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A note on closed G_2-structures and 3-manifolds

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Abstract: This article shows that given any orientable 3-manifold X, the 7-manifold $T^*X \times \mathbb{R}$ admits a closed G_2-structure $\varphi = \mathrm{Re}\,\Omega - \omega \wedge dt$ where Ω is a certain complex-valued 3-form on T^*X; next, given any 2-dimensional submanifold S of X, the conormal bundle N^*S of S is a 3-dimensional submanifold of $T^*X \times \mathbb{R}$ such that $\varphi|_{N^*S} \equiv 0$. A corollary of the proof of this result is that $N^*S\times\mathbb{R}$ is a 4-dimensional submanifold of $T^*X\times\mathbb{R}$ such that $\varphi|_{N^*S\times\mathbb{R}} \equiv 0$.

1. Introduction

Berger's classification of the possible holonomy groups for a given Riemannian manifold includes the exceptional Lie group G_2 as the holonomy group of a 7-dimensional manifold. On a given 7-dimensional manifold with holonomy group a subgroup of G_2, there is a nondegenerate differential 3-form φ that is torsion-free; that is, $\nabla\varphi = 0$. This torsion-free condition is equivalent to φ being closed and coclosed. Much work has been done to study manifolds with G_2-holonomy, e.g., [5, 16], but the condition φ being coclosed is a very rigid condition. If we drop the coclosed condition, then we are studying manifolds with a closed G_2-structure. In particular, manifolds with closed G_2-structures were studied in [6, 8]; these papers focused predominantly on the metric defined by the nondegenerate closed 3-form φ. We shift our focus to the form φ itself, and in particular to the results that depend on φ being nondegenerate and closed.

Links between Calabi–Yau geometry and G_2 geometry were explored in the context of mirror symmetry by Akbulut and Salur [2]. Of course, the connections between symplectic geometry and Calabi–Yau geometry are obvious; moreover, connections between symplectic and contact geometry have been explored for centuries. Thus, it seems completely natural to try to find connections between contact geometry and G_2 geometry. The study of these relationships is an ongoing project that begins with the work by Arikan et al. [3], and the purpose of the current article (and many upcoming articles) is to continue this study by examining the geometry of closed G_2-structures as an analogue of symplectic geometry.

Treating symplectic geometry and G_2 geometry as being analogues of one another is not new. In [4, 14], vector cross products (on manifolds) are studied in a general setting; in particular, symplectic geometry is the geometry of 1-fold vector cross products, better known as almost complex structures, and G_2 geometry is the geometry of 2-fold vector cross products in dimension 7. Furthermore, in all cases, one can show that using the metric, there is a nondegenerate differential form of degree $k+1$ associated to a k-fold vector cross product.

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This yields the symplectic form associated to almost complex structures and the G_2 3-form φ associated to 2-fold vector cross products in dimension 7. Examples of manifolds with G_2 structures satisfying various conditions (including closed G_2 structures) are studied and classified in [7, 10, 11, 12, 13].

This article is based on 2 elementary results from symplectic geometry: 1) the cotangent bundle T^*X of any n-dimensional manifold X admits a symplectic form, and 2) the conormal bundle N^*S of any k-dimensional submanifold S of X with $k < n$ is a Lagrangian submanifold of T^*X. It is a well-known result that any 7-manifold that is spin admits a G_2-structure [6]. We show that for an orientable 3-manifold X, $T^*X \times \mathbb{R}$ is spin and hence admits a G_2-structure. Next, we calculate an explicit formula in coordinates for a G_2-structure φ on $T^*X \times \mathbb{R}$. Using this, we prove:

Theorem 1 *Let S be a 2-dimensional submanifold of X, and let N^*S denote the conormal bundle of S in T^*X. Let $i : N^*S \hookrightarrow (T^*X \times \mathbb{R}, \varphi)$ be the inclusion map. Then $i^*\varphi = 0$.*

2. G_2 geometry

We begin with a description of G_2 geometry in flat space, and then we consider this geometry on 7-manifolds.

