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On semiparallel anti-invariant submanifolds of generalized Sasakian space forms

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Abstract: We consider minimal anti-invariant semiparallel submanifolds of generalized Sasakian space forms. We show that the submanifolds are totally geodesic under certain conditions.

Key words: Semiparallel submanifold, generalized Sasakian space form, Laplacian of the second fundamental form, totally geodesic submanifold

1. Introduction

Let (M,g) and $(N,\widetilde{g})$ be Riemannian manifolds and $f\colon\ M\to N$ an isometric immersion. Denote by σ and $\overline{\nabla}$ its second fundamental form and van der Waerden–Bortolotti connection, respectively. If $\overline{\nabla}\sigma=0$, then the submanifold M is said to have a parallel second fundamental form [6]. The act of $\overline{R}$ to the second fundamental form σ is defined by

$$\left(\overline{R}(X,Y)\cdot\sigma\right)(Z,W)=R^{\perp}(X,Y)h(Z,W)-\sigma(R(X,Y)Z,W)-\sigma(Z,R(X,Y)W)$$

$$=(\overline{\nabla}_X\overline{\nabla}_Y\sigma)(Z,W)-(\overline{\nabla}_Y\overline{\nabla}_X\sigma)(Z,W),\tag{1}$$

where $\overline{R}$ is the curvature tensor of the van der Waerden–Bortolotti connection $\overline{\nabla}$. Semiparallel submanifolds were introduced by Deprez in [7]. If $\overline{R}\cdot\sigma=0$, then f is called semiparallel. It is clear that if f has parallel second fundamental form, then it is semiparallel. Hence, a semiparallel submanifold can be considered as a natural generalization of a submanifold with a parallel second fundamental form. Semiparallel submanifolds have been studied by various authors; see, for example [3, 7, 8, 9, 13, 16] and the references therein. Recently, in [18], Yıldız et al. studied C-totally real pseudoparallel submanifolds of Sasakian space forms, which are generalizations of semiparallel submanifolds. In [5], Brasil et al. studied C-totally real pseudoparallel submanifolds of λ-Sasakian space forms. In [15], Sular, et al. studied anti-invariant pseudoparallel submanifolds of Kenmotsu space forms with ξ tangent to the submanifold. In [14], Sular studied pseudoparallel submanifolds of Kenmotsu space forms with ξ normal to the submanifold.

Motivated by the studies of the above authors, in the present paper, we study anti-invariant minimal semiparallel submanifolds of generalized Sasakian space forms.

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2. Generalized Sasakian space forms

Let $M^{2n+1} = M(\varphi, \xi, \eta, g)$ be an almost contact metric manifold. If $[\varphi, \varphi](X, Y) = -2d\eta(X, Y)\xi$ for all vector fields X, Y on M^{2n+1} then the almost contact metric structure is called *normal*, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion. If $d\eta(X, Y) = g(X, \varphi Y)$ for all vector fields X, Y on M, then the almost contact metric structure (φ, ξ, η, g) is a *contact metric structure*. In this case, the manifold M^{2n+1} with the contact metric structure (φ, ξ, η, g) is called a *contact metric manifold*. A normal contact metric manifold is called a *Sasakian manifold* [4]. An almost contact metric manifold M is called a *Kenmotsu manifold* [11] if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

where ∇ is the Levi-Civita connection. A Kenmotsu manifold is normal but not a contact manifold.

An almost contact metric manifold M is called a *cosymplectic manifold* [12] if $\nabla\varphi = 0$, which implies that $\nabla\xi = 0$. Hence, ξ is a Killing vector field for a cosymplectic manifold.

An almost contact metric manifold is called a λ-*Sasakian manifold* [10] if

$$(\nabla_X \varphi)Y = \lambda \left[g(X, Y)\xi - \eta(Y)X\right].$$

If $\lambda = 1$, a λ-*Sasakian manifold* is a Sasakian manifold.

The sectional curvature of a φ-section is called a φ-*sectional curvature*. A Sasakian (resp. Kenmotsu, cosymplectic, λ-Sasakian) manifold with constant φ-sectional curvature c is called a *Sasakian (resp. Kenmotsu, cosymplectic, λ-Sasakian) space form*; see [4, 11, 12, 10], respectively.

The notion of a generalized Sasakian space form was introduced by Alegre et al. in [1]. An almost contact metric manifold $M^{2n+1} = M(\varphi, \xi, \eta, g)$ whose curvature tensor satisfies

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z)\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2}$$

for certain differentiable functions f_1, f_2, and f_3 on M^{2n+1} is called a generalized Sasakian space form [1]. The natural examples of generalized Sasakian space forms with constant functions are a Sasakian space form ($f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$) [4], a Kenmotsu space form ($f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$) [11], and a cosymplectic space form ($f_1 = f_2 = f_3 = \frac{c}{4}$) [12]. If M is a λ-Sasakian space form then $f_1 = \frac{c+3\lambda}{4}$, $f_2 = f_3 = \frac{c-\lambda}{4}$ [10].

