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Norden structures of Hessian type

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Abstract: In this paper, we show that Kähler (para-Kähler) manifolds admit a Norden–Hessian metric $h = \nabla^2 f$ if the function f is holomorphic (para-holomorphic), and we further consider the existence condition of para-Kähler structures for Norden–Hessian metrics.

Key words: Hessian structure, Norden manifold, holomorphic (para-holomorphic) function, para-Kähler metric

1. Introduction

The Hessian Riemannian structures are intensively studied by famous scientists in the world. Hessian Riemannian structures, as well as being connected with important pure mathematical fields such as affine differential geometry, homogeneous spaces, and others, find applications in economic theory, in system modeling, and in statistical theory. Recent surveys on Hessian metrics were published by Duistermaat [3] and Shima and Yagi [16]. Shima and Yagi studied the geometry of the Euclidean space R^n endowed with Hessian metrics $h_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$, where $f : R^n \to R$ is a C^∞-class function. In [18], Udrişte and Bercu used pseudo-Riemannian Hessian metrics. Given an n-dimensional pseudo-Riemannian manifold (M_n, g) and a smooth function $f : M_n \to R$ whose Hessian with respect to g is nondegenerate and with constant signature, they introduced on M_n the associated pseudo-Riemannian Hessian metric $h = \nabla^2 f$ and studied the properties of the new pseudo-Riemannian manifold (M_n, h) in terms of local calculus, where ∇ is the Levi-Civita connection of g (see also [1]).

Almost Norden and almost para-Norden structures are among the most important geometrical structures that can be considered on a manifold. Let M_{2n} be a $2n$-dimensional differentiable manifold endowed with an almost (para-)complex structure φ and a pseudo-Riemannian metric g of signature (n, n) such that $g(\varphi X, Y) = g(X, \varphi Y)$ for arbitrary vector fields X and Y on M_{2n}. Then the metric g is called a Norden metric. Norden metrics are referred to as anti-Hermitian metrics or B-metrics. They find widespread application in mathematics as well as in theoretical physics.

The purpose of the present paper is to investigate Norden metrics of Hessian type $h = \nabla^2 f$. All manifolds, tensor fields, and other geometric objects considered throughout this paper are assumed to be differentiable of class C^∞ (i.e. smooth). We denote by $\Im^p_q(M_{2n})$ the set of all tensor fields of type (p, q) on M_{2n}.

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2. Preliminaries

2.1. Pseudo-Riemannian Hessian metrics

Let (M_{2n}, g) be a Riemannian manifold with a metric tensor g. The gradient $gradf$ of a function $f \in \Im_0^0(M_{2n})$ is the vector field metrically equivalent to the differential $df \in \Im_1^0(M_{2n})$. In terms of a coordinate system,

$$gradf = (g^{ij}\partial_i f)\partial_j .$$

Thus,

$$g(gradf, X) = g^{ij}(\partial_i f)X^k g_{jk} = Xf = (df)(X). \tag{2.1}$$

The Hessian of a function $f \in \Im_0^0(M_{2n})$ is its second covariant differential $h = \nabla(\nabla f) = \nabla^2 f$ with respect to the Levi-Civita connection of g, i.e. $h \in \Im_2^0(M_{2n})$. Since $\nabla_Y f = Yf = (df)(Y)$,

$$h(Y, X) = (\nabla(\nabla f))\,(Y, X) = (\nabla(df))(Y, X)$$

$$= X((df)(Y)) - (df)(\nabla_X Y) = XYf - (\nabla_X Y)\, f.$$

We easily see that h is a symmetric tensor field. Also, by virtue of (2.1), we have

$$g(\nabla_X(gradf), Y) = h(Y, X).$$

For the natural coordinates in Euclidean space, the components of h are just the second partials $\frac{\partial^2 f}{\partial x^i \partial x^j}$ (see [15, 16]).

Let us consider a differentiable function $f : M_{2n} \to \mathrm{R}$ such that its Hessian $\nabla^2 f$ is nondegenerate having constant signature (see [1, 18]). Hence, $h = \nabla^2 f$ defines a new metric on M_{2n} and is called a pseudo-Riemannian Hessian metric.

2.2. Norden metrics

Let (M_{2n}, φ) be an almost complex manifold with almost complex structure φ. Such a structure is said to be integrable if the matrix $\varphi = (\varphi_j^i)$ is reduced to the constant form in a certain holonomic natural frame in a neighborhood U_x of every point $x \in M_{2n}$. For an almost complex structure tensor φ to be integrable, it is necessary and sufficient that it be possible to introduce a torsion-free affine connection ∇ with respect to which the structure tensor φ is covariantly constant, i.e. $\nabla\varphi = 0$. It is also known that the integrability of φ is equivalent to the vanishing of the Nijenhuis tensor $N_\varphi \in \Im_2^1(M_{2n})$. If φ is integrable, then φ is a complex structure and, moreover, M_{2n} is a C-holomorphic manifold $X_n(\mathrm{C})$ whose transition functions are holomorphic mappings.

