H^{2N-2k-4}(G(2,N),\mathbb{Z})$. By $\int_{G(2,N-k)} \frac{1}{2}e^{N-k-2}(E(2,N)) = 1$ we know that $G(2,N-k) \in H_{2N-2k-4}(G(2,N),\mathbb{Z})$ is a generator and $D(e^k(E(2,N))) = G(2,N-k)$. This proves (1), (2), (3) of the Theorem.

Let $[S_1],[S_2]$ be generators of $H_{2n}(G(2,2n+2),\mathbb{Z})$ and harmonic forms ξ_1,ξ_2 be generators of $H_{2n}(G(2,2n+2),\mathbb{Z})$; they satisfy $\int_{S_i}\xi_j = \delta_{ij}$. There are integers a_{ij}, n_{ij} such that

$$\begin{pmatrix} e^n(E) \\ e(F) \end{pmatrix} = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad (\mathbb{C}P^n, \overline{\mathbb{C}P}^n) = (S_1, S_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then

$$\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} (-1)^n & (-1)^n \\ 1 & -1 \end{pmatrix},$$

and we have $\det(a_{ij}) = \pm 1$ or $\det(n_{ij}) = \pm 1$.

If $\det(n_{ij}) = \pm 1$, $e^n(E), e(F)$ are also the generators of $H^{2n}(G(2,2n+2),\mathbb{Z})$, we can assume $\xi_1 = e^n(E), \xi_2 = e(F)$. It is easy to see

$$(e^n(E), e(F)) = 0, \quad (e^n(E), e^n(E)) = (e(F), e(F)) = 2,$$

$$*e^n(E) = e^n(E), \quad *e(F) = e(F).$$

By Theorem 3.1, $\frac{1}{2}e^n(E)$, $\frac{1}{2}e(F)$ are also the generators of $H^{2n}(G(2,2n+2),\mathbb{Z})$. This contradicts the fact that $\int_{\mathbb{C}P^n} e^n(E) = (-1)^n$.

Then we must have $\det(n_{ij}) = \pm 2$ and $\det(a_{ij}) = \pm 1$. This shows $\mathbb{C}P^n, \overline{\mathbb{C}P}^n$ are generators of $H_{2n}(G(2,2n+2),\mathbb{Z})$, and $\frac{1}{2}\{(-1)^n e^n(E) + e(F)\}$, $\frac{1}{2}\{(-1)^n e^n(E) - e(F)\}$ are generators of $H^{2n}(G(2,2n+2),\mathbb{Z})$. The Poincaré duals of these generators are easy to compute. □

We give some applications to conclude this section.

Let $f\colon\ M \to \mathbb{R}^N$ be an immersion of an oriented compact surface M, $g\colon\ M \to G(2,N)$ the induced Gauss map, $g(p) = T_pM$. Then $e(M) = g^*(e(E(2,N))$ is the Euler class of M. Let $[M] \in H_2(M)$ be the fundamental class of M. When $N \neq 4$, we have

$$g_*[M] = \frac{1}{2}\chi(M)[G(2,3)] = \chi(M)[-\mathbb{C}P^1] \in H_2(G(2,N)).$$

In [10, 17], we have shown there is a fibre bundle $\tau\colon\ G(2,8) = G(6,8) \to S^6$ with fibres $\mathbb{C}P^3$, where $S^6 = \{v \in S^7 \mid v \perp \bar{e}_1 = (1,0,\cdots,0)\}$, $\tau^{-1}(\bar{e}_2) = \{v \wedge Jv \mid v \in S^7\}$, $\bar{e}_2 = (0,1,0,\cdots,0)$. On the other hand, the map $f(v) = \bar{e}_1 \wedge v$ gives a section of τ. Let $\mathrm{d}V$ be the volume form on S^6 such that $\int_{S^6} \mathrm{d}V = 1$. It is easy to see

$$[\tau^*\mathrm{d}V] = \frac{1}{2}e^3(E(2,8)) + \frac{1}{2}e(F(2,8)).$$

Let $\varphi\colon\ M \to \mathbb{R}^8$ be an immersion of an oriented compact 6-dimensional manifold, and $g\colon\ M \to G(6,8) = G(2,8)$ be the Gauss map. Then $e(M) = g^*e(F(2,8))$ is the Euler class of tangent bundle of M, and $e(T^\perp M) = g^*e(E(2,8))$ is the Euler class of normal bundle of M.

$$\int_M (\tau\circ g)^*\mathrm{d}V = \int_M g^*[\frac{1}{2}e^3(E(2,8)) + \frac{1}{2}e(F(2,8))] = \frac{1}{2}\int_M e^3(T^\perp M) + \frac{1}{2}\chi(M)$$

is the degree of the map $\tau\circ g\colon\ M \to S^6$. If φ is an imbedding, $e(T^\perp M) = 0$; see Milnor, Stasheff [13], p.120.

Let $J,\ \bar{J}$ be 2 complex structures on $\mathbb{R}^4$, with orthonormal basis e_1, e_2, e_3, e_4,

$$Je_1 = e_2,\ Je_2 = -e_1,\ Je_3 = e_4,\ Je_4 = -e_3;$$

$$\bar{J}e_1 = -e_2,\ \bar{J}e_2 = e_1,\ \bar{J}e_3 = e_4,\ \bar{J}e_4 = -e_3.$$

For any unit vector $v = \sum\ v_ie_i$, we have

$$vJv + *vJv = e_1e_2 + e_3e_4,$$

$$\begin{aligned} vJv - *vJv &= (v_1^2+v_2^2-v_3^2-v_4^2)(e_1e_2-e_3e_4) \\ &\quad +2(v_1v_3+v_2v_4)(e_1e_4-e_2e_3) + 2(v_2v_3-v_1v_4)(e_1e_3+e_2e_4); \end{aligned}$$

$$v\bar{J}v - *v\bar{J}v = -e_1e_2 + e_3e_4,$$

$$\begin{aligned} v\bar{J}v + *v\bar{J}v &= (-v_1^2-v_2^2+v_3^2+v_4^2)(e_1e_2+e_3e_4) \\ &\quad +2(v_1v_3-v_2v_4)(e_1e_4+e_2e_3) - 2(v_1v_4+v_2v_3)(e_1e_3-e_2e_4). \end{aligned}$$

This shows $\mathbb{C}P^1$, $\overline{\mathbb{C}P}^1$ are 2 spheres in $G(2,4) \approx S^2(\frac{\sqrt{2}}{2}) \times S^2(\frac{\sqrt{2}}{2})$ where the decomposition is given by star operator $*\colon\ G(2,4) \to G(2,4)$.

Let $f\colon\ M \to \mathbb{R}^4$ be an immersion of an oriented surface, and $g\colon\ M \to G(2,4)$ the Gauss map. Then we have $g_*[M] = a[G(2,3)] + b[G(1,3)]$, where $a = \frac{1}{2}\chi(M),\ b = \frac{1}{2}\int_M e(T^\perp M)$. If f is an imbedding,

$$g_*[M] = \frac{1}{2}\chi(M)[G(2,3)] = -\frac{1}{2}\chi(M)[\mathbb{C}P^1] - \frac{1}{2}\chi(M)[\overline{\mathbb{C}P}^1].$$

See also the work by Chern and Spanier [4].

6. The case of $G(3,6)$

The Poincaré polynomial of Grassmann manifold $G(3,6)$ is $p_t(G(3,6))=1+t^4+t^5+t^9$. To study the homology of $G(3,6)$ we need only consider the dimension $4,5$.