Let M be an n-dimensional submanifold of a Riemannian manifold $\widetilde{M}$. We denote by $\widetilde{\nabla}$, ∇ the Riemannian and induced Riemannian connections in $\widetilde{M}$ and M, respectively, and let σ be the second fundamental form of the submanifold. The equation of Gauss is given by

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) \\ &\quad -g\left(\sigma(X,W),\sigma(Y,Z)\right) + g\left(\sigma(X,Z),\sigma(Y,W)\right) \end{aligned} \tag{3}$$

for all vector fields X, Y, Z, W tangent to M, where $\widetilde{R}$ and R denote the curvature tensors of the connections $\widetilde{\nabla}$, ∇, respectively. The mean curvature vector field H is given by $H = \frac{1}{n}\operatorname{trace}(\sigma)$. The submanifold M is *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$ [6].

Using (3), the Gauss equation for the submanifold M^n of a generalized Sasakian space form $\widetilde{M}^{2m+1}$ is

$$\widetilde{R}(X,Y,Z,W) =$$
$$f_1\{g(Y,Z)X - g(X,Z)Y\}$$
$$+f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z)\}$$
$$+f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},$$
$$+g\left(\sigma(X,W),\sigma(Y,Z)\right) - g\ \left(\sigma(X,Z),\sigma(Y,W)\right). \tag{4}$$

A submanifold M of a generalized Sasakian space form $\widetilde{M}^{2m+1}$ is called *anti-invariant* if and only if $\varphi(T_xM)\ \subset T_x^{\perp}M$ for all $x\ \in M$ [2]. For more information about anti-invariant submanifolds we refer to [17].

3. Semiparallel anti-invariant submanifolds of a generalized Sasakian space form

In this section, we give the main results of the paper.

For an n-dimensional submanifold M of a $(2n+1)$-dimensional Riemannian manifold $\widetilde{M}^{2n+1}$, it is known that the Laplacian $\Delta\sigma_{ij}^{\alpha}$ of σ_{ij}^{α} is defined by

$$\Delta\sigma_{ij}^{\alpha} = \sum_{i,j,k=1}^{n} \sigma_{ijkk}^{\alpha}. \tag{5}$$

Then

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{2n+1} \sigma_{ij}^{\alpha}\sigma_{ijkk}^{\alpha} + \left\|\overline{\nabla}\sigma\right\|^2, \tag{6}$$

(see [17]), where

$$\|\sigma\|^2 = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{2n+1} (\sigma_{ij}^{\alpha})^2, \tag{7}$$

and

$$\left\|\overline{\nabla}\sigma\right\|^2 = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{2n+1} (\sigma_{ijkk}^{\alpha})^2 \tag{8}$$

are the square of the length of second and the third fundamental forms of M, respectively.

A simple calculation gives us the following proposition:

Proposition 1 *Let M be an n-dimensional minimal anti-invariant submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}^{2n+1}$ with ξ normal to M. Then we have*

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \left\|\overline{\nabla}\sigma\right\|^2 + (f_2 + nf_1)\left\|\sigma\right\|^2$$
$$-\left[\sum_{\alpha,\beta=n+1}^{2n+1} tr(A_\alpha \circ A_\beta)^2 + \|[A_\alpha, A_\beta]\|^2\right]. \tag{9}$$

Theorem 2 *Let M be an n-dimensional minimal anti-invariant semiparallel submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}^{2n+1}$ with ξ normal to M. If*

$$f_2 + nf_1 \leq 0,$$

then M is totally geodesic.

Proof Let $\{e_1, e_2, ..., e_n, \xi, \varphi e_1, \varphi e_2, ..., \varphi e_n\}$ be an orthonormal frame in $\widetilde{M}^{2n+1}$ such that $e_1, e_2, ..., e_n$ are tangent to M. By definition, the semiparallelity of M, for $1 \leq k, l \leq n$, gives us

$$\overline{R}(e_l, e_k) \cdot \sigma = 0. \tag{10}$$

By (1), we can write

$$(\overline{R}(e_l, e_k) \cdot \sigma)(e_i, e_j) = (\overline{\nabla}_{e_l} \overline{\nabla}_{e_k} \sigma)(e_i, e_j) - (\overline{\nabla}_{e_k} \overline{\nabla}_{e_l} \sigma)(e_i, e_j) = 0, \tag{11}$$

where $1 \leq i, j, k, l \leq n$.

