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On Joachimsthal's theorems in Riemann–Otsuki space $R - O_3$

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Abstract: In this paper we study the Joachimsthal theorem in Riemann–Otsuki space

Key words: Riemann–Otsuki spaces, Frenet formula, curvature strip

1. Introduction

The basis of the theory of Otsuki spaces was laid down by Otsuki and Moor. The metric used determined that the observed space is of Weyl–Otsuki or Riemann–Otsuki kind. Djerdji studied this subject in [2,3,4]. He observed the Frenet formula of the space considering the covariant and contravariant part of the connection; he also obtained auto-parallel curves of the Riemann–Otsuki space and, in another paper, he studied the subspaces of the relevant space.

Yıldırım and Bektaş gave the general properties of Bertrand curves and their characterizations in Riemann–Otsuki space in [21]. Later, in [11], Küçükarslan and Yıldırım studied the Mannheim partner curves in Riemann–Otsuki space and obtained some new characterizations for this curve.

The higher curvature of a curve in E^m was studied by Gluck [7] and by Sabuncuoğlu and Hacısalihoğlu [15]. The higher curvature of a strip in E^m was calculated by Sabuncuoğlu and Hacısalihoğlu [16].

Additionally, in [10], the higher curvatures of a strip in E^n were studied. The behavior of curvature lines near a principal cycle common to 2 orthogonal surfaces, as a complement of Joachimsthal theorem, was studied in [6].

Joachimsthal's theorems in Euclidean spaces E^m were given by Sabuncuoğlu [17].

In [9], cylindrical helix strips were investigated and a new definition and characterization of cylindrical helix strips were given. In addition the authors obtained a new characterization by using some characterizations of a general helix and the Terquem theorem (a Joachimsthal theorem for constant distances between 2 surfaces).

The theory of strips and the Joachimsthal theorem in 3-dimensional Lorentzian space were studied in [19].

Furthermore, the Joachimsthal theorem in n-dimensional Lorentzian space was generalized and investigated when the strip is time-like and space-like by Tutar and Sener [20]. Joachimsthal's theorems in semi-Euclidean spaces E_v^{n+1} were studied in [1].

In the present paper, first we give a short view of the basis of Riemann–Otsuki space, and then we study the Joachimsthal theorem in Riemann–Otsuki space.

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2. Riemann–Otsuki spaces and differentials

In the theory of affine or metrical points, the absolute derivation of the covariant and the contravariant coefficient is performed with the same connection coefficient. Otsuki [14] introduced a new connection type (general connection) in which this does not hold [8]. The notion of general connection extends the notion of an affine connection and is related to the almost product manifolds, web theory, and the theory of lightlike submanifolds [18]. As another example, one may focus on the modern theory of Finsler spaces based on the study of connection in fiber bundles [13].

In all Otsuki spaces, we have with respect to the local coordinates x^i of an n-dimensional differentiable manifold and a priori given tensor P^i_j such that det $\left\|P^i_j\right\| \neq 0$ holds and inverse tensor Q^i_j exists so that $P^i_j Q^j_r = \delta^i_r$.

In the metric Otsuki spaces the metric tensor g_{ij} (det $\|g_{ij}\| \neq 0$) is given so that in the W-O$_n$ Weyl-Otsuki space $\nabla_k g_{ij} = \gamma_k g_{ij}$, but in the R-O$_n$ (Riemann-Otsuki space)

$$\nabla_k g_{ij} = 0 \tag{2.1}$$

holds. In Otsuki spaces the covariant differential of the tensor T^i_j is defined by

$$DT^i_j = P^i_a P^j_b \overline{D} T^a_b = P^i_a P^j_b \left(\partial_k T^a_b +' \Gamma^a_{rk} T^r_b -'' \Gamma^r_{bk} T^a_r\right) dx^k. \tag{2.2}$$

The Leibnitz formula does not hold for this differential. The differential $\overline{D}$ is the basic covariant differential. The different coefficients of the connection are the characteristic of the Otsuki spaces, and here are

$$\delta^i_j\,|_k \; =' \Gamma^i_{jk} -'' \Gamma^i_{jk} \neq 0. \tag{2.3}$$

The coefficient of the connection $''\Gamma^i_{jk}$ was determined from the relation (2.1) and the coefficients of connection $'\Gamma^i_{jk}$ are obtained from

$$\partial_k P^i_j +'' \Gamma^i_{ak} P^a_j - P^{i'}_a \Gamma^a_{jk} = 0. \tag{2.4}$$

This relation is known as Otsuki's relation.