Let $i\colon\ G(2,4)\to G(3,6)$ be an inclusion defined naturally. It is easy to see that $i^*p_1(E(3,6))=p_1(E(2,4))=e^2(E(2,4))$; then

$$\int_{G(2,4)} p_1(E(3,6))=2.$$

As §2, let $SLAG=\{G(\bar{e}_1\bar{e}_3\bar{e}_5)\mid G\in SU(3)\subset SO(6)\}$ be a subspace of $G(3,6)$, and $e_i=G(\bar{e}_i), e_{i+1}=G(\bar{e}_{i+1})=Je_i$ be $SU(3)$-frame fields, $i=1,3,5$. Restricting the coframes $\omega_A^B=\langle \mathrm{d}e_A,e_B\rangle$ on $SLAG$ we have

$$\omega_i^j=\omega_{i+1}^{j+1},\ \omega_i^{j+1}=-\omega_{i+1}^j,\ i,j=1,3,5\ \text{ and }\ \omega_1^2+\omega_3^4+\omega_5^6=0.$$

By the proof of Proposition 2.4, we have $\mathrm{d}V_{SLAG}=2^{\frac{3}{2}}\sqrt{3}\,\omega_1^4\omega_1^6\omega_3^6\omega_1^2\omega_3^4$ and $V(SLAG)=\sqrt{\frac{3}{2}}\,\pi^3$. Let $G(3,6)$ be generated by $e_1e_3e_5$ locally, and the first Pontrjagin class of canonical vector bundle $E(3,6)$ is

$$p_1(E(3,6))=\frac{1}{4\pi^2}[(\Omega_{13})^2+(\Omega_{15})^2+(\Omega_{35})^2],$$

where $\Omega_{ij}=-\sum\limits_{\alpha}\ \omega_i^\alpha\wedge\omega_j^\alpha,\ \alpha=2,4,6$. By computation we have

$$*p_1(E(3,6))|_{SLAG}=\frac{\sqrt{6}}{4\pi^2}\mathrm{d}V_{SLAG},$$

$$a=(p_1(E(3,6)),p_1(E(3,6)))=\frac{3\cdot 4\cdot 3}{(2\pi)^4}V(G(3,6))=\frac{3}{2}\pi.$$

From $\int_{G(2,4)}p_1(E(3,6))=2$, we know that $p_1(E(3,6))$ or $\frac{1}{2}p_1(E(3,6))$ is a generator of $H^4(G(3,6),\mathbb{Z})$. If $p_1(E(3,6))$ is a generator, by Theorem 3.1, $\frac{1}{a}*p_1(E(3,6))$ is a generator of $H^5(G(3,6),\mathbb{Z})$, but

$$\int_{SLAG}\frac{1}{a}*p_1(E(3,6))=\int_{SLAG}\frac{1}{\sqrt{6}\pi^3}\mathrm{d}V_{SLAG}=\frac{1}{2}.$$

Then $\frac{1}{2}p_1(E(3,6))$ is a generator of $H^4(G(3,6),\mathbb{Z})$ and $\int_{SLAG}\frac{4}{a}*\frac{1}{2}p_1(E(3,6))=1$.

We have proved the following theorem

Theorem 6.1 *(1)* $\frac{1}{2}p_1(E(3,6))\in H^4(G(3,6),\mathbb{Z})$ *is a generator and its Poincaré dual* $[SLAG]$ *is a generator of* $H_5(G(3,6),\mathbb{Z})$*;*

(2) $\frac{4}{3\pi}*p_1(E(3,6))\in H^5(G(3,6),\mathbb{Z})$ *is a generator and its Poincaré dual* $[G(2,4)]$ *is a generator of* $H_4(G(3,6),\mathbb{Z})$.

Let $\bar{e}_1,\cdots,\bar{e}_6$ be a fixed orthonormal basis of $\mathbb{R}^6$, $G\in SO(3)$ acts on the subspace generated by $\bar{e}_4,\bar{e}_5,\bar{e}_6$, and denote $e_4=G(\bar{e}_4),e_5=G(\bar{e}_5),e_6=G(\bar{e}_6)$. As [7], let $PONT$ be the set of elements

$$(\cos t\bar{e}_1+\sin te_4)(\cos t\bar{e}_2+\sin te_5)(\cos t\bar{e}_3+\sin te_6),\quad t\in[0,\frac{\pi}{2}].$$

$PONT$ is a calibrated submanifold (except 2 points correspond to $t = 0, \frac{\pi}{2}$) of the first Pontrjagin form $p_1(E(3,6))$. By moving the frame we can show

$$\int_{PONT} p_1(E(3,6)) = 2 \text{ and } V(PONT) = \sqrt{\frac{2}{3}} V(G(2,4)) = \frac{4\sqrt{6}}{3}\pi^2.$$

Then 4-cycle $PONT$ is homologous to the 4-cycle $G(2,4)$ inside $G(3,6)$.

7. The case of $G(3,7)$

The Poincaré polynomial of $G(3,7)$ is $p_t(G(3,7)) = 1 + 2t^4 + 2t^8 + t^{12}$.

Let $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_8$ be a fixed orthonormal basis of $\mathbb{R}^8$ and $\mathbb{R}^7$ be a subspace generated by $\bar{e}_2, \cdots, \bar{e}_8$. The oriented Grassmann manifold $G(3,7)$ is the set of subspaces of $\mathbb{R}^7$.

Let $E = E(3,7)$ and $F = F(3,7)$. As §4, we can show

$$p_1(F) = -p_1(E), \; p_2(F) = e^2(F) = p_1^2(E), \; e(E \otimes F) = e(TG(3,7)) = 3e^3(F).$$

By $\int_{G(3,7)} e(TG(3,7)) = \chi(G(3,7)) = 6$ we have

$$\int_{G(3,7)} e^3(F) = 2.$$

By inclusion $G(2,6) \subset G(3,7)$, $\mathbb{C}P^2$ and $\overline{\mathbb{C}P}^2$ can be imbedded in $G(3,7)$.

Lemma 7.1 $\int_{\mathbb{C}P^2} p_1(E) = \int_{\overline{\mathbb{C}P}^2} p_1(E) = 1, \quad \int_{\mathbb{C}P^2} e(F) = -\int_{\overline{\mathbb{C}P}^2} e(F) = 1.$

Proof By $p_1(E(3,7))|_{\mathbb{C}P^2} = p_1(E(2,6))|_{\mathbb{C}P^2} = e^2(E(2,6))|_{\mathbb{C}P^2}$ and the results of §5, we have $\int_{\mathbb{C}P^2} p_1(E) = 1$. The other equalities can be proved similarly. □

Then

$$\int_{\mathbb{C}P^2} \frac{1}{2}(p_1(E) + e(F)) = \int_{\overline{\mathbb{C}P}^2} \frac{1}{2}(p_1(E) - e(F)) = 1,$$

$$\int_{\mathbb{C}P^2} \frac{1}{2}(p_1(E) - e(F)) = \int_{\overline{\mathbb{C}P}^2} \frac{1}{2}(p_1(E) + e(F)) = 0,$$

hence $\mathbb{C}P^2, \overline{\mathbb{C}P}^2 \in H_4(G(3,7))$ and $\frac{1}{2}(p_1(E) + e(F)), \; \frac{1}{2}(p_1(E) - e(F)) \in H^4(G(3,7))$ are generators.

Let $e_2, e_3, e_4, \cdots, e_8$ be oriented orthonormal frame fields on $\mathbb{R}^7$, and $G(3,7)$ be generated by $e_2 \wedge e_3 \wedge e_4$ locally. Euler class of F and first Pontrjagin class of E can be represented by

$$\begin{aligned} e(F) &= \frac{1}{2(4\pi)^2} \sum \varepsilon(\alpha_1\alpha_2\alpha_4\alpha_4)\Omega_{\alpha_1\alpha_2} \wedge \Omega_{\alpha_3\alpha_4} \\ &= \frac{1}{4\pi^2} \sum_{i,j=2}^{4} (\omega_5^i\omega_6^i\omega_7^j\omega_8^j - \omega_5^i\omega_7^i\omega_6^j\omega_8^j + \omega_5^i\omega_8^i\omega_6^j\omega_7^j), \end{aligned}$$

$$p_1(E) = \frac{1}{4\pi^2}[(\Omega_{23})^2 + (\Omega_{24})^2 + (\Omega_{34})^2].$$

Then we have

$$p_1(E)e^2(F) = p_1^3(E) = 0.$$

Lemma 7.2 *(1)* $*p_1(E) = \frac{4}{5}\pi^2 p_1(E)e(F),\ *e(F) = \frac{1}{2}\pi^2 e^2(F)$*;*

(2) $(p_1(E), e(F)) = 0,\ a = (p_1(E), p_1(E)) = \frac{8}{5}\pi^2,\ b = (e(F), e(F)) = \pi^2$.

Proof $*p_1(E),\ p_1(E)e(F)$ and $*e(F),\ e^2(F)$ are the harmonic forms on $G(3,7)$. $*p_1(E) = \frac{4}{5}\pi^2 p_1(E)e(F)$ follows from the equalities such as

$$\begin{aligned} *\omega_2^5\omega_3^5\omega_2^6\omega_3^6 &= \omega_2^7\omega_3^7\omega_2^8\omega_3^8\omega_5^4\omega_6^4\omega_7^4\omega_8^4 \\ &= \omega_3^7\omega_4^7\omega_3^8\omega_4^8\omega_5^4\omega_6^4\omega_7^2\omega_8^2 \\ &= \omega_2^7\omega_4^7\omega_2^8\omega_4^8\omega_5^4\omega_6^4\omega_7^3\omega_8^3. \end{aligned}$$

The proof of (2) is a direct computation. □

To study $G(3,7), G(3,8)$, and $G(4,8)$, we shall use Clifford algebras.