Hence, equation (6) turns into

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n} g((\overline{\nabla}_{e_k} \overline{\nabla}_{e_k} \sigma)(e_i, e_j), \sigma(e_i, e_j)) + \left\|\overline{\nabla}\sigma\right\|^2. \tag{12}$$

Furthermore, using equations (5) and (6), we have

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n} \sum_{\alpha=n+1}^{2n+1} \sigma_{ij}^{\alpha}(\overline{\nabla}_{e_i} \overline{\nabla}_{e_j} H^{\alpha}) + \left\|\overline{\nabla}\sigma\right\|^2. \tag{13}$$

Since M is minimal, equation (13) can be written as

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \left\|\overline{\nabla}\sigma\right\|^2 \tag{14}$$

(see [18]). Comparing (9) and (14), we find

$$-\left(f_2 + nf_1\right)\|\sigma\|^2$$

$$+ \sum_{\alpha,\beta=n+1}^{2n+1} tr(A_\alpha \circ A_\beta)^2 + \|[A_\alpha, A_\beta]\|^2 = 0.$$

From the assumption, if

$$f_2 + nf_1 \leq 0,$$

then $tr(A_\alpha \circ A_\beta) = 0$. In particular, $\|A_\alpha\|^2 = tr(A_\alpha \circ A_\alpha) = 0$, and thus $A_\alpha = 0$, which means that $\sigma = 0$. Then M is totally geodesic. □

Using Theorem 2, we have the following corollaries:

Corollary 3 [18] *Let M be an n-dimensional minimal anti-invariant semiparallel submanifold of a $(2n+1)$-dimensional Sasakian space form $\widetilde{M}^{2n+1}$ with ξ normal to M. If*

$$n(c+3)+c-1\leq 0,$$

then M is totally geodesic.

Corollary 4 *Let M be an n-dimensional minimal anti-invariant semiparallel submanifold of a $(2n+1)$-dimensional cosymplectic space form $\widetilde{M}^{2n+1}$ with ξ normal to M. If*

$$c\leq 0,$$

then M is totally geodesic.

Corollary 5 [5] *Let M be an n-dimensional minimal anti-invariant semiparallel submanifold of a $(2n+1)$-dimensional λ-Sasakian space form $\widetilde{M}^{2n+1}$ with ξ normal to M. If*

$$c-\lambda+n(c+3\lambda)\leq 0,$$

then M is totally geodesic.

If M is an $(n+1)$-dimensional minimal anti-invariant submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}^{2n+1}$ with ξ tangent to M, then we have the following proposition:

Proposition 6 *Let M be an $(n+1)$-dimensional minimal anti-invariant submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}^{2n+1}$ with ξ tangent to M. Then we have*

$$\frac{1}{2}\Delta(\|\sigma\|^2)=\left\|\overline{\nabla}\sigma\right\|^2+(f_2+(n+1)\,f_1-f_3)\,\|\sigma\|^2$$

$$-f_3\sum_{i=1}^{n+1}\|\sigma(e_i,\xi)\|^2-\left[\sum_{\alpha,\beta=n+2}^{2n+1}tr(A_\alpha\circ A_\beta)^2+\|[A_\alpha,A_\beta]\|^2\right]. \tag{15}$$

Theorem 7 *Let M be an $(n+1)$-dimensional minimal anti-invariant semiparallel submanifold of $(2n+1)$-dimensional generalized Sasakian space form $\widetilde{M}^{2n+1}$ with ξ tangent to M. If*

$$f_2+(n+1)\,f_1-f_3\leq 0$$

and

$$f_3\geq 0,$$

then M is totally geodesic.

Proof Let $\{e_1,e_2,...,e_n,\xi,\varphi e_1,\varphi e_2,...,\varphi e_n\}$ be an orthonormal frame in $\widetilde{M}^{2n+1}$ such that $e_1,e_2,...,e_n,\xi$ are tangent to M. Then for $1\leq i,j\leq n+1$ and $n+2\leq\alpha\leq 2n+1$. Similar to the proof of Theorem 2, using the minimality condition, we obtain

$$\frac{1}{2}\Delta(\|\sigma\|^2)=\left\|\overline{\nabla}\sigma\right\|^2. \tag{16}$$

Comparing (15) and (16) we find

$$-\left(f_2+(n+1)\,f_1-f_3\right)\|\sigma\|^2+f_3\sum_{i=1}^{n+1}\|\sigma(e_i,\xi)\|^2$$

$$+\sum_{\alpha,\beta=n+2}^{2n+1}tr(A_\alpha\circ A_\beta)^2+\|[A_\alpha,A_\beta]\|^2=0.$$

