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Two-weighted norm inequality on weighted Morrey spaces

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Abstract: Let u and ω be weight functions. We shall introduce the weighted Morrey spaces $L^{p,\kappa}(\omega)$ and investigate the sufficient condition and necessary condition about the 2-weighted boundedness of the Hardy–Littlewood maximal operator.

Key words: Weighted Morrey spaces, Hardy–Littlewood maximal operator, A_p weights

1. Introduction

Suppose $u(x)$ and $\omega(x)$ are weight functions on $\mathbb{R}^n$, and T is an operator taking suitable functions on $\mathbb{R}^n$. In his survey article [10], Muckenhoupt raised the general question of characterization when the weighted norm inequality

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^q \omega(x)dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p u(x)dx\right)^{\frac{1}{p}} \tag{1.1}$$

holds for any $1 \le p, q \le \infty$ and all appropriate f. In the case of one weight $u = \omega$, the inequality (1.1) can be characterized by remarkably simple conditions for many classical operators, e.g., the Hardy–Littlewood maximal operator, singular integral, and fractional operator (see [1, 9, 11]).

The case of different weights has been far less studied. Only for the Hardy–Littlewood maximal operator and other positive operators was this characterized in [13], while many classical operators are still open and only find sufficient conditions on weights for an operator to be bounded from $L^p(u)$ to $L^q(\omega)$. For the history of these results, we refer the reader to [2, 3, 5].

Weighted Morrey spaces $L^{p,\kappa}(\omega)$ were first introduced recently by Komori and Shirai [7], where the boundedness of many classical operators was established. Later, many authors found that the weighted Morrey spaces were also used in harmonic analysis [14, 15]. However, this only gives sufficient conditions for the boundedness of classical operators. The necessary condition associated with Hilbert transform in Morrey spaces was discussed by Samko [12].

In this paper, we concentrate our attention on the 2-weighted norm inequality associated with the Hardy–Littlewood maximal operator in weighted Morrey spaces. The same as the above cases, we only give a sufficient condition and a necessary condition, respectively.

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Throughout the paper cubes are assumed to have their sides parallel to the coordinate axes. Given cube $Q = Q(x,r)$ centered at x with side length r, $\omega(Q)$ denotes $\int_Q \omega(x)dx$ and the measure $\omega(x)dx$ is often abbreviated to ωdx. The Lebesgue measure of Q is denoted by $|Q|$ and the characteristic function of Q by χ_Q.

2. Some notations and lemmas

In this section, we introduce some basic definitions and lemmas.

Definition 2.1 *Let $1 < p < \infty$, $0 < \kappa < 1$, and w be a weight function. For any local integrable function f in $\mathbb{R}^n$, if it satisfies*

$$\|f\|_{L^{p,\kappa}(\omega)} := \sup_Q \left(\frac{1}{\omega(Q)^\kappa} \int_Q |f(x)|^p w(x)dx \right)^{\frac{1}{p}} < \infty.$$

Then f belongs to weighted Morrey spaces and $\|\cdot\|_{L^{p,\kappa}(\omega)}$ denotes the norm.

Note that if $\omega = 1$, $L^{p,\kappa}(\omega) = L^{p,\kappa}(\mathbb{R}^n)$ is the classical Morrey spaces; if $\kappa = 0$, $L^{p,0}(\omega) = L^p(\omega)$ is the weighted Lebesgue spaces.

Definition 2.2 *The Hardy–Littlewood maximal operator M is defined by*

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

and we define the maximal operator with respect to the measure $w(x)dx$ by

$$M_\omega f(x) = \sup_{x\in Q} \frac{1}{\omega(Q)} \int_Q |f(y)|\omega(y)dy.$$

Before the next definition, we recall that a dyadic cube is the product of the intervals that are divided by dyadic decomposition of the coordinate axis with side length 2^k, $k \in \mathbb{Z}$.