A pseudo-Riemannian metric G of signature (n, n) on M_{2n} is a Norden metric [9] if

$$G(\varphi X, \varphi Y) = -G(X, Y)$$

or equivalently

$$G(\varphi X, Y) = G(X, \varphi Y)$$

for any $X,\ Y \in \Im_0^1(M_{2n})$. Metrics of this type have also been studied under the names of pure, anti-Hermitian, and B-metrics (see [4, 7, 11, 17, 19, 21]). If (M_{2n}, φ) is an almost complex manifold with Norden metric G, we say that (M_{2n}, φ, G) is an almost Norden manifold. If φ is integrable, we say that (M_{2n}, φ, G) is a Norden manifold.

2.3. Holomorphic (almost holomorphic) tensor fields

Let $\overset{*}{t}$ be a complex tensor field on $X_n(\mathrm{C})$. The real model of such a tensor field is a tensor field t on M_{2n} of the same order such that the action of the structure tensor φ on t does not depend on which vector or covector argument of t that φ acts. Such tensor fields are said to be pure with respect to φ. They were studied by many authors (see, e.g., [7, 12, 13, 17, 19, 20, 21]). In particular, being applied to a (0,q)-tensor field ω, the purity means that for any $X_1, ..., X_q \in \Im_0^1(M_{2n})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, ..., X_q) = \omega(X_1, \varphi X_2, ..., X_q) = ... = \omega(X_1, X_2, ..., \varphi X_q).$$

We consider the operator

$$\Phi_\varphi : \Im_q^0(M_{2n}) \to \Im_{q+1}^0(M_{2n})$$

applied to a pure tensor field ω by (see [21])

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, ..., Y_q) = (\varphi X)(\omega(Y_1, Y_2, ..., Y_q)) - X(\omega(\varphi Y_1, Y_2, ..., Y_q)) \tag{2.2}$$

$$+\omega((L_{Y_1}\varphi)X, Y_2, ..., Y_q) + ... + \omega(Y_1, Y_2, ..., (L_{Y_q}\varphi)X),$$

where L_Y denotes the Lie differentiation with respect to Y.

When φ is a complex structure on M_{2n} and the tensor field $\Phi_\varphi \omega$ vanishes, the complex tensor field $\overset{*}{\omega}$ on $X_n(\mathrm{C})$ is said to be holomorphic (see [7, 17, 21]). Thus, a holomorphic tensor field $\overset{*}{\omega}$ on $X_n(\mathrm{C})$ is realized on M_{2n} in the form of a pure tensor field ω, such that

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, ..., Y_q) = 0$$

for any $X, Y_1, ..., Y_q \in \Im_0^1(M_{2n})$. Such a tensor field ω on M_{2n} is also called a holomorphic tensor field. When φ is an almost complex structure on M_{2n}, a tensor field ω satisfying $\Phi_\varphi \omega = 0$ is said to be almost holomorphic.

2.4. Holomorphic Norden (Kähler-Norden) metrics

On a Norden manifold, a Norden metric G is called holomorphic if

$$(\Phi_\varphi G)(X, Y, Z) = 0 \tag{2.3}$$

for any $X, Y, Z \in \Im_0^1(M_{2n})$.

By setting $X = \partial_k$, $Y = \partial_i$, $\mathrm{Z} = \partial_j$ in equation (2.3), we see that the components $(\Phi_\varphi G)_{kij}$ of $\Phi_\varphi G$ with respect to a local coordinate system $x^1, ..., x^n$ can be expressed as follows:

$$(\Phi_\varphi G)_{kij} = \varphi_k^m \partial_m G_{ij} - \varphi_i^m \partial_k G_{mj} + G_{mj}(\partial_i \varphi_k^m - \partial_k \varphi_i^m) + G_{im} \partial_j \varphi_k^m.$$

If (M_{2n}, φ, G) is a Norden manifold with a holomorphic Norden metric G, we say that (M_{2n}, φ, G) is a holomorphic Norden manifold.