Let $C\ell_8$ be the Clifford algebra associated with the Euclidean space $\mathbb{R}^8$. Let $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_8$ be a fixed orthonormal basis of $\mathbb{R}^8$, and the Clifford product be determined by the relations: $\bar{e}_B \cdot \bar{e}_C + \bar{e}_C \cdot \bar{e}_B = -2\delta_{BC},\ \ B, C = 1, 2, \cdots, 8$. Define the subspace $V = V^+ \oplus V^-$ of $C\ell_8$ by $V^+ = C\ell_8^{even} \cdot A, V^- = C\ell_8^{odd} \cdot A$, where

$$A = \frac{1}{16}\mathrm{Re}\,[(\bar{e}_1 + \sqrt{-1}\bar{e}_2)\cdots(\bar{e}_7 + \sqrt{-1}\bar{e}_8)(1 + \bar{e}_1\bar{e}_3\bar{e}_5\bar{e}_7)].$$

The space $V = V^+ \oplus V^-$ is an irreducible module over $C\ell_8$. The spaces V^+ and V^- are generated by $\bar{e}_1\bar{e}_B A$ and $\bar{e}_B A$ respectively, $B = 1, \cdots, 8$; see [16,17].

Let $Spin_7 = \{G \in SO(8) \mid G(A) = A\}$ be the isotropy group of $SO(8)$ acting on A. The group $Spin_7$ acts on $G(2,8), G(3,8)$ and S^7 transitively. $G_2 = \{G \in Spin_7 \mid G(\bar{e}_1) = \bar{e}_1\}$ is a subgroup of $Spin_7$.

The Grassmann manifold $G(k,8)$ can be viewed as a subset of Clifford algebra $C\ell_8$ naturally. Then, for any $\pi \in G(k,8)$, there is $v \in \mathbb{R}^8$ such that $\pi A = \bar{e}_1 v A$ or $\pi A = vA$ according to the number k being even or odd, $|v| = 1$. Thus we have maps $G(k,8) \to S^7,\ \pi \mapsto v$. Since $Spin_7$ acts on $G(3,8)$ transitively, from $\bar{e}_2\bar{e}_3\bar{e}_4 A = \bar{e}_1 A$ we have $G(\bar{e}_2\bar{e}_3\bar{e}_4)A = G(\bar{e}_1)A$ for any $G \in Spin_7$. This shows the map $\tau\colon\ G(3,8) \to S^7,\ \tau(\pi) = v,$ is a fibre bundle and $v \perp \pi$; see [10,17]. Let

$$ASSOC = \tau^{-1}(\bar{e}_1) = \{\pi \in G(3,8) \mid \tau(\pi) = \bar{e}_1\}$$

be the fibre over $\bar{e}_1$. The group G_2 acts on $ASSOC$ transitively, and we have $ASSOC = \{G(\bar{e}_2\bar{e}_3\bar{e}_4) \mid G \in G_2\}$. We can show the isotropy group $\{G(\bar{e}_2\bar{e}_3\bar{e}_4) = \bar{e}_2\bar{e}_3\bar{e}_4 \mid G \in G_2\}$ is isomorphic to the group $SO(4)$; then $ASSOC \approx G_2/SO(4)$.

Change the orientation of $\mathbb{R}^7$, and let $\tilde{A} = \frac{1}{16}\mathrm{Re}\,[(\bar{e}_1 - \sqrt{-1}\bar{e}_2)(\bar{e}_3 + \sqrt{-1}\bar{e}_4)\cdots(\bar{e}_7 + \sqrt{-1}\bar{e}_8)(1 + \bar{e}_1\bar{e}_3\bar{e}_5\bar{e}_7)]$. Define submanifold $\widetilde{ASSOC} = \{\pi \in G(3,8) \mid \pi\tilde{A} = \bar{e}_1\tilde{A}\}$, which is diffeomorphic to $ASSOC$.

Lemma 7.3 $V(ASSOC) = \frac{6}{5}\pi^4$.

Proof Let $\tilde{e}_1, \tilde{e}_2, \cdots, \tilde{e}_8$ be $Spin_7$ frame fields on $\mathbb{R}^8$, and the 1-forms $\omega_B^C = \langle \mathrm{d}\tilde{e}_B, \tilde{e}_C\rangle$ satisfy (for proof, see [10])

$$\omega_1^2+\omega_3^4+\omega_5^6+\omega_7^8=0,\quad \omega_1^3-\omega_2^4+\omega_6^8-\omega_5^7=0,$$
$$\omega_1^4+\omega_2^3+\omega_5^8+\omega_6^7=0,\quad \omega_1^5-\omega_2^6+\omega_3^7-\omega_4^8=0,$$
$$\omega_1^6+\omega_2^5-\omega_3^8-\omega_4^7=0,\quad \omega_1^7-\omega_2^8-\omega_3^5+\omega_4^6=0,$$
$$\omega_1^8+\omega_2^7+\omega_3^6+\omega_4^5=0.$$

Since $Spin_7$ acts on $G(3,8)$ transitively, $G(3,8)$ is locally generated by $\tilde{e}_2\tilde{e}_3\tilde{e}_4$. The volume element of $G(3,8)$ is $\mathrm{d}V_{G(3,8)}=\omega_2^1\omega_3^1\omega_4^1\omega_2^5\omega_3^5\omega_4^5\cdots\omega_2^8\omega_3^8\omega_4^8$.

Note that A can be represented by $Spin_7$ frames, that is

$$A=\frac{1}{16}\mathrm{Re}\,[(\tilde{e}_1+\sqrt{-1}\tilde{e}_2)\cdots(\tilde{e}_7+\sqrt{-1}\tilde{e}_8)(1+\tilde{e}_1\tilde{e}_3\tilde{e}_5\tilde{e}_7)].$$

Let $\tilde{e}_1=\bar{e}_1$ be a fixed vector, and $\bar{e}_1,\tilde{e}_2,\cdots,\tilde{e}_8$ be G_2 frame fields on $\mathbb{R}^8$; $ASSOC$ is locally generated by $\tilde{e}_2\tilde{e}_3\tilde{e}_4$ and

$$\begin{aligned}\mathrm{d}(\tilde{e}_2\tilde{e}_3\tilde{e}_4) &= \sum_{i=2}^{3}\sum_{\alpha=5}^{8}\omega_i^\alpha E_{i\alpha}\\ &= \omega_2^5(E_{25}+E_{47})+\omega_2^6(E_{26}-E_{48})+\omega_2^7(E_{27}-E_{45})+\omega_2^8(E_{28}+E_{46})\\ &\quad+\omega_3^5(E_{35}+E_{46})+\omega_3^6(E_{36}-E_{45})+\omega_3^7(E_{37}+E_{48})+\omega_3^8(E_{38}-E_{47}).\end{aligned}$$

The metric on $ASSOC$ is

$$\begin{aligned}\mathrm{d}s^2 &= 2(\omega_2^5)^2+2(\omega_3^8)^2-2\omega_2^5\omega_3^8+2(\omega_2^6)^2+2(\omega_3^7)^2-2\omega_2^6\omega_3^7\\ &\quad+2(\omega_2^7)^2+2(\omega_3^6)^2+2\omega_2^7\omega_3^6+2(\omega_2^8)^2+2(\omega_3^5)^2+2\omega_2^8\omega_3^5,\end{aligned}$$

with the volume form

$$\mathrm{d}V_{ASSOC}=9\ \omega_2^5\omega_3^5\ \cdots\ \omega_2^8\omega_3^8.$$

The normal space of $ASSOC$ in $G(3,8)$ at $\tilde{e}_2\tilde{e}_3\tilde{e}_4$ is generated by

$$E_{21},E_{31},E_{41},E_{25}-E_{47}-E_{38},E_{26}+E_{48}-E_{37},E_{27}+E_{45}+E_{36},E_{28}-E_{46}+E_{35}.$$

The sphere S^7 is generated by $\tilde{e}_1$, and $\mathrm{d}V_{S^7}=\omega_2^1\omega_3^1\cdots\omega_8^1$ is the volume form. From

$$(E_{27}+E_{45}+E_{36})A=-3\tilde{e}_8A,\ E_{21}A=-\tilde{e}_2A,\ \cdots,$$

we can compute the tangent map of $\tau\colon\ G(3,8)\to S^7$,

$$\tau_*(E_{27}+E_{45}+E_{36})=-3\tilde{e}_8,\ \tau_*(E_{21})=-\tilde{e}_2,\ \cdots.$$

Then we can compute the cotangent map τ^* and we have

$$\tau^*\mathrm{d}V_{S^7}=\omega_2^1\omega_3^1\omega_4^1(\omega_2^6-\omega_3^7+\omega_4^8)(-\omega_2^5+\omega_3^8+\omega_4^7)(\omega_2^8+\omega_3^5-\omega_4^6)(\omega_2^7+\omega_3^6+\omega_4^5),$$

and

$$\mathrm{d}V_{G(3,8)} = -\frac{1}{9}\tau^* \mathrm{d}V_{S^7} \cdot \mathrm{d}V_{\tau^{-1}(\tilde{e}_1)}.$$

From $V(G(3,8)) = \frac{2}{45}\pi^8,\ V(S^7) = \frac{1}{3}\pi^4$, we have $V(ASSOC) = \frac{6}{5}\pi^4$. □

It is easy to see that $ASSOC$ and $\widetilde{ASSOC}$ are submanifolds of $G(3,7)$. In the following lemma $E = E(3,7), F = F(3,7)$.

