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On semiparallel anti-invariant submanifolds of generalized Sasakian space forms

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Abstract: We consider minimal anti-invariant semiparallel submanifolds of generalized Sasakian space forms. We show that the submanifolds are totally geodesic under certain conditions.

Key words: Semiparallel submanifold, generalized Sasakian space form, Laplacian of the second fundamental form, totally geodesic submanifold

1. Introduction

Let (M, g) and (N, \tilde{g}) be Riemannian manifolds and $f: M \rightarrow N$ an isometric immersion. Denote by σ and $\bar{\nabla}$ its second fundamental form and van der Waerden–Bortolotti connection, respectively. If $\bar{\nabla}\sigma = 0$, then the submanifold M is said to have a parallel second fundamental form [6]. The act of \bar{R} to the second fundamental form σ is defined by

$$\begin{aligned}(\bar{R}(X, Y) \cdot \sigma)(Z, W) &= R^\perp(X, Y)h(Z, W) - \sigma(R(X, Y)Z, W) - \sigma(Z, R(X, Y)W) \\ &= (\bar{\nabla}_X \bar{\nabla}_Y \sigma)(Z, W) - (\bar{\nabla}_Y \bar{\nabla}_X \sigma)(Z, W),\end{aligned}\tag{1}$$

where \bar{R} is the curvature tensor of the van der Waerden–Bortolotti connection $\bar{\nabla}$. Semiparallel submanifolds were introduced by Deprez in [7]. If $\bar{R} \cdot \sigma = 0$, then f is called semiparallel. It is clear that if f has parallel second fundamental form, then it is semiparallel. Hence, a semiparallel submanifold can be considered as a natural generalization of a submanifold with a parallel second fundamental form. Semiparallel submanifolds have been studied by various authors; see, for example [3, 7, 8, 9, 13, 16] and the references therein. Recently, in [18], Yıldız et al. studied C -totally real pseudoparallel submanifolds of Sasakian space forms, which are generalizations of semiparallel submanifolds. In [5], Brasil et al. studied C -totally real pseudoparallel submanifolds of λ -Sasakian space forms. In [15], Sular, et al. studied anti-invariant pseudoparallel submanifolds of Kenmotsu space forms with ξ tangent to the submanifold. In [14], Sular studied pseudoparallel submanifolds of Kenmotsu space forms with ξ normal to the submanifold.

Motivated by the studies of the above authors, in the present paper, we study anti-invariant minimal semiparallel submanifolds of generalized Sasakian space forms.

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2. Generalized Sasakian space forms

Let $M^{2n+1} = M(\varphi, \xi, \eta, g)$ be an almost contact metric manifold. If $[\varphi, \varphi](X, Y) = -2d\eta(X, Y)\xi$ for all vector fields X, Y on M^{2n+1} then the almost contact metric structure is called *normal*, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion. If $d\eta(X, Y) = g(X, \varphi Y)$ for all vector fields X, Y on M , then the almost contact metric structure (φ, ξ, η, g) is a *contact metric structure*. In this case, the manifold M^{2n+1} with the contact metric structure (φ, ξ, η, g) is called a *contact metric manifold*. A normal contact metric manifold is called a *Sasakian manifold* [4]. An almost contact metric manifold M is called a *Kenmotsu manifold* [11] if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

where ∇ is the Levi-Civita connection. A Kenmotsu manifold is normal but not a contact manifold.

An almost contact metric manifold M is called a *cosymplectic manifold* [12] if $\nabla\varphi = 0$, which implies that $\nabla\xi = 0$. Hence, ξ is a Killing vector field for a cosymplectic manifold.

An almost contact metric manifold is called a λ -*Sasakian manifold* [10] if

$$(\nabla_X \varphi)Y = \lambda [g(X, Y)\xi - \eta(Y)X].$$

If $\lambda = 1$, a λ -*Sasakian manifold* is a Sasakian manifold.

The sectional curvature of a φ -section is called a φ -*sectional curvature*. A Sasakian (resp. Kenmotsu, cosymplectic, λ -Sasakian) manifold with constant φ -sectional curvature c is called a *Sasakian (resp. Kenmotsu, cosymplectic, λ -Sasakian) space form*; see [4, 11, 12, 10], respectively.