Consider the octonions $\mathbb{O}$ as an 8-dimensional real vector space. This becomes a normed algebra when equipped with the standard Euclidean inner product on $\mathbb{R}^8$. Furthermore, there is a cross product operation given by $u \times v = \mathrm{Im}(\bar{v}u)$ where $\bar{v}$ is the conjugate of v for $u, v \in \mathbb{O}$. This is an alternating form on $\mathrm{Im}\,\mathbb{O}$ since, for any $u \in \mathrm{Im}\,\mathbb{O}$, $u^2 \in \mathrm{Re}\,\mathbb{O}$. We now define a 3-form on $\mathrm{Im}\,\mathbb{O}$ by $\varphi(u, v, w) = \langle u \times v, w\rangle$. In terms of the standard orthonormal basis $\{e_1, \dots, e_7\}$ of $\mathrm{Im}\,\mathbb{O}$, $\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$, where $e^{ijk} = e^i \wedge e^j \wedge e^k$. Under the isomorphism $\mathbb{R}^7 \simeq \mathrm{Im}\,\mathbb{O}$, with coordinates on $\mathbb{R}^7$ given by $(x^1, \dots, x^7)$, we have $\varphi_0 = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356}$.

Definition 2.1 *Let M be a 7-dimensional manifold. M has a G_2-*structure *if there is a smooth 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$, the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$. A 7-manifold M has a* closed *G_2-structure if the 3-form φ is also closed,* $\mathrm{d}\varphi = 0$.

Equivalently, a smooth 7-dimensional manifold M has a G_2-structure if its tangent frame bundle reduces to a G_2-bundle. For a manifold with G_2-structure φ, there is a natural Riemannian metric and orientation induced by φ given by $(Y \lrcorner \varphi) \wedge (\tilde{Y} \lrcorner \varphi) \wedge \varphi = \langle Y, \tilde{Y}\rangle_\varphi dvol_M$. In particular, the 3-form φ is nondegenerate.

Remark 1 *One defines a G_2-manifold as a smooth 7-manifold with torsion-free G_2-structure, i.e. $\nabla\varphi = 0$ where ∇ is the Levi-Civita connection of the metric $\langle \cdot, \cdot \rangle_\varphi$. This means that (M, φ) has a holonomy group contained in G_2 and that $d\varphi = \mathrm{d}{*}\varphi = 0$. For our purposes, we do not assume that $\mathrm{d} * \varphi = 0$, so that (M, φ) will be a manifold with closed G_2-structure as defined above.*

3. The cotangent bundle

This material can be found in most introductions to symplectic geometry or topology, e.g., [9, 18].

Let X be any n-dimensional manifold (all manifolds under consideration are assumed to be C^∞); let $(U; x_1, \dots, x_n)$ be a coordinate chart for X, so that for each $1 \le i \le n$, $x_i : U \to \mathbb{R}$. For any point $x \in X$, the differentials $(\mathrm{d}x_1)_x, \dots, (\mathrm{d}x_n)_x$ form a basis for the cotangent space T_x^*X at x; hence, for any covector

$\xi \in T_x^*X$, $\xi = \sum_i \xi_i (\mathrm{d}x_i)_x$ with $\xi_i \in \mathbb{R}$. This yields a coordinate chart $(T^*U; x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ for the cotangent bundle of X associated to the coordinates $x_1, \dots, x_n$ on X.

Using these coordinates, the so-called tautological 1-form on T^*U is defined by $\alpha := \sum_i \xi_i \mathrm{d}x_i$. This definition is invariant under changes of coordinates: let $(T^*V; y_1, \dots, y_n, \eta_1, \dots, \eta_n)$ be an overlapping coordinate chart; then:

$$\eta_j = \sum_i \xi_i \frac{\partial x_i}{\partial y_j}$$

and

$$\mathrm{d}x_i = \sum_j \frac{\partial x_i}{\partial y_j} \mathrm{d}y_j.$$

Therefore, in the overlap,

$$\sum_i \xi_i \mathrm{d}x_i = \sum_i \xi_i (\sum_j \frac{\partial x_i}{\partial y_j} \mathrm{d}y_j) = \sum_{i,j} \xi_i \frac{\partial x_i}{\partial y_j} \mathrm{d}y_j = \sum_j \eta_j \mathrm{d}y_j.$$

Now define a 2-form by $\omega := -\mathrm{d}\alpha = \sum_i \mathrm{d}x_i \wedge \mathrm{d}\xi_i$; ω is also independent of the choice of coordinates, so ω is a symplectic form on the cotangent bundle T^*X, called the canonical symplectic form.