Lemma 7.4 $\int_{ASSOC} p_1^2(E) = \int_{ASSOC} p_1(E)e(F) = 1,\ \int_{\widetilde{ASSOC}} p_1^2(E) = -\int_{\widetilde{ASSOC}} p_1(E)e(F) = 1$. *Then* $ASSOC, \widetilde{ASSOC}$ *and* $p_1^2(E), p_1(E)e(F)$ *are generators of* $H_8(G(3,7))$ *and* $H^8(G(3,7))$*, respectively. Furthermore, we have*

$$[G(2,6)] = [ASSOC] + [\widetilde{ASSOC}].$$

Proof From

$$*\omega_2^5\omega_3^5\omega_2^6\omega_3^6|_{ASSOC} = \omega_2^7\omega_3^7\omega_2^8\omega_3^8\omega_5^4\omega_6^4\omega_7^4\omega_8^4|_{ASSOC} = \frac{1}{9}\mathrm{d}V_{ASSOC},$$

$$*\omega_2^7\omega_3^7\omega_2^8\omega_3^8|_{ASSOC} = *\omega_2^5\omega_3^5\omega_2^7\omega_3^7|_{ASSOC} = *\omega_2^6\omega_3^6\omega_2^8\omega_3^8|_{ASSOC} = \frac{1}{9}\mathrm{d}V_{ASSOC},$$

$$*\omega_2^6\omega_3^6\omega_2^7\omega_3^7|_{ASSOC} = \omega_2^5\omega_3^5\omega_2^8\omega_3^8\omega_5^4\omega_6^4\omega_7^4\omega_8^4|_{ASSOC} = 0,$$

$$*\omega_2^5\omega_3^5\omega_2^8\omega_3^8|_{ASSOC} = 0,$$

we have $\sum\limits_{\alpha,\beta} *\omega_2^\alpha\omega_3^\alpha\omega_2^\beta\omega_3^\beta|_{ASSOC} = 8 \cdot \frac{1}{9}\mathrm{d}V_{ASSOC}$,

$$*p_1(E)|_{ASSOC} = \frac{1}{4\pi^2} \cdot 3 \cdot 8 \cdot \frac{1}{9}\mathrm{d}V_{ASSOC}.$$

Then by $*p_1(E) = \frac{4}{5}\pi^2 p_1(E)e(F)$, we have

$$\int_{ASSOC} p_1(E)e(F) = \int_{ASSOC} \frac{5}{4\pi^2} * p_1(E) = 1.$$

The proof of $\int_{ASSOC} p_1^2(E) = 1$ is similar.

Change the orientation of $\mathbb{R}^7$, and we have Euclidean space $\widetilde{\mathbb{R}}^7$. Let $\widetilde{E}, \widetilde{F} \to G(3,7)$ be canonical vector bundles with respect to $\widetilde{\mathbb{R}}^7$. It is easy to see that $\widetilde{E} = E$, but the orientations of $\widetilde{F}$ and F are different. This shows

$$\int_{\widetilde{ASSOC}} p_1^2(E) = 1, \quad \int_{\widetilde{ASSOC}} p_1(E)e(F) = -1.$$

$$[G(2,6)] = [ASSOC] + [\widetilde{ASSOC}]$$

follows from $\int_{G(2,6)} p_1^2(E) = 2$ and $\int_{G(2,6)} p_1(E)e(F) = 0$. □

Theorem 7.5 *(1)* $\frac{1}{2}(p_1(E)+e(F))$, $\frac{1}{2}(p_1(E)-e(F))$ *are 2 generators of* $H^4(G(3,7),\mathbb{Z})$. *Their Poincaré duals are* $[ASSOC]$ *and* $[\widetilde{ASSOC}]$ *respectively;*

(2) $\frac{1}{2}(p_1(E)e(F)+e^2(F))$, $\frac{1}{2}(p_1(E)e(F)-e^2(F)) \in H^8(G(3,7),\mathbb{Z})$ *are generators and their Poincaré duals are* $[\mathbb{C}P^2]$, $[\overline{\mathbb{C}P}^2]$ *respectively;*

(3) $\frac{1}{2}(p_1(E)e(F)+e^2(F))$, $\frac{1}{2}(p_1(E)e(F)-e^2(F))$ *and* $[ASSOC]$, $[\widetilde{ASSOC}]$ *are dual basis with respect to the universal coefficients theorem.*

The proof is similar to Theorem 5.5.

8. The case of $G(3,8)$

The Poincaré polynomial of Grassmann manifold $G(3,8)$ is

$$p_t(G(3,8)) = (1+t^4+t^8)(1+t^7) = 1+t^4+t^7+t^8+t^{11}+t^{15}.$$

Let $E=E(3,8)$, $F=F(3,8)$. By $\int_{\mathbb{C}P^2} p_1(E)=1$, $\int_{ASSOC} p_1^2(E)=1$ we know that

$$[\mathbb{C}P^2]\in H_4(G(3,8),\mathbb{Z}),\ [ASSOC]\in H_8(G(3,8),\mathbb{Z}),$$

$$p_1(E)\in H^4(G(3,8),\mathbb{Z}),\ p_1^2(E)\in H^8(G(3,8),\mathbb{Z})$$

are all generators. By Theorem 3.1, to understand the structure of the homology groups of dimension $7, 11$, we need to compute the Poincaré duals of $p_1(E)$, $p_1^2(E)$.

It is not difficult to compute

$$a=(p_1(E),p_1(E))=\frac{15}{2\pi^4}V(G(3,8))=\frac{1}{3}\pi^4.$$

Then $\frac{1}{a}*p_1(E)\in H^{11}(G(3,8),\mathbb{Z})$ is a generator, and we look for a submanifold M such that $\int_M \frac{1}{a}*p_1(E)=1$.

In §7 we define fibre bundle $\tau\colon\ G(3,8)\to S^7$, $\tau(\pi)=v$ defined by $\pi A=vA$. As $v\perp\pi$, $v\pi\in G(4,8)$ and $v\pi A=-A$. Let $CAY=\{\pi\in G(4,8)\mid \pi A=-A\}$, called the Cayley submanifold of $G(4,8)$. Then we have a fibre bundle $\mu\colon\ G(3,8)\to CAY,\ \pi\mapsto v\pi$, with fibre $S^3=G(3,4)$. Let $ASSOC=\{\bar{e}_1e_2e_3e_4\mid e_2e_3e_4A=\bar{e}_1A\}$, where $\bar{e}_1=(1,0,\cdots,0)$. Then $M=\mu^{-1}(ASSOC)$ is a 11-dimensional submanifold of $G(3,8)$.

Let $\bar{e}_1,e_2,\cdots,e_8$ be G_2 frame fields on $\mathbb{R}^8$, and $ASSOC$ be generated by $\bar{e}_1e_2e_3e_4$. Represent the elements of $\mu^{-1}(\bar{e}_1e_2e_3e_4)$ by $\tilde{e}_2\tilde{e}_3\tilde{e}_4$, $\tau(\tilde{e}_2\tilde{e}_3\tilde{e}_4)=\tilde{e}_1$; then

$$\tilde{e}_1\tilde{e}_2\tilde{e}_3\tilde{e}_4=\bar{e}_1e_2e_3e_4.$$

Let $\mathrm{d}(\tilde{e}_2\tilde{e}_3\tilde{e}_4)=\sum\limits_{i=2}^{4}\tilde{\omega}_i^1\tilde{E}_{i1}+\sum\limits_{i=2}^{4}\sum\limits_{\alpha=5}^{8}\tilde{\omega}_i^\alpha\tilde{E}_{i\alpha}$, $\mathrm{d}s_M^2=\sum\limits_i(\tilde{\omega}_i^1)^2+\sum\limits_{i,\alpha}(\tilde{\omega}_i^\alpha)^2$ be the metric on M.