In some aspects, holomorphic Norden manifolds are similar to Kähler manifolds. The following theorem is analogous to the next known result: an almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

Theorem 2.1 *[5] For an almost complex manifold with Norden metric G, the condition $\Phi_\varphi G = 0$ is equivalent to $\nabla\varphi = 0$, where ∇ is the Levi–Civita connection of G.*

A Kähler–Norden manifold can be defined as a triple (M_{2n}, φ, G) that consists of a manifold M_{2n} endowed with an almost complex structure φ and a pseudo-Riemannian metric G such that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of G and the metric G is assumed to be a Norden one. Therefore, there exist a one-to-one correspondence between Kähler–Norden manifolds and Norden manifolds with holomorphic metrics. Recall that the Riemannian curvature tensor of such a manifold is pure and holomorphic, and the scalar curvature is locally holomorphic function (see [5, 11]).

We know that the integrability of an almost complex structure φ is equivalent to the existence of a torsion-free affine connection with respect to which the equation $\nabla\varphi = 0$ holds. Since the Levi-Civita connection ∇ of G is a torsion-free affine connection, we have: if $\Phi_\varphi G = 0$, then φ is integrable. Thus, almost Norden manifolds with conditions $\Phi_\varphi G = 0$ and $N_\varphi \neq 0$, i.e. *almost holomorphic Norden manifolds (analogues of the almost Kähler manifolds with closed Kähler form),* do not exist.

Remark 2.2 *By similar devices, we can introduce para-Kähler–Norden (or para-holomorphic Norden) manifolds (see [14]).*

3. Holomorphic (para-holomorphic) functions

Let (M_{2n}, φ, g) be a Kähler (para-Kähler or para-Kähler–Norden) manifold.

Theorem 3.1 *A necessary and sufficient condition for an exact 1-form df, $f \in \Im_0^0(M_{2n})$ to be holomorphic (para-holomorphic), i.e. $\Phi_\varphi(df) = 0$, is that an associated 1-form $df \circ \varphi$ be closed, i.e. $d(df \circ \varphi) = 0$.*

Proof Using

$$(d\omega)(X,Y) = \frac{1}{2}\left\{X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\right\}, X,\ Y \in \Im_0^1(M_{2n}), \omega \in \Im_1^0(M_{2n})$$

for $(\omega \circ \varphi)(X) = \omega(\varphi(X))$, we have

$$\begin{aligned}(d\omega)(Y,\varphi X) &= \frac{1}{2}\left\{Y(\omega(\varphi X)) - (\varphi X)(\omega(Y)) - \omega([Y,\varphi X])\right\} \\ &= \frac{1}{2}\left\{Y(\omega(\varphi X)) - (\varphi X)(\omega(Y)) + \omega([\varphi X, Y])\right\} \\ &= \frac{1}{2}\{Y(\omega(\varphi X)) - (\varphi X)(\omega(Y)) + \omega([\varphi X, Y] \\ &\qquad -\varphi[X,Y]) + \omega(\varphi[X,Y])\}.\end{aligned} \tag{3.4}$$

From (2.2), we have

$$\begin{aligned}(\Phi_\varphi\omega)(X,Y) &= (\varphi X)(\omega(Y)) - X(\omega(\varphi Y)) + \omega((L_Y\varphi)(X)) \\ &= (\varphi X)(\omega(Y)) - X(\omega(\varphi Y)) - \omega([\varphi X, Y] - \varphi[X,Y]).\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4), we obtain

$$(d\omega)(Y,\varphi X) = \frac{1}{2}\left\{-(\Phi_\varphi\omega)(X,Y) + Y(\omega(\varphi X)) - X(\omega(\varphi Y)) + \omega(\varphi[X,Y])\right\}$$

$$= -\frac{1}{2}\{(\Phi_\varphi \omega)(X,Y) + Y((\omega\circ\varphi)(X)) - X((\omega\circ\varphi)(Y)) - (\omega\circ\varphi)([Y,X])\}$$

$$= -\frac{1}{2}(\Phi_\varphi \omega)(X,Y) + (d(\omega\circ\varphi))(Y,X).$$

From this we see that the equation $\Phi_\varphi \omega = 0$ is equivalent to

$$(d(\omega\circ\varphi))(Y,X) = (d\omega)(Y,\varphi X). \tag{3.6}$$

For $\omega = df$, equation (3.6) turns into the following simple form:

$$(d(df\circ\varphi))(Y,X) = (d^2 f)(Y,\varphi X) = 0,$$

i.e.

$$d(df\circ\varphi) = 0. \tag{3.7}$$

Thus, Theorem 3.1 is proven. □

If there exists a function $\overset{*}{f}$ on a Kähler (para-Kähler or para-Kähler–Norden) manifold such that $df\circ\varphi = d\overset{*}{f}$ for a function f, then we shall call f a holomorphic (para-holomorphic) function and $\overset{*}{f}$ its associated function. If such a function f is defined locally, then we call it a locally holomorphic (para-holomorphic) function.