Now let S be any k-dimensional submanifold of X with $k < n$. Recall that the conormal space N_x^*S at $x \in S$ is given by

$$N_x^*S := \{\xi \in T^*X : \xi(v) = 0 \text{ for all } v \in T_xS\},$$

and the conormal bundle N^*S of S by

$$N^*S = \{(x, \xi) \in T^*X : x \in S, \xi \in N_x^*S\}.$$

Let $(U; x_1, \dots, x_n)$ be a coordinate chart on X such that $S \cap U$ is given by the equations

$$x_{k+1} = \cdots = x_n = 0.$$

In the associated cotangent bundle coordinate chart $(T^*U; x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, any $\xi \in N^*S \cap T^*U$ is given by

$$\xi = \sum_{i=1}^{k} \xi_i \mathrm{d}x_i$$

since $x_{k+1} = \cdots = x_n = 0$ on S; furthermore, since $\xi \in N_x^*S$ and T_xS is spanned by

$$(\frac{\partial}{\partial x_1})_x, \dots, (\frac{\partial}{\partial x_k})_x,$$

we find that

$$0 = \xi((\frac{\partial}{\partial x_i})_x) = \xi_i, \text{ for all } 1 \le i \le k.$$

Hence, $N^*S \cap T^*U$ is described by the equations $x_{k+1} = \cdots = x_n = \xi_1 = \cdots = \xi_k = 0$, so N^*S is an n-dimensional submanifold of T^*X; furthermore, $\alpha = \sum_{i=1}^n \xi_i \mathrm{d}x_i$ when restricted to N^*S is zero, so $\omega|_{N^*S} \equiv 0$. Thus, N^*S is a Lagrangian submanifold of T^*X.

4. G_2-structures on $T^*X \times \mathbb{R}$

References for this section are [1, 6, 17, 19].

Recall that for each $n \geq 3$, the Lie group $SO(n)$ is connected, and it has a double-covering map $\iota : Spin(n) \to SO(n)$ where the Lie group $Spin(n)$ is a compact, connected, simply connected Lie group. An oriented Riemannian manifold (X, g) has $SO(n)$ as its structure group on the tangent bundle. A spin structure on (X, g) is a $Spin(n)$-principal bundle over X, together with a bundle map $\pi : P_{Spin(n)}X \to P_{SO(n)}X$ such that $\pi(pg) = \pi(p)\iota(g)$ for $p \in P_{Spin(n)}X, g \in Spin(n)$. A spin manifold is an oriented Riemannian manifold with a spin structure on its tangent bundle.

Theorem 4.1 *For any oriented Riemannian 3-manifold X, $T^*X \times \mathbb{R}$ has a G_2-structure.*

Proof Let X be an oriented Riemannian 3-manifold. We know that every orientable 3-manifold is parallelizable, and since a framing on a bundle gives a spin structure, X is a spin manifold.

Now we check that T^*X is itself a spin manifold for a spin manifold X. Recall that T^*X carries a canonical 1-form $\sum p_i dx^i$, where p_i are coordinates in T^*X and x^i are coordinates on X. One can define the map $T^*X \times Spin(3) \to T^*X$ by $(p_i, g) \mapsto p_i g|_{x^i} = p_i(v_i g)$ for $v_i \in T_{x^i}X$; it is well defined because of the spin structure on X. This implies that T^*X has an induced spin structure from X. For another way to see this, one can find a bundle map of the principal bundles $b : P_{Spin}T^*X \to P_{Spin}TX$ over X because the map $\mathfrak{F} : \mathcal{E}(T^*X) \to \mathcal{E}(TX)$ is linear, where $\mathcal{E}$ is the set of sections of the bundle. This gives a map $P_{Spin(3)}X \to P_{SO(3)}X$ for which the following diagram commutes:

$$\begin{array}{ccc} P_{Spin(3)}T^*X & \to & P_{SO(3)}T^*X \\ \downarrow & & \downarrow \\ P_{Spin(3)}TX & \to & P_{SO(3)}TX \ . \end{array}$$

Let $E(\xi)$ be the total space of ξ with base space X. A principal $Spin(3)$-bundle $E(\xi) \times Spin(3) \to X$ induces a principal $Spin(4)$-bundle $E(\xi) \times Spin(4) \to X$, which is itself induced from $Spin(3)(\simeq SU(2) \simeq S^3) \to Spin(4)(\simeq SU(2) \times SU(2))$; hence, a spin structure on ξ gives a spin structure on $\xi \oplus \varepsilon$, so $T^*X \times \mathbb{R}$ admits a spin structure. By [1, p. 4], we conclude that $T^*X \times \mathbb{R}$ is a smooth 7-dimensional manifold with a G_2-structure. □