Let $\tilde{e}_1=\lambda_1\bar{e}_1+\sum\limits_{i=2}^{4}\lambda_ie_i$, $\mathrm{d}\tilde{e}_1=\sum\limits_{i=2}^{4}\tilde{\omega}_1^i\tilde{e}_i+\sum\limits_{\alpha=5}^{8}\tilde{\omega}_1^\alpha e_\alpha$ and $\mathrm{d}(e_2e_3e_4)=\sum\limits_{i,\alpha}\omega_i^\alpha E_{i\alpha}$. It is easy to see $\tilde{\omega}_1^\alpha=\sum\limits_{i=2}^{4}\lambda_i\omega_i^\alpha$. $\sum\limits_{i=2}^{4}(\tilde{\omega}_1^i)^2$ is the metric on the fibres of $\mu\colon\ M\to ASSOC$ and $\langle\mathrm{d}(e_2e_3e_4),\mathrm{d}(e_2e_3e_4)\rangle=\sum(\omega_i^\alpha)^2$

is the metric on $ASSOC$. From $\langle \bar{e}_1 E_{i\alpha}, e_\beta \tilde{e}_2\tilde{e}_3\tilde{e}_4\rangle = \lambda_i\delta_{\alpha\beta}$ we have

$$\langle \bar{e}_1 \mathrm{d}(e_2e_3e_4), (\mathrm{d}\tilde{e}_1)\tilde{e}_2\tilde{e}_3\tilde{e}_4\rangle = \langle \sum \omega_i^\alpha \bar{e}_1 E_{i\alpha}, \sum \lambda_j\omega_j^\beta e_\beta\tilde{e}_2\tilde{e}_3\tilde{e}_4\rangle = \sum_\alpha (\sum_i \lambda_i\omega_i^\alpha)^2.$$

Hence

$$\begin{aligned}\sum (\tilde{\omega}_i^\alpha)^2 &= \langle \tilde{e}_1\mathrm{d}(\tilde{e}_2\tilde{e}_3\tilde{e}_4), \tilde{e}_1\mathrm{d}(\tilde{e}_2\tilde{e}_3\tilde{e}_4)\rangle\\ &= \langle \bar{e}_1\mathrm{d}(e_2e_3e_4) - (\mathrm{d}\tilde{e}_1)\tilde{e}_2\tilde{e}_3\tilde{e}_4, \bar{e}_1\mathrm{d}(e_2e_3e_4) - (\mathrm{d}\tilde{e}_1)\tilde{e}_2\tilde{e}_3\tilde{e}_4\rangle\\ &= \sum_{i,\alpha}(\omega_i^\alpha)^2 + \sum(\tilde{\omega}_1^\alpha)^2 - 2\sum_\alpha(\sum_i \lambda_i\omega_i^\alpha)^2\\ &= \sum_{i,\alpha}(\omega_i^\alpha)^2 - \sum_\alpha(\sum_i \lambda_i\omega_i^\alpha)^2.\end{aligned}$$

Then the metric on M can be represented by

$$\mathrm{d}s_M^2 = \sum_{i,\alpha}(\omega_i^\alpha)^2 - \sum_\alpha(\sum_i \lambda_i\omega_i^\alpha)^2 + \sum_{i=2}^4(\tilde{\omega}_1^i)^2.$$

For fixed $\tilde{e}_2\tilde{e}_3\tilde{e}_4$ of M, we can choose G_2 frame fields $\bar{e}_1, e_2, \cdots, e_8$ such that $\tilde{e}_1 = \lambda_1\bar{e}_1 + \lambda_2 e_2$. By

$$\omega_4^5 = -\omega_2^7 - \omega_3^6,\ \omega_4^6 = \omega_2^8 + \omega_3^5,\ \omega_4^7 = \omega_2^5 - \omega_3^8,\ \omega_4^8 = -\omega_2^6 + \omega_3^7,$$

we have

$$\mathrm{d}V_M = \frac{1}{9}(1+2\lambda_1^2)^2\tilde{\omega}_1^2\tilde{\omega}_1^3\tilde{\omega}_1^4\mu^*\mathrm{d}V_{ASSOC}.$$

We can show that $\int_{S^3}(1+2\lambda_1^2)^2\mathrm{d}V_{S^3} = 5\pi^2$; then $V(M) = \frac{2}{3}\pi^6$.

Lemma 8.1 $\int_M \frac{3}{\pi^4} {*p_1(E)} = 1$. *Then* $[M]$ *and* $\frac{3}{\pi^4}{*p_1(E)}$ *are dual generators of* $H_8(G(3,8),\mathbb{Z})$, $H^8(G(3,8),\mathbb{Z})$ *respectively.*

Proof By Theorem 3.1, the integration $\int_M \frac{1}{a} * p_1(E)$ is an integer. On the other hand, $2\pi^2 p_1(E)$ is a calibration on $G(3,8)$ with comass $\frac{4}{3}$; see [7]; then we have

$$\left|\int_M \frac{1}{a} * p_1(E)\right| \le \frac{2}{3a\pi^2}V(M) = \frac{4}{3}.$$

We need to show that $\int_M \frac{1}{a} * p_1(E) \neq 0$.

Let $i\colon\ CAY \to G(4,8)$ be the inclusion, $\mu\colon\ G(3,8) \to CAY$ be a sphere bundle associated with the induced vector bundle $i^*E(4,8) \to CAY$, and $e(i^*E(4,8)) = i^*e(E(4,8)) \in H^4(CAY,\mathbb{Z})$ be the Euler class. The induced bundle $(i\circ\mu)^*E(4,8) \to G(3,8)$ has a nonzero section, $(i\circ\mu)^*e(E(4,8)) = 0$. By Gysin sequence for the sphere bundle $G(3,8) \to CAY$, we can show $\mu_*(\frac{1}{a}{*p_1(E)}), (i^*e(E(4,8)))^2$ are 2 generators of $H^8(CAY)$, where $\mu_*\colon\ H^{11}(G(3,8)) \to H^8(CAY)$ is the integration along the fibre. From $e^2(E(4,8)) = p_2(E(4,8))$, we have $e^2(E(4,8))|_{G(3,7)} = 0$; then $\int_{ASSOC}(i^*e(E(4,8)))^2 = 0$. If we also have $\int_{ASSOC}\mu_*(\frac{1}{a} * p_1(E)) = 0$, then

$[ASSOC]=0$ in $H_8(CAY)$. This contradicts the fact that $i_*[ASSOC]\neq 0$. Then with a suitable choice of orientation on M, we have

$$\int_M \frac{1}{a}*p_1(E)=\int_{ASSOC}\mu_*(\frac{1}{a}*p_1(E))=1.$$

□

Finally, we study the $H_7(G(3,8))$ and $H^7(G(3,8))$. Let I,J,K be the quaternion structures on $\mathbb{R}^8=\mathbb{H}^2$ and $Sp(2)$ the symplectic group. As we know $Sp(2)$ is a subgroup of $SU(4)\subset SO(8)$; hence $Sp(2)$ is a subgroup of $Spin_7$. Let $f\colon\ S^7\to G(3,8), and\ f(v)=IvJvKv$. By $I\bar{e}_1J\bar{e}_1K\bar{e}_1A=\bar{e}_2\bar{e}_3\bar{e}_4A=\bar{e}_1A$ and $Sp(2)$ acting on S^7 transitively, we have $IvJvKvA=vA$ for any $v\in S^7$, $\tau(f(v))=v$.

Lemma 8.2 $b=(p_1^2(E),p_1^2(E))=\frac{5}{2}$.

Proof By computation, we have

$$\begin{aligned} p_1^2(E) &= \frac{1}{2\pi^4}\{\sum_{i<j}\sum_{\alpha<\beta<\gamma<\tau} 3\omega_i^\alpha\omega_j^\alpha\omega_i^\beta\omega_j^\beta\omega_i^\gamma\omega_j^\gamma\omega_i^\tau\omega_j^\tau \\ &\quad +\sum_{j\neq k}\sum_{\alpha<\beta,\gamma<\tau}\omega_i^\alpha\omega_j^\alpha\omega_i^\beta\omega_j^\beta\omega_i^\gamma\omega_k^\gamma\omega_i^\tau\omega_k^\tau\}, \end{aligned}$$

$$i,j,k=2,3,4,\ \ \alpha,\beta,\gamma,\tau=1,5,6,7,8.$$

Then

$$b=(p_1^2(E),p_1^2(E))=\frac{1}{4\pi^8}(9\cdot 3\cdot C_5^4+3\cdot C_5^2\cdot C_3^2)V(G(3,8))=\frac{5}{2}.$$

□

$p_1^2(E)\in H^8(G(3,8),\mathbb{Z})$ is a generator, by Theorem 3.1, $\frac{1}{b}*p_1^2(E)\in H^7(G(3,8),\mathbb{Z})$ is a generator.