In Otsuki spaces it is possible to determine the covariant differentials D and $\overline{D}$ with respect only to the co-resp. contravariant part of the connection. Thus,

$$'\overline{D} T^i_j =' \nabla_k T^i_j dx^k = \left(\partial_k T^i_j +' \Gamma^i_{rk} T^r_j -' \Gamma^r_{jk} T^i_r\right) dx^k \tag{2.5}$$

holds. For this basic covariant differential the Leibnitz formula holds.The basic covariant differential $''\overline{D}$ can be defined in the same way.

It is characteristic that the basic covariant differential $'\overline{D}$ is identical in the case of contravariant indices with the basic covariant differential $\overline{D}$; similarly, in the case of covariant indices, the basic covariant differential $''\overline{D}$ is identical to the basic covariant differential $\overline{D}$.

In the following, we shall use the relations

$$'\overline{D}g_{ij} = dg_{ij} - \left({}'\Gamma^r_{ik}g_{rj} + {}'\Gamma^r_{jk}g_{ir}\right)dx^k, \tag{2.6}$$

$$''\overline{D}g_{ij} = dg_{ij} - \left({}''\Gamma^r_{ik}g_{rj} + {}''\Gamma^r_{jk}g_{ir}\right)dx^k = 0, \tag{2.7}$$

$$'\overline{D}g^{ra} = -g^{ia}g^{jr}\left({}'\overline{D}g_{ij}\right), \tag{2.8}$$

$$''\overline{D}g^{ra} = 0. \tag{2.9}$$

3. The Frenet formula with respect to the contravariant componenets of the vectors

Let $C: \ x^3(s)$ be the curve of an $R-O_3$ space and $v_l, \ l=0,1,2$ be mutually orthogonal unit vectors that satisfy the relation

$$D_{v_l} \ j = P^j_i(-\kappa_l v^i_{l-1} + \kappa_{l+1}v^i_{l+1}) + v^q_l Q^b_q D\delta^j_b, \tag{3.1}$$

If v_4 is orthogonal to all before and $\kappa_0 = \kappa_3 = 0$ holds, then vector v_3 also satisfies Eq. (3.1).

Thus, we obtain the Frenet trihedron as follows:

$$\begin{aligned} D_{v_0} \ j &= P^j_i(\kappa_1 v^i_1) + v^q_0 Q^b_q D\delta^j_b \\ D_{v_1} \ j &= P^j_i(-\kappa_1 v^i_0 + \kappa_2 v^i_2) + v^q_1 Q^b_q D\delta^j_b \\ D_{v_2} \ j &= P^j_i(-\kappa_2 v^i_1) + v^q_2 Q^b_q D\delta^j_b \end{aligned}$$

or

$$\begin{bmatrix} D_{v_0} \ j \\ D_{v_1} \ j \\ D_{v_2} \ j \end{bmatrix} = P^j_i \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & \kappa_2 \\ 0 & -\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} v^i_0 \\ v^i_1 \\ v^i_2 \end{bmatrix} + \begin{bmatrix} v^q_0 Q^b_q D\delta^j_b \\ v^q_1 Q^b_q D\delta^j_b \\ v^q_2 Q^b_q D\delta^j_b \end{bmatrix}$$

where $\kappa_1 = \kappa_1(s)$ and $\kappa_2 = \kappa_2(s)$ [3].

Remark 3.1 One can see that the difference between (3.1) and the known formula of $\mathbb{R}^3$ is the covariant differential of the Kronecker-δ. In such a case, but not only such special ones where this differential is zero, formula (3.1) reduces to the known Frenet formula multiplied by the tensor P^j_i. Hence, $\mathbb{R}^3$ can be given as an example of $R-O_3$ under special circumstances [3].

The main goal of this paper is to introduce the Joachimsthal theorem for surfaces of 3-dimensional Riemann–Otsuki space $R-O_3$. To this end, it is convenient to recall some basic definitions. See [5] for further information.

Let M denote a regular orientable surface of $R-O_3$; for each $p \in M$ there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(M)$ such that

$$\begin{aligned} dN_p(e_1) &= -k_1 e_1 \\ dN_p(e_2) &= -k_2 e_2. \end{aligned}$$

Moreover, k_1 and k_2 $(k_1 \geq k_2)$ are the maximum and minimum of the second fundamental form II_p, which we call extreme values of the normal curvature at p.