Definition 2.3 *Supposing that $\mathcal{F}$ is the collection of the dyadic cubes, we define $M_t^\star f(x)$ with translation operator τ_t as follows (see [4], p. 112, or [6], p. 431):*

$$M_t^\star f(x) = (\tau_{-t} \circ M^\star \circ \tau_t) f(x) = M^\star(\tau_t f)(x+t).$$

In the definition, $M^\star f(x)$ is a dyadic maximal operator (see [4], p. 111), which is defended by

$$M^\star f(x) = \sup_{x\in Q\in\mathcal{F}} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

The following definition was considered by Fefferman and Stein (see [4], p. 112):

Definition 2.4 *Let $\ell(Q)$ be the side length of a cube Q. For a positive real number N, we define the locally maximal operator by*

$$\bar{M}_N f(x) = \sup_{\substack{x\in Q \\ \ell(Q)\le N}} \frac{1}{|Q|} \int_Q |f(y)|dy$$

and the locally dyadic maximal operator by

$$\bar{M}_N^{\star} f(x) = \sup_{\substack{x\in Q\in\mathcal{F}\\ \ell(Q)\leq N}} \frac{1}{|Q|}\int_Q |f(y)|dy.$$

Definition 2.5 *A weight function ω satisfies the A_p condition with $1 < p < \infty$, if there exists a constant $C \geq 1$ such that for any cube Q,*

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} \leq C,$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

The definition of 2.5 can be found in [8] on page 21. In fact, the A_p weights have the following important lemma (see [8], p. 22):

Lemma 2.1 *Given a weight function $w \in A_p$, $1 < p < \infty$, it also satisfies the doubling condition Δ_2: for any cube Q, there exists a constant $C > 0$ such that $w(2Q) \leq Cw(Q)$.*

The next 2 definitions have a relation with the 2-weighted inequality in weighted Morrey spaces.

Definition 2.6 *A weight ω is called a (p,κ)-permission weight if for every cube Q, the inequality*

$$\|\chi_Q\|_{L^{p,\kappa}(\omega)} < \infty$$

holds. Furthermore, a weight ω is called a (p,κ)-specific permission weight if it is a (p,κ)-permission weight and for every cube Q

$$\|\chi_Q\sigma\|_{L^{p,\kappa}(\omega)} < \infty,$$

where $\sigma = \omega^{1-p'}$.

Definition 2.7 *We say $(u,\omega) \in \mathcal{S}_{p,\kappa}$ if u is a (p,κ)-permission weight and ω is a (p,κ)-specific permission weight, such that the following inequalities hold:*

$$\sup_Q \frac{\|\chi_Q\|_{L^{p,\kappa}(u)}}{\|\chi_{3Q}\|_{L^{p,\kappa}(\omega)}} < \infty \quad \text{and} \quad \sup_Q \frac{\sigma(3Q)}{|Q|} \times \frac{\|\chi_Q\|_{L^{p,\kappa}(u)}}{\|\chi_{3Q}\sigma\|_{L^{p,\kappa}(\omega)}} < \infty.$$

The following lemmas play an important role in our proofs.

Lemma 2.2 *Let $1 < p < \infty$ and $\omega \in A_p$; then there exists an index r: $1 < r < p$, such that $\omega \in A_r$.*

This lemma was first obtained by Muckenhoupt in [9], page 214. One can also find a clear statement in [8], page 26.

Lemma 2.3 *Let $1 < p < \infty$. σ is a nonnegative locally integrable weight. Then $M_\sigma^{\star}$ is bounded in $L^p(\sigma)$.*

Lemma 2.10 would be found in [6], page 426. In fact, $M_\sigma^{\star}$ is of weak type (1,1) and bounded in $L^\infty(\sigma)$. By using the Marcinkiewicz interpolation theorem we can get this result.

Lemma 2.4 *Suppose f is a locally integrable function in $\mathbb{R}^n$; then for every integer k and $x \in \mathbb{R}^n$ we have*

$$M_{2^k} f(x) \leq 2^{1-kn} \int_{Q(0,2^{k+3})} M_t^\star f(x) dt,$$

where $Q(0, 2^{k+3})$ means the cube centered at 0 with side length 2^{k+3}.

As to the proof of Lemma 2.11, we refer to [6], page 431. Note that the notation $Q\left(0, 2^{k+2}\right)$ in [6] means a cube centered at 0 with half side length 2^{k+2}, which differs from our argument.

3. A sufficient condition of 2-weighted norm inequalities in weighted Morrey spaces

In this section we give a sufficient condition of 2-weighted boundedness of the Hardy–Littlewood maximal operator. The statement is the following theorem.