Lemma 8.3 $\frac{2}{5}\int_{S^7}f^**p_1^2(E)=1$. *Then* $[f(S^7)]$ *and* $\frac{2}{5}*p_1^2(E)$ *are dual generators of* $H_7(G(3,8),\mathbb{Z})$ *and* $H^7(G(3,8),\mathbb{Z})$ *respectively.*

Proof Let $e_1,e_2=Ie_1,e_3=Je_1,e_4=Ke_1,e_5,e_6=Ie_5,e_7=Je_5,e_8=Ke_5$ be $Sp(2)$ frame fields on $\mathbb{R}^8$, $f(e_1)=e_2e_3e_4$. The 1 forms $\omega_i^\alpha=\langle \mathrm{d}e_i,e_\alpha\rangle$, $i=1,2,3,4$, $\alpha=5,6,7,8$, satisfy

$$\omega_1^5=\omega_2^6=\omega_3^7=\omega_4^8,\ \ \omega_1^6=-\omega_2^5=-\omega_3^8=\omega_4^7,$$

$$\omega_1^7=\omega_2^8=-\omega_3^5=-\omega_4^6,\ \ \omega_1^8=-\omega_2^7=\omega_3^6=-\omega_4^5.$$

We have

$$*\omega_2^5\omega_3^5\omega_2^6\omega_3^6\omega_2^7\omega_3^7\omega_2^8\omega_3^8|_{f(S^7)}=\omega_2^1\omega_3^1\omega_4^1\omega_4^5\omega_4^6\omega_4^7\omega_4^8|_{f(S^7)}=\omega_2^1\omega_3^1\omega_4^1\omega_5^1\omega_6^1\omega_7^1\omega_8^1=\mathrm{d}V_{S^7},$$

$$*\omega_2^1\omega_3^1\omega_2^6\omega_3^6\omega_2^7\omega_3^7\omega_2^8\omega_3^8|_{f(S^7)}=-\omega_2^5\omega_3^5\omega_4^5\omega_4^1\omega_4^6\omega_4^7\omega_4^8|_{f(S^7)}=0,$$

$$*\omega_2^5\omega_3^5\omega_2^6\omega_3^6\omega_2^7\omega_4^7\omega_2^8\omega_4^8|_{f(S^7)}=\omega_2^1\omega_3^1\omega_4^1\omega_4^5\omega_4^6\omega_3^7\omega_3^8|_{f(S^7)}=\mathrm{d}V_{S^7},$$

$$*\omega_2^1\omega_3^1\omega_2^6\omega_3^6\omega_2^7\omega_4^7\omega_2^8\omega_4^8|_{f(S^7)}=-\omega_2^5\omega_3^5\omega_4^5\omega_4^1\omega_4^6\omega_3^7\omega_3^8|_{f(S^7)}=0,\ \cdots.$$

Then

$$\frac{1}{b}f^* * p_1^2(E) = \frac{2}{5}\cdot\frac{1}{2\pi^4}(3\cdot 3+3\cdot 2)\mathrm{d}V_{S^7} = \frac{3}{\pi^4}\mathrm{d}V_{S^7},$$

$$\frac{2}{5}\int_{S^7} f^* * p_1^2(E) = 1.$$

□

Obviously, we also have $\int_{f(S^7)}\tau^*\frac{3}{\pi^4}\mathrm{d}V_{S^7} = 1$; then

$$[\tau^*\frac{3}{\pi^4}\mathrm{d}V_{S^7}] = \frac{2}{5} * p_1^2(E) \in H^7(G(3,8)).$$

Theorem 8.4 *(1) The Poincaré dual of* $p_1(E)$ *is* $[M]$ *and the Poincaré dual of* $\frac{3}{\pi^4} * p_1(E)$ *is* $[\mathbb{C}P^2]$*;*

(2) The Poincaré dual of $p_2(F) = p_1^2(E)$ *is* $[f(S^7)]$ *and the Poincaré dual of* $\frac{2}{5} * p_1^2(E)$ *is* $[ASSOC]$*.*

9. The case of $G(4,8)$

Let $E = E(4,8)$, $F = F(4,8)$ be canonical vector bundles on Grassmann manifold $G(4,8)$. We have

$$p_t(G(4,8)) = 1 + 3t^4 + 4t^8 + 3t^{12} + t^{16},$$

$$e(E)e(F) = 0,\ p_1(E) = -p_1(F),\ p_2(E) = e^2(E),\ p_2(F) = e^2(F),$$

$$p_1^2(E) = p_2(E) + p_2(F),\ p_1(E)p_2(E) = p_1(E)p_2(F) = \frac{1}{2}p_1^3(E),$$

$$p_1^2(E)e(E) = e^3(E),\ p_1^2(F)e(F) = e^3(F).$$

By the method used in §4, we can show $e(E\otimes F) = 6e^4(E) = 6e^4(F)$. Then by $\int_{G(4,8)} e(E\otimes F) = \chi(G(4,8)) = 12$, we have

$$\int_{G(4,8)} e^4(E) = \int_{G(4,8)} e^4(F) = 2.$$

We first study the cases of 4 and 12. Under the star operator $*\colon\ G(4,8)\to G(4,8)$, $*\mathbb{C}P^2$ is a submanifold of $G(4,8)$. From $\mathbb{C}P^2 = G_{\mathbb{C}}(1,3)\subset G(4,8)$, we have $*\mathbb{C}P^2 = G_{\mathbb{C}}(2,3)$. The following table computes the integration of the characteristic classes on the submanifolds of $G(4,8)$.

	$\mathbb{C}P^2$	$*\mathbb{C}P^2$	$G(2,4)$	$G(1,5)$	$G(4,5)$	$\overline{\mathbb{C}P}^2$	$\mathbb{H}P^1$
$e(E)$	0	1	0	0	2	0	-1
$e(F)$	1	0	0	2	0	-1	1
$p_1(E)$	1	-1	2	0	0	1	2

Note that $\det\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 1 & -1 & 2\end{pmatrix} = -2$, as proof of Theorem 5.5, we can show $e(E), e(F), p_1(E) \in H^4(G(4,8),\mathbb{Z})$ or $\mathbb{C}P^2, *\mathbb{C}P^2, G(2,4)\in H_4(G(4,8),\mathbb{Z})$ are the generators.

By Proposition 2.2, $V(G(4,8)) = \frac{8}{135}\pi^8$ and we can compute

$$a = (e(E), e(E)) = (e(F), e(F)) = \frac{1}{2}(p_1(E), p_1(E)) = \frac{4}{15}\pi^4,$$

$$(e(E), e(F)) = (e(E), p_1(E)) = (e(F), p_1(E)) = 0.$$

In the last section we have Cayley submanifold $CAY = \{\pi \in G(4,8) \mid \pi A = -A\}$ of $G(4,8)$. The Lie group $Spin_7$ acts on CAY transitively. Let $e_1, e_2, \cdots, e_8$ be $Spin_7$ frame fields on $\mathbb{R}^8$ and then CAY be generated by $e_1e_2e_3e_4$. By the equations listed in the proof of Lemma 7.3, we have

$$\begin{aligned}
&\mathrm{d}(e_1e_2e_3e_4) = \sum \omega_i^\alpha E_{i\alpha} \\
&= \omega_1^5(E_{15} + E_{48}) + \omega_1^6(E_{16} + E_{47}) + \omega_1^7(E_{17} - E_{46}) + \omega_1^8(E_{18} - E_{45}) \\
&\quad + \omega_2^5(E_{25} + E_{47}) + \omega_2^6(E_{26} - E_{48}) + \omega_2^7(E_{27} - E_{45}) + \omega_2^8(E_{28} + E_{46}) \\
&\quad + \omega_3^5(E_{35} + E_{46}) + \omega_3^6(E_{36} - E_{45}) + \omega_3^7(E_{37} + E_{48}) + \omega_3^8(E_{38} - E_{47}).
\end{aligned}$$

Then the induced metric is

$$\begin{aligned}
\mathrm{d}s^2_{CAY} \quad = \quad & (\omega_1^5 - \omega_2^6)^2 + (\omega_1^5 + \omega_3^7)^2 + (\omega_2^6 - \omega_3^7)^2 \\
& + (\omega_1^6 + \omega_2^5)^2 + (\omega_1^6 - \omega_3^8)^2 + (\omega_2^5 - \omega_3^8)^2 \\
& + (\omega_1^7 - \omega_3^5)^2 + (\omega_1^7 - \omega_2^8)^2 + (\omega_2^8 + \omega_3^5)^2 \\
& + (\omega_1^8 + \omega_2^7)^2 + (\omega_1^8 + \omega_3^6)^2 + (\omega_2^7 + \omega_3^6)^2.
\end{aligned}$$

By $(\omega_1^5 - \omega_2^6)(\omega_1^5 + \omega_3^7)(\omega_2^6 - \omega_3^7) = 2\omega_1^5\omega_3^7\omega_2^6$, $\cdots$, we get

$$\mathrm{d}V_{CAY} = 16\omega_1^5\omega_2^5\omega_3^5 \ \cdots \ \omega_1^8\omega_2^8\omega_3^8.$$