Let X be an orientable 3-dimensional manifold with symplectic cotangent bundle $(M = T^*X, \omega := -\mathrm{d}\alpha)$ where α is the tautological 1-form on M. If $x_1, x_2, x_3, \xi_1, \xi_2, \xi_3$ are the standard cotangent bundle coordinates associated to the coordinates x_1, x_2, x_3 on X, define a complex-valued $(3,0)$-form on M by

$$\Omega = (\mathrm{d}x_1 + i\mathrm{d}\xi_1) \wedge (\mathrm{d}x_2 + i\mathrm{d}\xi_2) \wedge (\mathrm{d}x_3 + i\mathrm{d}\xi_3).$$

Consider $M \times \mathbb{R}$. This is a 7-manifold with coordinates $x_1, x_2, x_3, \xi_1, \xi_2, \xi_3, t$ where t is the $\mathbb{R}$ coordinate. Finally, define $\varphi = \mathrm{Re}(\Omega) + \omega \wedge \mathrm{d}t$.

We show that this defines a G_2 structure on $M \times \mathbb{R}$; that is, we exhibit an isomorphism of $(T_{(p,t)}(M \times \mathbb{R}), \varphi_{(p,t)})$ with $(\mathbb{R}^7, \varphi_0)$. We first calculate

$$\Omega = (\mathrm{d}x_1 + i\mathrm{d}\xi_1) \wedge ((\mathrm{d}x_2 \wedge \mathrm{d}x_3 - \mathrm{d}\xi_2 \wedge \mathrm{d}\xi_3) + i(\mathrm{d}x_2 \wedge \mathrm{d}\xi_3 - \mathrm{d}x_3 \wedge \mathrm{d}\xi_2))$$

$$= (\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - \mathrm{d}x_1 \wedge \mathrm{d}\xi_2 \wedge \mathrm{d}\xi_3 + \mathrm{d}x_2 \wedge \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_3 - \mathrm{d}x_3 \wedge \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2)$$

$$+i(\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\xi_3 - \mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\xi_2 + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\xi_1 - \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 \wedge \mathrm{d}\xi_3),$$

so we find that

$$\begin{aligned}\varphi &= \mathrm{Re}\,\Omega + \omega \wedge \mathrm{d}t \\ &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - \mathrm{d}x_1 \wedge \mathrm{d}\xi_2 \wedge \mathrm{d}\xi_3 + \mathrm{d}x_2 \wedge \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_3 - \mathrm{d}x_3 \wedge \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 \\ &\quad + \mathrm{d}x_1 \wedge \mathrm{d}\xi_1 \wedge \mathrm{d}t + \mathrm{d}x_2 \wedge \mathrm{d}\xi_2 \wedge \mathrm{d}t + \mathrm{d}x_3 \wedge \mathrm{d}\xi_3 \wedge \mathrm{d}t.\end{aligned}$$

Now, for $p = (x, \xi) \in T^*U$, let

$$\{(\frac{\partial}{\partial x_1})_p, (\frac{\partial}{\partial x_2})_p, (\frac{\partial}{\partial x_3})_p, (\frac{\partial}{\partial \xi_1})_p, (\frac{\partial}{\partial \xi_2})_p, (\frac{\partial}{\partial \xi_3})_p\}$$

be the basis for the tangent space T_pM at p with respect to the cotangent coordinates on M; let $(\frac{\partial}{\partial t})_q$ be the basis for $T_q\mathbb{R}$ with respect to the coordinate t on $\mathbb{R}$. Let $(x_1, \ldots, x_7)$ be the standard Euclidean coordinates on $\mathbb{R}^7$. Define an isomorphism of the tangent vector spaces by $\Phi : T_0\mathbb{R}^7 \to T_{(p,q)}(M \times \mathbb{R})$ by

$$\Phi(\frac{\partial}{\partial x_1})_0 = -(\frac{\partial}{\partial x_3})_p,\ \Phi(\frac{\partial}{\partial x_2})_0 = (\frac{\partial}{\partial x_2})_p,\ \Phi(\frac{\partial}{\partial x_3})_0 = (\frac{\partial}{\partial x_1})_p$$

$$\Phi(\frac{\partial}{\partial x_4})_0 = (\frac{\partial}{\partial \xi_1})_p,\ \Phi(\frac{\partial}{\partial x_5})_0 = (\frac{\partial}{\partial \xi_2})_p,\ \Phi(\frac{\partial}{\partial x_6})_0 = (\frac{\partial}{\partial \xi_3})_p$$