Theorem 3.1 *Suppose $1 < p < \infty$, $0 < \kappa < 1$, (u, ω) is a couple of weights, $\sigma = \omega^{1-p'}$ and $\omega \in A_p$. Then the Hardy–Littlewood maximal operator M is bounded from $L^{p,\kappa}(\omega)$ to $L^{p,\kappa}(u)$ if there exists a constant $C > 0$, such that for any cubes Q and Q'*

$$\frac{1}{u(Q)^\kappa} \int_{Q'} M(\chi_{Q'}\sigma)(x)^p u dx \leq \frac{C}{\omega(Q)^\kappa} \int_{Q'} \sigma dx < \infty.$$

To prove Theorem 3.1, we need an auxiliary proposition as follows:

Proposition 3.1 *Let $1 < p < \infty, 0 < \kappa < 1$. If (u, ω) is a couple of weights and $\sigma = \omega^{1-p'}$ is locally integrable, then the following statements are equivalent:*

(a) There exists a constant $C > 0$, such that for any cube Q

$$\frac{1}{u(Q)^\kappa} \int_{\mathbb{R}^n} (Mf(x))^p \, u dx \leq \frac{C}{\omega(Q)^\kappa} \int_{\mathbb{R}^n} |f(x)|^p \omega dx;$$

(b) There exists a constant $C > 0$, such that for any cube Q and Q'

$$\frac{1}{u(Q)^\kappa} \int_{Q'} (M(\chi_{Q'}\sigma)(x))^p \, u dx \leq \frac{C}{\omega(Q)^\kappa} \int_{Q'} \sigma dx < \infty.$$

Proof The idea follows from [13] and [6]. Once having chosen $f = \sigma(x)\chi_Q(x)$, we can easily draw the conclusion $(a) \Rightarrow (b)$. To verify the opposite, we partition it into 3 steps. First, it suffices to prove the result for the dyadic maximal operator $M^\star$; second, by using the first step, we show the result for the translation dyadic maximal operator $M_t^\star$; and, third, by using Lemma 2.11, we complete the proof for the maximal operator M.

We first check the case of the dyadic maximal operator. Since (b) is satisfied by $M^\star$, let us consider the locally dyadic maximal operator $\bar{M}_N^\star$. Recall the definition of $\bar{M}_N^\star f(x)$: it takes the supremum over all dyadic cubes Q that contain x with side length of less than N; therefore, under the condition $\bar{M}_N^\star f(x) > 2^k$, $k \in \mathbb{Z}$, $x \in \mathbb{R}^n$, we get a family of countable such dyadic cubes $\left\{Q_l^k\right\}_l$ satisfying

$$2^k < \frac{1}{|Q_l^k|} \int_{Q_l^k} |f(y)| dy. \tag{3.1}$$

For any 2 dyadic cubes, either 1 is contained in the other or they are disjoint. Hence, we can choose the maximum ones in the family $\{Q_l^k\}_l$. The collection of these maximum dyadic cubes is denoted by $\{Q_j^k\}_j$. They satisfy the same inequality as in (3.1). Moreover, for any dyadic cube $Q \supsetneqq Q_j^k$ with side length $\ell(Q) \leq N$, we have

$$\frac{1}{|Q|}\int_Q |f(y)|\,dy \leq 2^k.$$

Obviously

$$\left\{x \in \mathbb{R}^n | \bar{M}_N^\star f(x) > 2^k\right\} = \bigcup_j Q_j^k.$$

Let $E_j^k = Q_j^k \backslash \left\{x \in \mathbb{R}^n | \bar{M}_N^\star f(x) > 2^{k+1}\right\}$. For $k_1 \neq k_2$ or $j_1 \neq j_2$, it is easy to check that $E_{j_1}^{k_1}$ and $E_{j_2}^{k_2}$ are disjoint and

$$\bigcup_{j,k} E_j^k = \bigcup_{j,k} Q_j^k.$$

Hence, for any cube Q, we have

$$\begin{aligned}
\frac{1}{u(Q)^\kappa}\int_{\mathbb{R}^n} (\bar{M}_N^\star f(x))^p u dx &= \frac{1}{u(Q)^\kappa}\sum_{j,k}\int_{E_j^k} (\bar{M}_N^\star f(x))^p u dx \\
&\leq \frac{2^p}{u(Q)^\kappa}\sum_{j,k} u(E_j^k)\left(\frac{1}{|Q_j^k|}\int_{Q_j^k}\sigma dx\right)^p \left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k}|f(x)|\omega^{\frac{p'}{p}}\sigma dx\right)^p \\
&= \frac{C}{u(Q)^\kappa}\sum_{j,k}\gamma_j^k\left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} g(x)\sigma dx\right)^p, \qquad (3.2)
\end{aligned}$$

where $g = |f|\omega^{\frac{p'}{p}}$ and

$$\gamma_j^k = u(E_j^k)\left(\frac{1}{|Q_j^k|}\int_{Q_j^k}\sigma dx\right)^p.$$

Next we define the measure γ on the measure space $\mathcal{M}$ where $\mathcal{M} = \mathbb{Z}\times\mathbb{Z}_+$. Let $\mathcal{M}_0 = \{(k,j) \in \mathcal{M}|$ k, j is the index of $Q_j^k\}$, and

$$\widetilde{g}(k,j) = \begin{cases} \left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} g(x)\sigma dx\right)^p, & (k,j)\in\mathcal{M}_0 \\ 0, & otherwise. \end{cases}$$