The notion of a generalized Sasakian space form was introduced by Alegre et al. in [1]. An almost contact metric manifold $M^{2n+1} = M(\varphi, \xi, \eta, g)$ whose curvature tensor satisfies

$$\begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &+ f_2\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned} \tag{2}$$

for certain differentiable functions f_1, f_2 , and f_3 on M^{2n+1} is called a generalized Sasakian space form [1]. The natural examples of generalized Sasakian space forms with constant functions are a Sasakian space form ($f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$) [4], a Kenmotsu space form ($f_1 = \frac{c-3}{4}, f_2 = f_3 = \frac{c+1}{4}$) [11], and a cosymplectic space form ($f_1 = f_2 = f_3 = \frac{c}{4}$) [12]. If M is a λ -Sasakian space form then $f_1 = \frac{c+3\lambda}{4}, f_2 = f_3 = \frac{c-\lambda}{4}$ [10].

Let M be an n -dimensional submanifold of a Riemannian manifold \widetilde{M} . We denote by $\widetilde{\nabla}, \nabla$ the Riemannian and induced Riemannian connections in \widetilde{M} and M , respectively, and let σ be the second fundamental form of the submanifold. The equation of Gauss is given by

$$\begin{aligned} \widetilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) \\ &-g(\sigma(X, W), \sigma(Y, Z)) + g(\sigma(X, Z), \sigma(Y, W)) \end{aligned} \tag{3}$$

for all vector fields X, Y, Z, W tangent to M , where \widetilde{R} and R denote the curvature tensors of the connections $\widetilde{\nabla}, \nabla$, respectively. The mean curvature vector field H is given by $H = \frac{1}{n} \text{trace}(\sigma)$. The submanifold M is *totally geodesic* in \widetilde{M} if $\sigma = 0$, and *minimal* if $H = 0$ [6].

Using (3), the Gauss equation for the submanifold M^n of a generalized Sasakian space form \widetilde{M}^{2m+1} is

$$\begin{aligned} \widetilde{R}(X, Y, Z, W) = & \\ & f_1\{g(Y, Z)X - g(X, Z)Y\} \\ & + f_2\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \\ & + g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)). \end{aligned} \tag{4}$$

A submanifold M of a generalized Sasakian space form \widetilde{M}^{2m+1} is called *anti-invariant* if and only if $\varphi(T_x M) \subset T_x^\perp M$ for all $x \in M$ [2]. For more information about anti-invariant submanifolds we refer to [17].

3. Semiparallel anti-invariant submanifolds of a generalized Sasakian space form

In this section, we give the main results of the paper.

For an n -dimensional submanifold M of a $(2n + 1)$ -dimensional Riemannian manifold \widetilde{M}^{2n+1} , it is known that the Laplacian $\Delta\sigma_{ij}^\alpha$ of σ_{ij}^α is defined by

$$\Delta\sigma_{ij}^\alpha = \sum_{i,j,k=1}^n \sigma_{ijkk}^\alpha. \tag{5}$$

Then

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^n \sum_{\alpha=n+1}^{2n+1} \sigma_{ij}^\alpha \sigma_{ijkk}^\alpha + \|\overline{\nabla}\sigma\|^2, \tag{6}$$

(see [17]), where

$$\|\sigma\|^2 = \sum_{i,j,k=1}^n \sum_{\alpha=n+1}^{2n+1} (\sigma_{ij}^\alpha)^2, \tag{7}$$

and

$$\|\overline{\nabla}\sigma\|^2 = \sum_{i,j,k=1}^n \sum_{\alpha=n+1}^{2n+1} (\sigma_{ijkk}^\alpha)^2 \tag{8}$$

are the square of the length of second and the third fundamental forms of M , respectively.