Definition 1 *The maximum normal curvature* k_1 *and the minimum normal curvature* k_2 *are called principal curvatures at* p; *the corresponding directions, that is, the directions given by the eigenvectors* e_1, e_2, *are called principal directions at* p.

Definition 2 *Suppose* α *is a regular connected curve on* M ; *if the tangent of this curve at each point is in a principal direction at that point, it is called a line of curvature. In other words, a principal curvature line is a regular curve (parametrized by arc length parameter* s)

$$\gamma : (a, b) \to M$$

such that for all $s \in (a, b)$ *we have that* γ *is a principal direction* [6].

The knowledge of the principal curvatures at p allows us to compute the normal curvature along a given direction of $T_p(M)$. In fact, let $v \in T_p(M)$ with $|v|$=1. Since e_1 and e_2 form an orthonormal basis of $T_p(M)$,

then we have

$$v = e_1 \cos\theta + e_2 \sin\theta,$$

where θ is the angle from e_1 to v in the orientation of $T_p(M)$. Hence, the normal curvature along v is given by

$$\begin{aligned} k_n &= II_p(v) = -g_{ij}(dN_p(v), v) \\ &= k_1 \cos^2\theta + k_2 \sin^2\theta. \end{aligned}$$

This equation is known as the Euler formula, and, as is seen, it is just the expression of the second fundamental form in the basis $\{e_1, e_2\}$.

Now let us give the classical statement of the Joachimsthal theorem.

Theorem 3.1 (Joachimsthal theorem) *Suppose that* M_1 *and* M_2 *intersect along a regular curve* α *and make an angle* $\theta(p)$, $p \in \alpha$. *Then* α *is a line of curvature of* M_1 *if and only if it is a curvature line of* M_2.

In this paper, we will define strips for Riemann–Otsuki space $R - O_3$ *and prove this well-known theorem with their help.*

Definition 4 *Suppose* M *is a surface in* $R - O_3$ *and* α *is a curve on* M. *The points of* α *and the geometrical shape formed by the tangents vectors to the surface at these points are called surface strips* $(\alpha\ ,\ M)$ *of* M *along* α. *Hence:*

$$g_{ij}(N\mid_{\alpha(t)}, \alpha^{'}(t)) = 0.$$

Definition 5 *Let* α *be a curve in* $M \subset R - O_3$. *If* $\frac{dx^i}{ds} = v_0^i$ *and* v_1^i , v_2^i *are mutually orthogonal unit vectors satisfying relation (3.1). The orthonormal vector field system* $\left\{v_0^i, v_1^i, v_2^i\right\}$ *is then called Frenet vectors on the stripe and is equivalent to:*

$$\begin{aligned} D_{v_0}\ i &= t_{11} v_0^i + t_{12} v_1^i + t_{13} v_2^i \\ D_{v_1}\ i &= t_{21} v_0^i + t_{22} v_1^i + t_{23} v_2^i \\ D_{v_2}\ i &= t_{31} v_0^i + t_{32} v_1^i + t_{33} v_2^i. \end{aligned}$$

Furthermore,

$$\begin{aligned} g\left(v_o^i, v_o^i\right) &= g\left(v_1^i, v_1^i\right) = g\left(v_2^i, v_2^i\right) = 1, \\ g\left(v_o^i, v_1^i\right) &= g\left(v_0^i, v_2^i\right) = g\left(v_1^i, v_2^i\right) = 0. \end{aligned}$$

Differentiating these equation according to v_0^i, *we can express*

$$\begin{bmatrix} D_{v_0}\ i \\ D_{v_1}\ i \\ D_{v_2}\ i \end{bmatrix} = P_i^j \begin{bmatrix} 0 & t_{12} & t_{13} \\ -t_{12} & 0 & t_{23} \\ -t_{13} & -t_{23} & 0 \end{bmatrix} \begin{bmatrix} v_0^i \\ v_1^i \\ v_2^i \end{bmatrix} + Q_q^b D\delta_b^j \begin{bmatrix} v_0^q \\ v_1^q \\ v_2^q \end{bmatrix}.$$

One may take $t_{12} = c,\ t_{13} = -b,\ t_{23} = a$. *These values are called differential invariants of* $(\alpha\ ,\ M)$. *Some authors use* $a = t_r,\ -b = k_n$, *and* $c = k_g$ *notations, called geodesic torsion, normal curvature, and geodesic curvature of the strip, respectively.*