As shown in [7], $2\pi^2 p_1(E)$ is a calibration on $G(4,8)$ with comass $\frac{3}{2}$ and CAY is a calibrated submanifold of $2\pi^2 * p_1(E)$; then

$$2\pi^2 * p_1(E)|_{CAY} = \frac{3}{2}\mathrm{d}V_{CAY}.$$

By triality transformation, we can show that CAY is isometric to $G(3,7)$; then $V(CAY) = V(G(3,7)) = \frac{16\pi^6}{45}$; see [14]. This shows

$$\int_{CAY} \frac{1}{2a} * p_1(E) = \int_{CAY} \frac{45}{32\pi^6}\mathrm{d}V_{CAY} = \frac{1}{2}.$$

By Theorem 3.1, $e(E), e(F), p_1(E)$ cannot be the generators of $H^4(G(4,8),\mathbb{Z})$. We have proved

Lemma 9.1 *$\mathbb{C}P^2, *\mathbb{C}P^2, G(2,4)$ are the generators of $H_4(G(4,8),\mathbb{Z})$, and the dual generators of $H^4(G(4,8),\mathbb{Z})$ are $e(F), e(E), \frac{1}{2}(p_1(E) + e(E) - e(F))$, respectively.*

The inner product of $e(E), e(F), \frac{1}{2}(p_1(E) + e(E) - e(F))$ forms a matrix

$$A = \frac{4\pi^4}{15}\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}, \quad A^{-1} = \frac{15}{4\pi^4}\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{3}{2} & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Lemma 9.2 *(1)* $*e(E) = \frac{2\pi^4}{15} e^3(E),\ *e(F) = \frac{2\pi^4}{15} e^3(F),\ *p_1(E) = \frac{2\pi^4}{15} p_1^3(E)$;

(2) $\frac{1}{2}e^3(E) - \frac{1}{4}p_1^3(E), \frac{1}{2}e^3(F) + \frac{1}{4}p_1^3(E), \frac{1}{2}p_1^3(E) \in H^{12}(G(4,8),\mathbb{Z})$ *are the generators.*

Proof $*e(E), *e(F), *p_1(E)$ and $e^3(E), e^3(F), p_1^3(E)$ are 2 generators of the cohomology group $H^{12}(G(4,8))$, and they are all the harmonic forms on $G(4,8)$. Then $e^3(E), e^3(F), p_1^3(E)$ can be represented by $*e(E), *e(F), *p_1(E)$. Assuming $e^3(E) = \lambda * e(E) + \mu * e(F) + \nu * p_1(E)$, by $e(F) \wedge *e(E) = 0,\ e(F) \wedge *p_1(E) = 0$, we have $\mu = \nu = 0$,

$$2 = \int_{G(4,8)} e(E) \wedge e^3(E) = \lambda \int_{G(4,8)} e(E) \wedge *e(F) = \lambda \frac{4\pi^4}{15}.$$

Then $*e(E) = \frac{2\pi^4}{15} e^3(E)$. The other 2 equalities can be proved similarly.

Then

$$\begin{aligned}&(*e(E), *e(F), *\frac{1}{2}(p_1(E) + e(E) - e(F)))A^{-1}\\&= (\frac{1}{2}(e^3(E) - \frac{1}{2}p_1^3(E)), \frac{1}{2}(e^3(F) + \frac{1}{2}p_1^3(E)), \frac{1}{2}p_1^3(E)).\end{aligned}$$

□

Lemma 9.3 *The following table computes the integration of the characteristic classes on the submanifolds of* $G(4,8)$ *in dimension 12.*

	$G(4,7)$	$G(3,7)$	CAY
$e^3(E)$	2	0	-1
$e^3(F)$	0	2	1
$p_1^3(E)$	0	0	2

Proof The second column follows from $\int_{G(4,7)} e^3(E) = \int_{G(3,7)} e^3(F) = 2$ and $e(F)|_{G(4,7)} = 0,\ p_1^3(E) = 2p_1(E)p_2(F) = 2p_1(E)e^2(F)$. The third column can be proved similarly. For the fourth column, we have proved $\int_{CAY} \frac{1}{2a} * p_1(E) = \frac{1}{2}$; then $\int_{CAY} p_1^3(E) = 2$. By computing $*e(E)|_{CAY},\ *e(F)|_{CAY}$, we can show $\int_{CAY} e^3(E) = -1$ and $\int_{CAY} e^3(F) = 1$. □

Theorem 9.4 *(1)* $e(E), e(F), \frac{1}{2}(p_1(E) + e(E) - e(F)) \in H^4(G(4,8),\mathbb{Z})$ *are the generators, and their dual generators are* $[\mathbb{C}P^2], [*\mathbb{C}P^2], [G(2,4)] \in H_4(G(4,8),\mathbb{Z})$*;*

(2) $\frac{1}{2}e^3(E), \frac{1}{2}e^3(F), \frac{1}{2}p_1^3(E)$ *and* $[G(4,7)], [G(3,7)], [CAY]$ *are the generators of* $H^{12}(G(4,8),\mathbb{Z})$ *and* $H_{12}(G(4,8),\mathbb{Z})$*, respectively;*

(3) The Poincaré duals of $e(E), e(F), \frac{1}{2}(p_1(E) + e(E) - e(F))$ *are*

$$[G(4,7)],\ [G(3,7)],\ [CAY] + [G(4,7)] - [G(3,7)]$$

respectively.

Proof By Lemma 9.2, $\frac{1}{2}e^3(E)-\frac{1}{4}p_1^3(E), \frac{1}{2}e^3(F)+\frac{1}{4}p_1^3(E), \frac{1}{2}p_1^3(E)$ are the generators of $H^{12}(G(4,8),\mathbb{Z})$. Then $\frac{1}{2}e^3(E), \frac{1}{2}e^3(F), \frac{1}{2}p_1^3(E)$ are also the generators of $H^{12}(G(4,8),\mathbb{Z})$. □

By Theorem 3.1, we can compute the Poincaré duals of

$$\frac{1}{2}e^3(E)-\frac{1}{4}p_1^3(E), \frac{1}{2}e^3(F)+\frac{1}{4}p_1^3(E), \frac{1}{2}p_1^3(E).$$

By Theorem 9.4, $\frac{1}{2}(p_1(E)+e(E)-e(F))e(E)=\frac{1}{2}(p_1(E)e(E)+e^2(E))$ and $\frac{1}{2}(p_1(E)e(F)-e^2(F))$, $\frac{1}{2}(p_1(E)e(E)-e^2(E))$, $\frac{1}{2}(p_1(E)e(F)+e^2(F))$ are integral cocycles. The submanifolds $ASSOC$, $\widetilde{ASSOC}$ defined in §7 are also the submanifolds of $G(4,8)$; then $*ASSOC$, $*\widetilde{ASSOC}$ are submanifolds of $G(4,8)$. The following table can be proved by Lemma 7.4.

	$ASSOC$	$\widetilde{ASSOC}$	$*ASSOC$	$*\widetilde{ASSOC}$
$\frac{1}{2}e^2(F)+\frac{1}{2}p_1(E)e(F)$	1	0	0	0
$\frac{1}{2}e^2(F)-\frac{1}{2}p_1(E)e(F)$	0	1	0	0
$\frac{1}{2}e^2(E)+\frac{1}{2}p_1(E)e(E)$	0	0	1	0
$\frac{1}{2}e^2(E)-\frac{1}{2}p_1(E)e(E)$	0	0	0	1

Theorem 9.5 *The characteristic classes*

$$\frac{1}{2}e^2(F)+\frac{1}{2}p_1(E)e(F),\ \frac{1}{2}e^2(F)-\frac{1}{2}p_1(E)e(F),$$

$$\frac{1}{2}e^2(E)+\frac{1}{2}p_1(E)e(E),\ \frac{1}{2}e^2(E)-\frac{1}{2}p_1(E)e(E)$$

are the generators of $H^8(G(4,8),\mathbb{Z})$. Their Poincaré duals are

$$[ASSOC],\ [\widetilde{ASSOC}],\ [*ASSOC],\ [*\widetilde{ASSOC}]$$

respectively.