A simple calculation gives us the following proposition:

Proposition 1 *Let M be an n -dimensional minimal anti-invariant submanifold of a $(2n + 1)$ -dimensional generalized Sasakian space form \widetilde{M}^{2n+1} with ξ normal to M . Then we have*

$$\begin{aligned} \frac{1}{2}\Delta(\|\sigma\|^2) = & \|\overline{\nabla}\sigma\|^2 + (f_2 + nf_1)\|\sigma\|^2 \\ & - \left[\sum_{\alpha,\beta=n+1}^{2n+1} tr(A_\alpha \circ A_\beta)^2 + \|[A_\alpha, A_\beta]\|^2 \right]. \end{aligned} \tag{9}$$

Theorem 2 Let M be an n -dimensional minimal anti-invariant semiparallel submanifold of a $(2n + 1)$ -dimensional generalized Sasakian space form \widetilde{M}^{2n+1} with ξ normal to M . If

$$f_2 + nf_1 \leq 0,$$

then M is totally geodesic.

Proof Let $\{e_1, e_2, \dots, e_n, \xi, \varphi e_1, \varphi e_2, \dots, \varphi e_n\}$ be an orthonormal frame in \widetilde{M}^{2n+1} such that e_1, e_2, \dots, e_n are tangent to M . By definition, the semiparallelity of M , for $1 \leq k, l \leq n$, gives us

$$\overline{R}(e_l, e_k) \cdot \sigma = 0. \tag{10}$$

By (1), we can write

$$(\overline{R}(e_l, e_k) \cdot \sigma)(e_i, e_j) = (\overline{\nabla}_{e_l} \overline{\nabla}_{e_k} \sigma)(e_i, e_j) - (\overline{\nabla}_{e_k} \overline{\nabla}_{e_l} \sigma)(e_i, e_j) = 0, \tag{11}$$

where $1 \leq i, j, k, l \leq n$.

Hence, equation (6) turns into

$$\frac{1}{2} \Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^n g((\overline{\nabla}_{e_k} \overline{\nabla}_{e_k} \sigma)(e_i, e_j), \sigma(e_i, e_j)) + \|\overline{\nabla} \sigma\|^2. \tag{12}$$

Furthermore, using equations (5) and (6), we have

$$\frac{1}{2} \Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^n \sum_{\alpha=n+1}^{2n+1} \sigma_{ij}^\alpha (\overline{\nabla}_{e_i} \overline{\nabla}_{e_j} H^\alpha) + \|\overline{\nabla} \sigma\|^2. \tag{13}$$

Since M is minimal, equation (13) can be written as

$$\frac{1}{2} \Delta(\|\sigma\|^2) = \|\overline{\nabla} \sigma\|^2 \tag{14}$$

(see [18]). Comparing (9) and (14), we find

$$-(f_2 + nf_1) \|\sigma\|^2$$

$$+ \sum_{\alpha,\beta=n+1}^{2n+1} tr(A_\alpha \circ A_\beta)^2 + \|[A_\alpha, A_\beta]\|^2 = 0.$$

From the assumption, if

$$f_2 + nf_1 \leq 0,$$

then $tr(A_\alpha \circ A_\beta) = 0$. In particular, $\|A_\alpha\|^2 = tr(A_\alpha \circ A_\alpha) = 0$, and thus $A_\alpha = 0$, which means that $\sigma = 0$. Then M is totally geodesic. \square

Using Theorem 2, we have the following corollaries:

Corollary 3 [18] *Let M be an n -dimensional minimal anti-invariant semiparallel submanifold of a $(2n + 1)$ -dimensional Sasakian space form \widetilde{M}^{2n+1} with ξ normal to M . If*

$$n(c + 3) + c - 1 \leq 0,$$

then M is totally geodesic.

Corollary 4 *Let M be an n -dimensional minimal anti-invariant semiparallel submanifold of a $(2n + 1)$ -dimensional cosymplectic space form \widetilde{M}^{2n+1} with ξ normal to M . If*

$$c \leq 0,$$

then M is totally geodesic.

Corollary 5 [5] *Let M be an n -dimensional minimal anti-invariant semiparallel submanifold of a $(2n + 1)$ -dimensional λ -Sasakian space form \widetilde{M}^{2n+1} with ξ normal to M . If*

$$c - \lambda + n(c + 3\lambda) \leq 0,$$

then M is totally geodesic.