$$\Phi(\frac{\partial}{\partial x_7})_0 = -(\frac{\partial}{\partial t})_q.$$

This induces an isomorphism of the cotangent vector spaces $\Phi^* : T^*_{(p,q)}(M \times \mathbb{R}) \to T^*_0\mathbb{R}^7$ where

$$\Phi^*(\mathrm{d}x_1)_p = (\mathrm{d}x_3)_0,\ \Phi^*(\mathrm{d}x_2)_p = (\mathrm{d}x_2)_0,\ \Phi^*(\mathrm{d}x_3)_p = -(\mathrm{d}x_1)_0$$

$$\Phi^*(\mathrm{d}\xi_1)_p = (\mathrm{d}x_4)_0,\ \Phi^*(\mathrm{d}\xi_2)_p = (\mathrm{d}x_5)_0,\ \Phi^*(\mathrm{d}\xi_3)_p = (\mathrm{d}x_6)_0$$

$$\Phi^*(\mathrm{d}t)_t = -(\mathrm{d}x_7)_0.$$

Then

$$\begin{aligned}\Phi^*\varphi &= \Phi^*\mathrm{d}x_1 \wedge \Phi^*\mathrm{d}x_2 \wedge \Phi^*\mathrm{d}x_3 - \Phi^*\mathrm{d}x_1 \wedge \Phi^*\mathrm{d}\xi_2 \wedge \Phi^*\mathrm{d}\xi_3 + \Phi^*\mathrm{d}x_2 \wedge \Phi^*\mathrm{d}\xi_1 \wedge \Phi^*\mathrm{d}\xi_3 - \Phi^*\mathrm{d}x_3 \wedge \Phi^*\mathrm{d}\xi_1 \wedge \Phi^*\mathrm{d}\xi_2 \\ &\quad + \Phi^*\mathrm{d}x_1 \wedge \Phi^*\mathrm{d}\xi_1 \wedge \Phi^*\mathrm{d}t + \Phi^*\mathrm{d}x_2 \wedge \Phi^*\mathrm{d}\xi_2 \wedge \Phi^*\mathrm{d}t + \Phi^*\mathrm{d}x_3 \wedge \Phi^*\mathrm{d}\xi_3 \wedge \Phi^*\mathrm{d}t \\ &= \mathrm{d}x_3 \wedge \mathrm{d}x_2 \wedge (-\mathrm{d}x_1) - \mathrm{d}x_3 \wedge \mathrm{d}x_5 \wedge \mathrm{d}x_6 + \mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_6 - (-\mathrm{d}x_1) \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 \\ &\quad + \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge -(\mathrm{d}x_7) + \mathrm{d}x_2 \wedge \mathrm{d}x_5 \wedge (-\mathrm{d}x_7) + (-\mathrm{d}x_1) \wedge \mathrm{d}x_6 \wedge (-\mathrm{d}x_7) \\ &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 + \mathrm{d}x_1 \wedge \mathrm{d}x_6 \wedge \mathrm{d}x_7 + \mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_6 \\ &\quad - \mathrm{d}x_2 \wedge \mathrm{d}x_5 \wedge \mathrm{d}x_7 - \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_7 - \mathrm{d}x_3 \wedge \mathrm{d}x_5 \wedge \mathrm{d}x_6 = \varphi_0.\end{aligned}$$

In order to show that φ is independent of the choice of coordinates, it is enough prove that our coordinate definition of Ω on T^*X is independent of the choice of coordinates on T^*X. This calculation follows as in [15]. Let L be an oriented 3-dimensional subspace of T^*X. Let $\{f_1, f_2, f_3\}$ be any oriented linearly independent subset of L, let $\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}$ denote the standard basis for $\mathbb{R}^6$, and let A be the map defined by

$e_j \mapsto f_j$ and $Je_j \mapsto Jf_j$. Now $A \in GL(3;\mathbb{C})$; that is, A is complex linear and $A(e_1 \wedge e_2 \wedge e_3) = f_1 \wedge f_2 \wedge f_3$. Let $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$ be an oriented orthonormal basis for L, and let B be the map defined by $\tilde{f}_j \mapsto f_j$. Then $f_1 \wedge f_2 \wedge f_3 = (detB)\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3$. Since $\Omega(A(e_1 \wedge e_2 \wedge e_3)) = det_{\mathbb{C}}A$ and $\Omega = \operatorname{Re}\Omega + i \operatorname{Im}\Omega$, we have

$$(detB)[\operatorname{Re}\Omega(\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3)] = \operatorname{Re}(det_{\mathbb{C}}A)$$

and

$$(detB)[\operatorname{Im}\Omega(\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3)] = \operatorname{Im}(det_{\mathbb{C}}A).$$