Then we have

$$\begin{aligned}
&\frac{C}{u(Q)^\kappa}\sum_{j,k}\gamma_j^k\left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} g(x)\sigma dx\right)^p \\
&= \frac{C}{u(Q)^\kappa}\int_{\mathcal{M}}\widetilde{g}(k,j)d\gamma \\
&= C\int_0^\infty \frac{\gamma(S_\lambda)}{u(Q)^\kappa}d\lambda, \qquad (3.3)
\end{aligned}$$

where

$$S_\lambda = \left\{ (k,j) \in \mathcal{M}_0 \mid \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} g(x)\sigma dx \right)^p > \lambda \right\}.$$

Note that all the cubes in $\left\{Q_j^k\right\}_{j,k}$ have side length of at most N. For the same reason, we can choose maximum dyadic cubes in $\left\{Q_j^k : (k,j) \in S_\lambda\right\}$. These maximum dyadic cubes are relabeled by $\{Q_i^\lambda\}$. Thus:

$$\bigcup_i Q_i^\lambda \subseteq \{x \in \mathbb{R}^n | (M_\sigma^\star g(x))^p > \lambda\}.$$

Joining (3.2) and (3.3) and by using condition (b) and Lemma 2.10, we have

$$\begin{aligned}
&\frac{1}{u(Q)^\kappa} \int_{\mathbb{R}^n} \left(\bar{M}_N^\star f(x)\right)^p u dx \leq C \int_0^\infty \frac{\gamma(S_\lambda)}{u(Q)^\kappa} d\lambda \\
&= C \int_0^\infty \frac{1}{u(Q)^\kappa} \sum_i \sum_{\substack{Q_j^k \subseteq Q_i^\lambda \\ (k,j)\in\mathcal{M}_0}} u(E_j^k) \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} \sigma \right)^p d\lambda \\
&\leq C \int_0^\infty \left(\sum_i \frac{1}{u(Q)^\kappa} \int_{Q_i^\lambda} (M^\star(\chi_{Q_i^\lambda}\sigma)(x))^p u dx \right) d\lambda \\
&\leq C \int_0^\infty \frac{1}{\omega(Q)^\kappa} \sum_i \sigma(Q_i^\lambda) d\lambda \\
&\leq \frac{C}{\omega(Q)^\kappa} \int_0^\infty \sigma\left(\{x \in \mathbb{R}^n | (M_\sigma^\star g(x))^p > \lambda\} \right) d\lambda \\
&= \frac{C}{\omega(Q)^\kappa} \int_{\mathbb{R}^n} \left(M_\sigma^\star \left(\frac{f}{\sigma} \right)(x) \right)^p \sigma dx \\
&\leq \frac{C}{\omega(Q)^\kappa} \int_{\mathbb{R}^n} |f(x)|^p \omega dx.
\end{aligned}$$

Letting N tend to ∞, we get

$$\frac{1}{u(Q)^\kappa} \int_{\mathbb{R}^n} (M^\star f(x))^p u dx \leq \frac{C}{\omega(Q)^\kappa} \int_{\mathbb{R}^n} |f(x)|^p \omega dx.$$

Now we prove the case of maximal operator M. Note that $(\tau_t u, \tau_t \omega)$ is also a couple of weights and $\tau_t u(Q) = u(Q-t)$. Then for 2 arbitrary cubes Q and Q', we have