Proof To see that the Poincaré dual of $\xi=\frac{1}{2}(e^2(F)+p_1(E)e(F))$ is $ASSOC$, we want to show that for any $\eta\in H^8(G(4,8))$ we have $\int_{G(4,8)}\xi\wedge\eta=\int_{ASSOC}\eta$. We can take $\eta=\frac{1}{2}(e^2(F)\pm p_1(E)e(F))$, $\frac{1}{2}(e^2(E)\pm p_1(E)e(E))$ to verify this equation. □

By $\mathbb{R}^8=\mathbb{R}^3\oplus\mathbb{R}^5$, we see the product Grassmann $G(2,3)\times G(2,5), G(1,3)\times G(3,5)$ can imbedded in $G(4,8)$ and we have

	$G(4,6)$	$G(2,6)$	$G(2,3)\times G(2,5)$	$G(1,3)\times G(3,5)$
$e^2(E)$	2	0	0	0
$e^2(F)$	0	2	0	0
$p_1(E)e(E)$	0	0	4	0
$p_1(E)e(F)$	0	0	0	4

Then

$$G(4,6),\ G(2,6),\ G(2,3)\times G(2,5),\ G(1,3)\times G(3,5)\in H_8(G(4,8),\mathbb{R})$$

and

$$e^2(E),\ e^2(F),\ p_1(E)e(E),\ p_1(E)e(F) \in H^8(G(4,8),\mathbb{R})$$

are also the generators.

As an application, we consider the immersion $f\colon\ M \to \mathbb{R}^8$ of a compact oriented 4-dimensional manifold, with $g\colon\ M \to G(4,8)$ as its Gauss map. We have

$$g_*[M] = \frac{1}{2}\chi(M)[G(4,5)] + \lambda[G(1,5)] + \frac{3}{2}\tau(M)[G(2,4)],$$

where $\lambda = \frac{1}{2}\int_M g^*e(F(4,8))$ and $\tau(M) = \frac{1}{3}\int_M g^*p_1(E(4,8)) = \frac{1}{3}\int_M p_1(TM)$ is the signature of M. $\lambda = 0$ if f is an imbedding.

If g is the Gauss map of immersion M in $\mathbb{R}^7$ or $\mathbb{R}^6$, we have

$$g_*[M] = \frac{1}{2}\chi(M)[G(4,5)] + \frac{3}{2}\tau(M)[G(2,4)].$$

10. The cohomology groups on $ASSOC$

The submanifold $ASSOC \approx G_2/SO(4)$ of Grassmann manifold $G(3,7)$ is important in the theory of calibrations; see [7,9]. In [6] Borel and Hirzebruch studied the characteristic classes on homogeneous spaces, and they computed the cohomology of $ASSOC$. In what follows we use Gysin sequence to study the cohomology of $ASSOC$.

As §7, let $G(2,7)$ and $G(3,7)$ be Grassmann manifolds on $\mathbb{R}^7 \subset \mathbb{R}^8$ generated by $\bar{e}_2, \cdots, \bar{e}_8$, and $S^6 \subset \mathbb{R}^7$ the unit sphere. There is a fibre bundle $\tau_1\colon\ G(2,7) \to S^6$ defined by $\pi A = \bar{e}_1 v A,\ \tau_1(\pi) = v$, where $A \in C\ell_8$ is defined in §7. For any $G \in G_2$, we have the following commutative diagram

$$\begin{array}{ccc} G(2,7) & \xrightarrow{G} & G(2,7) \\ \tau_1 \downarrow & & \downarrow \tau_1 \\ S^6 & \xrightarrow{G} & S^6. \end{array}$$

The fibre $\tau_1^{-1}(\bar{e}_2) = \{v \wedge Jv \mid v \in S^6,\ v \perp \bar{e}_2\} \approx \mathbb{C}P^2$; see [10].

Then for any $\pi \in G(2,7),\ v = \tau_1(\pi),\ v \wedge \pi \in ASSOC$. This defines the map

$$\tau_2\colon\ G(2,7) \to ASSOC,\quad \pi \mapsto v \wedge \pi.$$

For any $e_2e_3e_4 \in ASSOC,\ e_2e_3e_4A = \bar{e}_1 A$, then $\tau_2(e_3e_4) = e_2e_3e_4$. This shows

Lemma 10.1 *$\tau_2\colon\ G(2,7) \to ASSOC$ is a fibre bundle with fibre $G(2,3) = S^2$.*

Let $i\colon\ ASSOC \to G(3,7)$ be an inclusion. It is easy to see $G(2,7)$ is isomorphic to the sphere bundle $S(\tilde{E}) = \{v \in \tilde{E},\ |v| = 1\}$ of the induced bundle $\tilde{E} = i^*E(3,7)$. Let $e(E(3,7)) \in H^3(G(3,7),\mathbb{Z})$ be the Euler class of $E(3,7)$, $2e(E(3,7)) = 0$; see [13] p. 95–103. Then $e(\tilde{E}) = i^*e(E(3,7)) \in H^3(ASSOC,\mathbb{Z})$ is the Euler class of the induced bundle $\tilde{E}$. There is a Gysin exact sequence for the sphere bundle $G(2,7) \to ASSOC$,

$$\longrightarrow H^q(ASSOC) \xrightarrow{\tau_2^*} H^q(G(2,7)) \xrightarrow{\tau_{2*}} H^{q-2}(ASSOC)$$

$$\xrightarrow{\wedge e(\tilde{E})} H^{q+1}(ASSOC) \xrightarrow{\tau_2^*} H^{q+1}(G(2,7)) \longrightarrow,$$

where τ_{2*} is the integration along the fibre. The coefficients of the cohomology groups can be $\mathbb{R}, \mathbb{Z}$, or $\mathbb{Z}_2$.

Lemma 10.2 $e(\tilde{E}) = i^* e(E(3,7)) \neq 0$.

Proof The map $\tau_{2*}\colon\ H^q(G(2,7),\mathbb{Z}) \to H^{q-2}(ASSOC,\mathbb{Z})$ is the integration along the fibre. Let $\bar{e}_1, e_2, e_3, \cdots, e_8$ be G_2 frame fields, and $G(2,7)$ is generated by e_3e_4 and $\tau_2(e_3e_4) = e_2e_3e_4$. Then the Euler class of vector bundle $E(2,7)$ can be represented by

$$e(E(2,7)) = \frac{1}{2\pi}\omega_3^2 \wedge \omega_4^2 + \frac{1}{2\pi}\sum_{\alpha=5}^{8} \omega_3^\alpha \wedge \omega_4^\alpha$$

and $\omega_3^2 \wedge \omega_4^2$ is the volume element of the fibre at e_3e_4. Then $\tau_{2*}(e(E(2,7)) = 2$.

By Gysin sequence, the map $\tau_{2*}\colon\ H^2(G(2,7)) \to H^0(ASSOC)$ is surjective if $e(\tilde{E}) = 0$. This contradicts the fact that $\tau_{2*}(e(E(2,7)) = 2$ and $e(E(2,7)) \in H^2(G(2,7),\mathbb{Z})$ is a generator. □

Then $e(E(3,7)) \in H^3(G(3,7),\mathbb{Z})$ is a nonzero torsion.

Theorem 10.3 *The cohomology groups of* $ASSOC$ *are*

$$H^q(ASSOC,\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q \neq 1,7, \\ 0, & q = 1,7; \end{cases}$$

$$H^q(ASSOC,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0,4,8, \\ \mathbb{Z}_2, & q = 3,6, \\ 0, & q = 1,2,5,7; \end{cases}$$

$$H^q(ASSOC,\mathbb{R}) = \begin{cases} \mathbb{R}, & q = 0,4,8, \\ 0, & q \neq 0,4,8. \end{cases}$$

Proof $G(2,7)$ is a Kähler manifold, and the cohomology of $G(2,7)$ is generated by Euler class $e(E(2,7))$. We prove the case of $\mathbb{Z}_2$ coefficients; the other cases are left to the reader. By Gysin sequence, we have

$$0 = H^{-2}(ASSOC) \xrightarrow{\wedge e(\tilde{E})} H^1(ASSOC) \xrightarrow{\tau_2^*} H^1(G(2,7)) = 0,$$

$$0 = H^{-1}(ASSOC) \xrightarrow{\wedge e(\tilde{E})} H^2(ASSOC) \xrightarrow{\tau_2^*} H^2(G(2,7)) \xrightarrow{\tau_{2*}=0} H^0(ASSOC)$$
$$\xrightarrow{\wedge e(\tilde{E})} H^3(ASSOC) \xrightarrow{\tau_2^*} H^3(G(2,7)) = 0.$$

This shows $H^1(ASSOC) = 0$ and $H^2(ASSOC) \cong H^2(G(2,7))$, $H^0(ASSOC) \cong H^3(ASSOC)$. By

$$0 = H^1(ASSOC) \xrightarrow{\wedge e(\tilde{E})} H^4(ASSOC) \xrightarrow{\tau_2^*} H^4(G(2,7)) \xrightarrow{\tau_{2*}} H^2(ASSOC)$$

and $\tau_{2*} = 0\colon\ H^4(G(2,7),\mathbb{Z}_2) \to H^2(ASSOC,\mathbb{Z}_2)$, we have

$$H^4(ASSOC) \cong H^4(G(2,7)).$$

The cases of $q = 5, \cdots, 8$ can be proved similarly. □

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