If M is an $(n + 1)$ -dimensional minimal anti-invariant submanifold of a $(2n + 1)$ -dimensional generalized Sasakian space form \widetilde{M}^{2n+1} with ξ tangent to M , then we have the following proposition:

Proposition 6 *Let M be an $(n + 1)$ -dimensional minimal anti-invariant submanifold of a $(2n + 1)$ -dimensional generalized Sasakian space form \widetilde{M}^{2n+1} with ξ tangent to M . Then we have*

$$\begin{aligned} \frac{1}{2}\Delta(\|\sigma\|^2) &= \|\overline{\nabla}\sigma\|^2 + (f_2 + (n + 1)f_1 - f_3)\|\sigma\|^2 \\ &- f_3 \sum_{i=1}^{n+1} \|\sigma(e_i, \xi)\|^2 - \left[\sum_{\alpha, \beta=n+2}^{2n+1} \text{tr}(A_\alpha \circ A_\beta)^2 + \|[A_\alpha, A_\beta]\|^2 \right]. \end{aligned} \tag{15}$$

Theorem 7 *Let M be an $(n + 1)$ -dimensional minimal anti-invariant semiparallel submanifold of $(2n + 1)$ -dimensional generalized Sasakian space form \widetilde{M}^{2n+1} with ξ tangent to M . If*

$$f_2 + (n + 1)f_1 - f_3 \leq 0$$

and

$$f_3 \geq 0,$$

then M is totally geodesic.

Proof Let $\{e_1, e_2, \dots, e_n, \xi, \varphi e_1, \varphi e_2, \dots, \varphi e_n\}$ be an orthonormal frame in \widetilde{M}^{2n+1} such that $e_1, e_2, \dots, e_n, \xi$ are tangent to M . Then for $1 \leq i, j \leq n + 1$ and $n + 2 \leq \alpha \leq 2n + 1$. Similar to the proof of Theorem 2, using the minimality condition, we obtain

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \|\overline{\nabla}\sigma\|^2. \tag{16}$$

Comparing (15) and (16) we find

$$\begin{aligned}
 & - (f_2 + (n + 1) f_1 - f_3) \|\sigma\|^2 + f_3 \sum_{i=1}^{n+1} \|\sigma(e_i, \xi)\|^2 \\
 & + \sum_{\alpha, \beta=n+2}^{2n+1} \text{tr}(A_\alpha \circ A_\beta)^2 + \|[A_\alpha, A_\beta]\|^2 = 0.
 \end{aligned}$$

From the assumption, if

$$f_2 + (n + 1) f_1 - f_3 \leq 0$$

and

$$f_3 \geq 0,$$

then $\text{tr}(A_\alpha \circ A_\beta) = 0$. Similar to the proof of Theorem 2, this gives us $\sigma = 0$. Then M is totally geodesic. \square

Using Theorem 7, we have the following corollaries:

Corollary 8 *Let M be an $(n + 1)$ -dimensional minimal anti-invariant semiparallel submanifold of a $(2n + 1)$ -dimensional Sasakian space form \widetilde{M}^{2n+1} with ξ tangent to M . If*

$$c \in (-\infty, -3] \cup [1, \infty),$$

then M is totally geodesic.

Corollary 9 *Let M be an $(n + 1)$ -dimensional minimal anti-invariant semiparallel submanifold of a $(2n + 1)$ -dimensional cosymplectic space form \widetilde{M}^{2n+1} with ξ tangent to M . If $c = 0$, then M is totally geodesic.*

Corollary 10 [15] *Let M be an $(n + 1)$ -dimensional minimal anti-invariant semiparallel submanifold of a $(2n + 1)$ -dimensional Kenmotsu space form \widetilde{M}^{2n+1} with ξ tangent to M . If $c \in [-1, 3]$, then M is totally geodesic.*

Corollary 11 *Let M be an $(n + 1)$ -dimensional minimal anti-invariant semiparallel submanifold of a $(2n + 1)$ -dimensional λ -Sasakian space form \widetilde{M}^{2n+1} with ξ tangent to M .*

i) If λ is a positive function on M and

$$c \in (-\infty, -3\lambda] \cup [\lambda, \infty)$$

or

ii) If λ is a negative function on M and

$$c \in [\lambda, -3\lambda],$$

then M is totally geodesic.

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