$$
\begin{aligned}
&\frac{1}{\tau_t u(Q)^\kappa}\int_{Q'}(M^\star((\tau_t\sigma)\chi_{Q'})(x))^p\tau_t u(x)dx\\
&\qquad=\frac{1}{\tau_t u(Q)^\kappa}\int_{Q'}(M^\star(\tau_t(\sigma\chi_{Q'-t}))(x))^p u(x-t)dx\\
&\qquad=\frac{1}{\tau_t u(Q)^\kappa}\int_{Q'-t}(M_t^\star(\sigma\chi_{Q'-t})(y))^p u(y)dy\\
&\qquad\leq\frac{1}{\tau_t u(Q)^\kappa}\int_{Q'-t}(M(\sigma\chi_{Q'-t})(x))^p u(x)dx\\
&\qquad=\frac{1}{\tau_t u(Q)^\kappa}\int_{Q'}(M(\tau_t\sigma\chi_{Q'})(x))^p\tau_t u(x)dx\\
&\qquad\leq\frac{C}{\tau_t\omega(Q)^\kappa}\int_{Q'}\tau_t\sigma dx.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
&\frac{1}{u(Q)^\kappa}\int_{\mathbb{R}^n}(M_t^\star f(x))^p u(x)dx\\
&=\frac{1}{\tau_t u(Q+t)^\kappa}\int_{\mathbb{R}^n}(M^\star(\tau_t f)(x))^p(\tau_t u)dx\\
&\leq\frac{C}{\tau_t\omega(Q+t)^\kappa}\int_{\mathbb{R}^n}|\tau_t f(x)|^p\tau_t\omega dx\\
&=\frac{C}{\omega(Q)^\kappa}\int_{\mathbb{R}^n}|f(x)|^p\omega.
\end{aligned}
$$

Using Lemma 2.11, for each $k\in\mathbb{Z}$, we have

$$
\begin{aligned}
&\left(\frac{1}{u(Q)^\kappa}\int_{\mathbb{R}^n}(M_{2^k}f(x))^p udx\right)^{\frac{1}{p}}\\
&\leq\frac{2^{1-kn}}{u(Q)^{\frac{\kappa}{p}}}\left(\int_{\mathbb{R}^n}\left(\int_{Q(0,2^{k+3})}M_t^\star f(x)dt\right)^p udx\right)^{\frac{1}{p}}\\
&\leq 2^{1-kn}\int_{Q(0,2^{k+3})}\left(\frac{1}{u(Q)^\kappa}\int_{\mathbb{R}^n}(M_t^\star f(x))^p\,udx\right)^{\frac{1}{p}}dt\\
&\leq C\left(\frac{1}{\omega(Q)^\kappa}\int_{\mathbb{R}^n}|f(x)|^p\omega dx\right)^{\frac{1}{p}}.
\end{aligned}
$$

Letting k tend to ∞, we get

$$
\frac{1}{u(Q)^\kappa}\int_{\mathbb{R}^n}(Mf(x))^p udx\leq\frac{C}{\omega(Q)^\kappa}\int_{\mathbb{R}^n}|f(x)|^p\omega dx.
$$

This completes the proof. □

Next we shall prove Theorem 3.1.

Proof Suppose $f = f\chi_{3Q} + f\chi_{(3Q)^c} \triangleq f_1 + f_2$. Since $\omega \in A_p$, σ is locally integrable, by Proposition 3.2:

$$\left(\frac{1}{u(Q)^\kappa}\int_Q (Mf_1(x))^p u dx\right)^{\frac{1}{p}} \leq \left(\frac{C}{\omega(Q)^\kappa}\int_{3Q}|f(x)|^p \omega dx\right)^{\frac{1}{p}}$$
$$\leq C\|f\|_{L^{p,\kappa}(\omega)}.$$

On the other hand, from [7] we know that, for every $x \in Q$,

$$M_\omega f_2(x) \leq \sup_{R:Q\subseteq 3R}\left(\frac{1}{\omega(R)}\int_R |f(x)|\omega dx\right). \tag{3.4}$$

Noting that $\omega \in A_p$, by Lemma 2.9, there exists an index $r : 1 < r < p$, such that $\omega \in A_r$, and then $Mf_2(x) \leq C(M_\omega|f_2|^r(x))^{\frac{1}{r}}$. By inequality (3.4), for every $x \in Q$, we have

$$\begin{aligned}
&Mf_2(x) \leq C\left(M_\omega|f_2|^r(x)\right)^{\frac{1}{r}}\\
&\leq C\sup_{R:Q\subseteq 3R}\left(\frac{1}{\omega(R)}\int_R |f(x)|^r\omega dx\right)^{\frac{1}{r}}\\
&\leq C\sup_{R:Q\subseteq 3R}\left(\frac{1}{\omega(R)^\kappa}\int_R |f(y)|^p\omega dy\right)^{\frac{1}{p}}\omega(R)^{\frac{\kappa-1}{p}}\\
&\leq C\|f\|_{L^{p,\kappa}(\omega)}\omega(Q)^{\frac{\kappa-1}{p}}.
\end{aligned}$$