Remark 3.2 *If (M_{2n},φ) is a complex (or para-complex) manifold, then in terms of real coordinates $(x^i, x^{\bar{i}})$, $i=1,...,n$; $\bar{i}=n+1,...,2n$, the equation $df\circ\varphi = d\overset{*}{f}$ reduces to*

$$\begin{cases} \partial_{\bar{i}} f = \partial_i \overset{*}{f} \quad (\partial_{\bar{i}} f = \partial_i \overset{*}{f}), \\ \partial_i f = -\partial_{\bar{i}} \overset{*}{f} \quad (\partial_i f = \partial_{\bar{i}} \overset{*}{f}), \end{cases}$$

which are the Cauchy–Riemann (or para-Cauchy–Riemann) equations for the complex (or para-complex) function $F = \overset{*}{f} + if$ (see [6, p. 122], [2]).

Remark 3.3 *We notice that equation (3.7) is equivalent to $df\circ\varphi = d\overset{*}{f}$ only locally. Hence, the condition for f to be locally holomorphic (para-holomorphic) also is given by*

$$(\Phi_\varphi df)_{ij} = \varphi_i^m \partial_m \partial_j f - \partial_i(\varphi_j^m \partial_m f) + (\partial_j \varphi_i^m)\partial_m f = 0.$$

4. Norden–Hessian metrics

If we assume that f is holomorphic (para-holomorphic), then, from (2.2), we have

$$(\Phi_\varphi(df))(X,Y) = (\varphi X)((df)(Y)) - X((df)(\varphi Y)) + (df)((L_Y\varphi)(X))$$

$$= \varphi(X)((df)(Y)) - X((df)(\varphi Y)) + (df)([Y,\varphi X] - \varphi([Y,X]))$$

$$= \varphi(X)((df)(Y)) - X((df)(\varphi Y))$$

$$+(df)(\nabla_Y \varphi X - \nabla_{\varphi X} Y - \varphi(\nabla_Y X - \nabla_X Y) - \nabla_X \varphi Y + \nabla_X \varphi Y)$$

$$= (\nabla_{\varphi X} df)(Y) - (\nabla_X df)(\varphi Y) - (df)(\nabla\varphi)(Y, X) + (df)(\nabla\varphi)(X, Y) = 0. \tag{4.8}$$

We now consider a holomorphic (para-holomorphic) function f on a Kähler (para-Kähler or para-Kähler–Norden) manifold (M_{2n}, φ, g). Let f be a smooth function such that the Hessian $\nabla^2 f$ is nondegenerate [18]. On a Kähler (para-Kähler or para-Kähler–Norden) manifold (M_{2n}, φ, g), $(\nabla\varphi = 0)$, equation (4.8) is equivalent to the following equation:

$$(\nabla^2 f)(Y, \varphi X) = (\nabla^2 f)(\varphi Y, X),$$

i.e. a manifold $(M_{2n}, \varphi, h = \nabla^2 f)$ is an almost Norden manifold, where h is a metric of signature (n, n). Thus, h naturally defines a Norden metric on Kähler (para-Kähler or para-Kähler–Norden) manifold (M_{2n}, φ, g). We call it a Norden–Hessian metric. Thus, we have the next theorem.

Theorem 4.1 *Let (M_{2n}, φ, g) be a Kähler (para-Kähler or para-Kähler–Norden) manifold and f be a smooth function such that the Hessian $\nabla^2 f$ is nondegenerate. Then, M_{2n} admits a Norden–Hessian structure $(\varphi, h = \nabla^2 f, g)$ if $f \in \Im_0^0(M_{2n})$ is holomorphic (para-holomorphic), where ∇ is the Levi-Civita connection of g.*

5. Para-Kähler metrics on Norden–Hessian manifolds

Let (M_{2n}, J, g) be a locally decomposable Riemannian manifold with integrable para-complex structure

$$J = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix},$$

E being an $(n \times n)$-unit matrix. In such manifolds, g is pure with respect to J, and moreover $\nabla J = 0$, i.e. a triple (M_{2n}, J, g) is a para-Kähler–Norden manifold. Additionally, g is para-holomorphic and the curvature tensor field R of g is pure with respect to the structure J [14]. Para-Kähler (hybrid) metrics for the case of para-complex algebras were introduced and studied in [10].