Definition 6 *A strip* $(\alpha\ ,\ M)$ *in* $R - O_3$ *is called a curvature stripe if its geodesic torsion, i.e.* $= a$, *is zero.*

Theorem 7 *i) Let* $M_1 \subset R - O_3$ *and* $M_2 \subset R - O_3$ *be 2 regular and oriented surfaces such that* $M_1 \cap M_2 = \alpha$; *if* (α, M_1) *and* (α, M_2) *are curvature strips, then the angle between* (α, M_1) *and* (α, M_2) *is constant.*

ii) If the curvature strip rotates under a θ *angle, another curvature strip is obtained.*

iii) If the angle between (α, M_1) *and* (α, M_2) *is constant, then they have same geodesic torsion.*

Proof *i)* Suppose that the surfaces M_1 and M_2 intersect along α. Take v_{12}^j and v_{22}^j as unit normal vector fields of M_1 and M_2,respectively. Then we derive:

$$\begin{bmatrix} D_{v_0}\ j \\ D_{v_{11}}\ j \\ D_{v_{21}}\ j \end{bmatrix} = P_i^j \begin{bmatrix} 0 & c_1 & -b_1 \\ -c_1 & 0 & a_1 \\ b_1 & -a_1 & 0 \end{bmatrix} \begin{bmatrix} v_0^j \\ v_{11}^j \\ v_{21}^j \end{bmatrix} + Q_q^b D\delta_b^j \begin{bmatrix} v_0^q \\ v_{11}^q \\ v_{21}^q \end{bmatrix} \tag{3.2}$$

and

$$\begin{bmatrix} D_{v_0}\ j \\ D_{v_{12}}\ j \\ D_{v_{22}}\ j \end{bmatrix} = P_i^j \begin{bmatrix} 0 & c_2 & -b_2 \\ -c_2 & 0 & a_2 \\ b_2 & -a_2 & 0 \end{bmatrix} \begin{bmatrix} v_0^j \\ v_{12}^j \\ v_{22}^j \end{bmatrix} + Q_q^b D\delta_b^j \begin{bmatrix} v_0^q \\ v_{12}^q \\ v_{22}^q \end{bmatrix}. \tag{3.3}$$

Recalling that $(\alpha\ ,\ M_1)$ and $(\alpha\ ,\ M_2)$ are curvature strips, we can find $a_1 = a_2 = 0$. Then:

$$\begin{aligned} D_{v_{21}}\ j &= P_i^j \left(b_1 v_0^j\right) + Q_q^b D\delta_b^j v_{21}^{1q}, \\ D_{v_{22}}\ j &= P_i^j \left(b_2 v_0^j\right) + Q_q^b D\delta_b^j v_{22}^{1q}. \end{aligned}$$

Since v_{21}^j and v_{22}^j are orthonormal vector fields,

$$g_i^j \left(v_{21}^j, v_{22}^j\right) = \cos\theta.$$

Taking the derivative along α and noting that θ is constant, we obtain:

$$g_{ij}\left(Dv_{21}^j, v_{22}^j\right) + g_{ij}\left(v_{21}^j, Dv_{22}^j\right) = -\sin\theta d\theta.$$

Using (3.2) and (3.3), we obtain:

$$g_{ij}\left(P_i^j\left(b_1 v_0^j\right)+Q_q^b D\delta_b^j v_{21}^{1q}, v_{22}^j\right)+g_{ij}\left(v_{21}^j, P_i^j\left(b_2 v_0^j\right)+Q_q^b D\delta_b^j v_{22}^{1q}\right)=-\sin\theta d\theta.$$

This can be rewritten as:

$$P_i^j b_1 g_{ij}\left(v_0^j, v_{22}^j\right)+g_{ij}\left(Q_q^b D\delta_b^j v_{21}^{1q}, v_{22}^j\right)$$

$$+P_i^j b_2 g_{ij}\left(v_{21}^j, v_0^j\right)+g_{ij}\left(v_{21}^j, Q_q^b D\delta_b^j v_{22}^{1q}\right)=-\sin\theta d\theta.$$