Hence:

$$\left(\frac{1}{u(Q)^\kappa}\int_Q (Mf_2(x))^p u dx\right)^{\frac{1}{p}} \leq Cu(Q)^{\frac{1-\kappa}{p}}\omega(Q)^{\frac{\kappa-1}{p}}\|f\|_{L^{p,\kappa}(\omega)}.$$

Using Proposition 3.2 again, let $f = \chi_Q$; then for every $x \in Q^o$, $M(\chi_Q)(x) \equiv 1$. We have

$$u(Q)^{\frac{1-\kappa}{p}} = \left(\frac{1}{u(Q)^\kappa}\int_Q (M(\chi_Q)(x))^p u dx\right)^{\frac{1}{p}} \leq C\omega(Q)^{\frac{1-\kappa}{p}},$$

and then

$$\left(\frac{1}{u(Q)^\kappa}\int_Q (Mf_2(x))^p u dx\right)^{\frac{1}{p}} \leq C\|f\|_{L^{p,\kappa}(\omega)}.$$

Therefore, we complete the proof of Theorem 3.1. □

4. A necessary condition of 2-weighted norm inequalities in weighted Morrey spaces

In this section we give a necessary condition of 2-weighted boundedness of the Hardy–Littlewood maximal operator. The idea goes back to Samko [12].

Theorem 4.1 *If u, ω, and $\sigma = \omega^{1-p'}$ are respectively (p,κ)-permission weight, (p,κ)-specific permission weight, and a doubling weight, then $(u,\omega) \in \mathcal{S}_{p,\kappa}$ is the necessary condition of $\|Mf\|_{L^{p,\kappa}(u)} \leq C\|f\|_{L^{p,\kappa}(\omega)}$.*

Proof Suppose $Q_1, Q_2, ..., Q_{2^n}$ are any neighboring cubes that have the same edge length but no intersecting interior whose union is a new big cube, which is denoted by Q_0. Let $x \in Q_i$, $i \in \{1, 2, ..., 2^n\}$. Then for $j \neq i$:

$$M(\chi_{Q_j})(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q \chi_{Q_j}(y) dy \right) \geq \frac{1}{|Q_0|} \int_{Q_j} dy = \frac{1}{2^n}.$$

Hence:

$$\begin{aligned} \sup_Q \left(\frac{1}{u(Q)^\kappa} \int_Q \chi_{Q_i}(y) u dy \right) &\leq 2^{np} \sup_Q \left(\frac{1}{u(Q)^\kappa} \int_{Q \bigcap Q_i} (M(\chi_{Q_j})(y))^p u dy \right) \\ &\leq 2^{np} C^p \|\chi_{Q_j}\|^p_{L^{p,\kappa}(\omega)}. \end{aligned}$$

Note that $Q_j \subseteq 3Q_i$,

$$\|\chi_{Q_i}\|_{L^{p,\kappa}(u)} \leq 2^n C \|\chi_{Q_j}\|_{L^{p,\kappa}(\omega)} \leq C \|\chi_{3Q_i}\|_{L^{p,\kappa}(\omega)}.$$

On the other hand, for every $x \in Q_j$, we have

$$M(\chi_{Q_i}\sigma)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q \bigcap Q_i} \sigma dx \geq \frac{1}{|Q_0|} \int_{Q_i} \sigma dx = \frac{1}{2^n |Q_i|} \int_{Q_i} \sigma dx.$$

Then

$$\begin{aligned} &\left(\frac{1}{|Q_i|} \int_{Q_i} \sigma dx \right)^p \sup_Q \frac{1}{u(Q)^\kappa} \int_{Q \bigcap Q_j} u dy \\ &\leq 2^{np} \sup_Q \frac{1}{u(Q)^\kappa} \int_{Q \bigcap Q_j} (M(\chi_{Q_i}\sigma)(y))^p u dy \\ &\leq C \|\chi_{Q_i}\sigma\|^p_{L^{p,\kappa}(\omega)}. \end{aligned}$$

Since σ is a doubling weight and $3Q_i \subseteq 5Q_j$, we have $\sigma(3Q_i) \leq \sigma(5Q_j) \leq C\sigma(Q_j)$ and

$$\frac{\|\chi_{Q_i}\|_{L^{p,\kappa}(u)}}{\|\chi_{3Q_i}\sigma\|_{L^{p,\kappa}(\omega)}} \leq C \frac{|3Q_i|}{\sigma(Q_j)} \leq C \frac{|Q_i|}{\sigma(3Q_i)}.$$

This completes the proof. □

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