From the assumption, if

$$f_2+(n+1)\,f_1-f_3\leq 0$$

and

$$f_3\geq 0,$$

then $tr(A_\alpha\circ A_\beta)=0$. Similar to the proof of Theorem 2, this gives us $\sigma=0$. Then M is totally geodesic. □

Using Theorem 7, we have the following corollaries:

Corollary 8 *Let M be an $(n+1)$-dimensional minimal anti-invariant semiparallel submanifold of a $(2n+1)$-dimensional Sasakian space form $\widetilde{M}^{2n+1}$ with ξ tangent to M. If*

$$c\in(-\infty,-3]\cup[1,\infty),$$

then M is totally geodesic.

Corollary 9 *Let M be an $(n+1)$-dimensional minimal anti-invariant semiparallel submanifold of a $(2n+1)$-dimensional cosymplectic space form $\widetilde{M}^{2n+1}$ with ξ tangent to M. If $c=0$, then M is totally geodesic.*

Corollary 10 *[15] Let M be an $(n+1)$-dimensional minimal anti-invariant semiparallel submanifold of a $(2n+1)$-dimensional Kenmotsu space form $\widetilde{M}^{2n+1}$ with ξ tangent to M. If $c\in[-1,3]$, then M is totally geodesic.*

Corollary 11 *Let M be an $(n+1)$-dimensional minimal anti-invariant semiparallel submanifold of a $(2n+1)$-dimensional λ-Sasakian space form $\widetilde{M}^{2n+1}$ with ξ tangent to M.*

i) If λ is a positive function on M and

$$c\in(-\infty,-3\lambda]\cup[\lambda,\infty)$$

or

ii) If λ is a negative function on M and

$$c\in[\lambda,-3\lambda],$$

then M is totally geodesic.

References

[1] Alegre P, Blair DE, Carriazo A. Generalized Sasakian space-forms. Israel J Math 2004; 141: 157–183.

[2] Alegre P, Carriazo A. Submanifolds of generalized Sasakian space forms. Taiwanese J Math 2009; 13: 923–941.

[3] Asperti AC, Mercuri F. Semi-parallel immersions into space forms. Boll Un Mat Ital 1994; 8B: 883–895.

[4] Blair DE. Riemannian Geometry of Contact and Symplectic Manifolds. Boston: Birkhauser, 2002.

[5] Brasil A, Lobos GA, Mariano M. C-totally real submanifolds with parallel mean curvature in λ-Sasakian space forms. Mat Contemp 2008; 34: 83–102.

[6] Chen BY. Geometry of Submanifolds. Pure and Applied Mathematics. New York: Marcel Dekker, 1973.

[7] Deprez J. Semiparallel hypersurfaces. Rend Sem Mat Univ Politec Torino 1986; 44: 303–316.

[8] Deprez J. Semi-parallel immersions. In: Morvan JM, Verstraelen L, editors. Geometry and Topology of Submanifolds. Singapore: World Scientific, 1989, pp. 73–88.

[9] Dillen F. Semi-parallel hypersurfaces of a real space form. Israel J Math 1991; 75: 193–202.

[10] Janssens D, Vanhecke L. Almost contact structures and curvature tensors. Kodai Math J 1981; 4: 1–27.

[11] Kenmotsu K. A class of almost contact Riemannian manifolds. Tohoku Math J 1972; 24: 93–103.

[12] Ludden GD. Submanifolds of cosymplectic manifolds. J Differential Geometry 1970; 4: 237–244.

[13] Özgür C, Murathan C. On invariant submanifolds of Lorentzian para-Sasakian manifolds. Arab J Sci Eng Sect A Sci 2009; 34: 177–185.

[14] Sular S. Kenmotsu manifolds and their some submanifolds. PhD, Balıkesir University, Balıkesir, Turkey, 2009.

[15] Sular S, Özgür C, Murathan C. Pseudoparallel anti-invariant submanifolds of Kenmotsu manifolds. Hacettepe J Math Stat 2010; 39: 535–543.

[16] Van der Veken J, Vrancken L. Parallel and semi-parallel hypersurfaces of $S^n \times \mathbb{R}$. Bull Braz Math Soc (NS) 2008; 39: 355–370.

[17] Yano K, Kon M. Structures on Manifolds. Series in Pure Mathematics. Singapore: World Scientific, 1984.

[18] Yıldız A, Murathan C, Arslan K, Ezentaş R. C-totally real pseudo-parallel submanifolds of Sasakian space forms. Monatsh- Math 2007; 151: 247-256.