Let $(M_{2n}, J, h = \nabla^2 f)$ be a Hessian–Norden structure, which exists on a para-complex decomposable Riemannian manifold. Then

$$(\nabla^2 f)(JX, Y) = (\nabla^2 f)(X, JY),$$

from which we have

$$(\nabla^3 f)(JX, Y, Z) = (\nabla^3 f)(X, JY, Z). \tag{5.9}$$

Using the Ricci equation, from (5.9) we obtain

$$(\nabla^3 f)(X, JY, Z) = \nabla_Z(\nabla_{JY}(\nabla_X f))$$

$$= \nabla_{JY}(\nabla_Z(\nabla_X f)) - (df)(R(Z, JY)X)$$

$$= (\nabla^3 f)(X, Z, JY) - (df)(R(Z, JY)X) \tag{5.10}$$

and

$$(\nabla^3 f)(JX, Y, Z) = \nabla_Z(\nabla_Y(\nabla_{JX} f))$$

$$= \nabla_Y(\nabla_Z(\nabla_{JX} f)) - (df)(R(Z, Y)JX)$$

$$= (\nabla^3 f)(JX, Z, Y) - (df)(R(Z, Y)JX). \tag{5.11}$$

Since h is symmetric and the curvature tensor R of g is pure with respect to J, from (5.10) and (5.11) we have

$$(\nabla^3 f)(Z, JX, Y) = (\nabla^3 f)(Z, X, JY), \tag{5.12}$$

i.e. a tensor field $\nabla^3 f$ is pure in all arguments, where $\nabla^3 f$ is a higher-order Hessian structure on M_{2n} (see [8]).

On the other hand,

$$\begin{aligned}(\Phi_J h)(X, Z_1, Z_2) &= J(X)(h(Z_1, Z_2)) - X(h(JZ_1, Z_2)) - h(\nabla_{JX} Z_1, Z_2)\\ &+h((\nabla J)(X, Z_1), Z_2) + h(Z_1, (\nabla J)(X, Z_2)) - h(Z_1, \nabla_{JX} Z_2)\\ &+h(J(\nabla_X Z_1), Z_2) + h(JZ_1, \nabla_X Z_2)\\ &= (\nabla_{JX} h)(Z_1, Z_2) - (\nabla_X h)(JZ_1, Z_2) + h((\nabla J)(X, Z_1), Z_2)\\ &+h(Z_1, (\nabla J)(X, Z_2)).\end{aligned} \tag{5.13}$$

Substituting $h(Z_1, Z_2) = \nabla_{Z_1} \nabla_{Z_2} f$ and $\nabla J = 0$ in (5.13), by virtue of (5.12) we have

$$\begin{aligned}(\Phi_J h)(X, Z_1, Z_2) = (\Phi_J h)(X, Z_2, Z_1) &= (\nabla_{JX}(\nabla^2 f))(Z_2, Z_1) - (\nabla_X(\nabla^2 f))(Z_2, JZ_1)\\ &= (\nabla^3 f)(Z_2, Z_1, JX) - (\nabla^3 f)(Z_2, JZ_1, X) = 0,\end{aligned}$$

i.e. h is para-holomorphic. Then, using Theorem 2.1, we see that ${}^h\nabla J = 0$, where ${}^h\nabla$ is the Levi-Civita connection of h. Thus, we have the following theorem.

Theorem 5.1 *Let (M_{2n}, J, g) be a para-complex decomposable Riemannian manifold and f be a smooth function such that the Hessian $\nabla^2 f$ is nondegenerate. If f is para-holomorphic, then $(M_{2n}, J, h = \nabla^2 f, g)$ is a para-Kähler-Norden–Hessian manifold, where ∇ is the Levi-Civita connection of g.*

Remark 5.2 *The above proof is an alternative proof of the well-known result of holomorphic manifolds [20, p. 184]: the purity of the covariant derivative $\nabla h = \nabla^3 f$ in a torsion-free connection ∇ preserving the structure J ($\nabla J = 0$) is necessary and sufficient for the para-holomorphity of a pure tensor field $h = \nabla^2 f$. Moreover, if a function f on (M_{2n}, J, g) is para-holomorphic, then $h = \nabla^2 f$ is pure. Since the Levi-Civita connection ∇ of the para-holomorphic manifold (M_{2n}, J, g) is para-holomorphic with respect to J, it follows that the metric $h = \nabla^2 f$ is obviously para-holomorphic.*

Remark 5.3 *For Kähler manifold (M_{2n}, J, g), the curvature tensor R of the Hermitian metric g is not pure in all arguments. Therefore, Kähler manifolds may not always locally admit any Kähler–Norden–Hessian metric.*

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