Hence,

$$\begin{aligned}
&(\operatorname{Re}\Omega(f_1 \wedge f_2 \wedge f_3))^2 + (\operatorname{Im}\Omega(f_1 \wedge f_2 \wedge f_3))^2 \\
&= |det_{\mathbb{C}}A|^2 = det_{\mathbb{R}}A = vol(A(e_1 \wedge Je_1 \wedge e_2 \wedge Je_2 \wedge e_3 \wedge Je_3)) \\
&= vol(A(e_1 \wedge e_2 \wedge e_3 \wedge Je_1 \wedge Je_2 \wedge Je_3)) = vol(f_1 \wedge f_2 \wedge f_3 \wedge Jf_1 \wedge Jf_2 \wedge Jf_3) \\
&= (detB)^2 vol(\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3 \wedge J\tilde{f}_1 \wedge J\tilde{f}_2 \wedge J\tilde{f}_3).
\end{aligned}$$

5. The conormal bundle

Let X be a 3-dimensional manifold as in the previous section. Then $T^*X \times \mathbb{R}$ admits the G_2-structure $\varphi = \operatorname{Re}\Omega + \omega \wedge \mathrm{d}t$. Let S be a 2-dimensional submanifold of X, and let (U, x_1, x_2, x_3) be a coordinate chart on X such that $S \cap U$ is given by the equation $x_3 = 0$. For the associated cotangent coordinate chart $(T^*U, x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)$, any $\xi \in N^*S \cap T^*U$ is given by

$$\xi = \xi_1 \mathrm{d}x_1 + \xi_2 \mathrm{d}x_2$$

since $x_3 = 0$ on S; furthermore, since $\xi \in N^*_x S$ and $T_x S$ is spanned by

$$(\frac{\partial}{\partial x_1})_x, (\frac{\partial}{\partial x_2})_x,$$

we find that

$$0 = \xi((\frac{\partial}{\partial x_1})_x) = \xi_1$$

and

$$0 = \xi((\frac{\partial}{\partial x_2})_x) = \xi_2.$$

Hence, $N^*S \cap T^*U$ is described by the equations $x_3 = \xi_1 = \xi_2 = 0$. Thus, N^*S is a 3-dimensional submanifold of T^*X.

Proposition 5.1 *Let $i : N^*S \hookrightarrow (T^*X \times \mathbb{R}, \varphi)$ be the inclusion. Then $i^*\varphi = 0$.*

Proof In this case, $N^*S \cap (T^*U \times \mathbb{R})$ is given by the equations $x_3 = \xi_1 = \xi_2 = t = 0$ where t is the $\mathbb{R}$ coordinate. Recall that in this coordinate system, we have

$$\begin{aligned}
\varphi = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - \mathrm{d}x_1 \wedge \mathrm{d}\xi_2 \wedge \mathrm{d}\xi_3 + \mathrm{d}x_2 \wedge \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_3 - \mathrm{d}x_3 \wedge \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 \\
+ \mathrm{d}x_1 \wedge \mathrm{d}\xi_1 \wedge \mathrm{d}t + \mathrm{d}x_2 \wedge \mathrm{d}\xi_2 \wedge \mathrm{d}t + \mathrm{d}x_3 \wedge \mathrm{d}\xi_3 \wedge \mathrm{d}t.
\end{aligned}$$

Hence, for $p \in N^*S$, we have $(i^*\varphi)_p = \varphi_p|_{T_p(N^*S)} = 0$ since every term contains a factor that on N^*S is zero. □

Corollary 5.2 *Let* $i : N^*S \times \mathbb{R} \hookrightarrow (T^*X \times \mathbb{R}, \varphi)$ *be the inclusion. Then* $i^*\varphi = 0$.

Proof Note that $(N^*S \times \mathbb{R})$ is a 4-dimensional submanifold of T^*X, given by the equations $x_3 = \xi_1 = \xi_2 = 0$ on $(N^*S \times \mathbb{R}) \cap (T^*U \times \mathbb{R})$. □

Note that in contrast to the symplectic case, S must be 2-dimensional; if S is 1-dimensional, then $\varphi|_{N^*S} \neq 0$.

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