Since

$$g_{ij}\left(v_0^j, v_{22}^j\right)=g_{ij}\left(v_{21}^j, Q_q^b D\delta_b^j v_{22}^{1q}\right)=0,$$

we get $\sin\theta=0$ or $d\theta=0$. If $\sin\theta=0$ we get $\theta=0$ or $\theta=\pi$, which means that $(\alpha\ ,\ M_1)$ and $(\alpha\ ,\ M_2)$ are congruent. Thus, we conclude that $d\theta=0$. Solving this equation implies that $\theta=c\ \ e^t$. *ii*) Let $(\alpha\ ,\ M_1)$ be a curvature strip and $\theta=const$. Suppose that the curvature strip $(\alpha\ ,\ M_2)$ is obtained by a rotation through the angle θ. Using the normal vectors of M_1 and M_2, we may write $g_{ij}\left(v_{21}^j, v_{22}^j\right)=\cos\theta$. Taking the derivative along α, we get

$$g_{ij}\left(Dv_{21}^j, v_{22}^j\right)+g_{ij}\left(v_{21}^j, Dv_{22}^j\right)=0.$$

Noting that $(\alpha\ ,\ M_1)$ is a curvature strip, we have:

$$g_{ij}\left(v_{21}^j, Dv_{22}^j\right)=0,$$

which means that $Dv_{22}^j\in T_{M_1}(\alpha(s))$.

From the well-known theorem of linear algebra,

$$\dim\left(T_{M_1}\alpha(s)\cap T_{M_2}\alpha(s)\right)=1,$$

which means that the intersection space is spanned by a unique vector. Recall that the only common vector of $T_{M_1}\alpha(s)$ and $T_{M_2}\alpha(s)$ is v_0^j. Hence:

$$Dv_{22}^j=\mu v_0^j.$$

From the last equation, the projection of Dv_{22}^j onto v_{22}^j equals zero, i.e. $a_2=0$. Therefore, $(\alpha\ ,\ M_1)$ is a curvature strip.

iii) Let us suppose that the angle between 2 strips $(\alpha\ ,\ M_1)$ and $(\alpha\ ,\ M_2)$ is constant. Then we may write

$$g_{ij}\left(v_{21}^j, Dv_{22}^j\right)=\cos\theta.$$

Differentiating both sides, we get

$$g_{ij}\left(Dv_{21}^j, v_{22}^j\right)+g_{ij}\left(v_{21}^j, Dv_{22}^j\right)=0.$$

From (3.2), we get

$$g_{ij}\left(P_j^i\left(-a_1 v_{11}^j\right)+Q_b^a D\delta_b^j v_{21}^q, v_{22}^j\right)+g_{ij}\left(v_{21}^j, P_j^i\left(-a_2 v_{12}^j\right)+Q_a^b D\delta_b^j v_{22}^q\right)=0$$

$$-a_2 P_j^i \; g_{ij}\left(v_{12}^j, v_{21}^j\right) + g_{ij}\left(Q_a^b D\delta_b^j v_{22}^q, v_{21}^j\right) = 0$$

or

$$-a_1 P_j^i \; g_{ij}\left(v_{11}^j, v_{22}^j\right) + g_{ij}\left(Q_b^a D\delta_b^j v_{21}^q, v_{22}^j\right).$$

Supposing that

$$g_{ij}\left(v_{11}^j, v_{22}^j\right) = -\sin\theta \quad \text{and } g_{ij}\left(v_{12}^j, v_{21}^j\right) = \sin\theta,$$

then it is not hard to show that

$$a_1 P_j^i \; \sin\theta - a_2 P_j^i \; \sin\theta + g_{ij}\left(Q_b^a D\delta_b^j v_{21}^q, v_{22}^j\right) + g_{ij}\left(Q_a^b D\delta_b^j v_{22}^q, v_{21}^j\right) = 0.$$

□

Theorem 8 *Let M_1 and M_2 be 2 surfaces in $R - O_3$, α be a nonplanar curve in M_1, and β be a curve in M_2. Then any 2 of the following conditions imply the third.*

i) If all points α and β correspond one-to-one to an ε plane that rolls over M_1 and M_2, then the distance between the corresponding points is constant.

ii) $(\alpha\ ,\ M_1)$ is a curvature strip.

iii) $(\beta\ ,\ M_2)$ is a curvature strip.

Proof $(i, ii) \Rightarrow (iii)$ Let λ be the distance between 2 corresponding points $P \in M_1$ and $Q \in M_2$, respectively. From (ii), $(\alpha\ ,\ M_1)$ is a curvature strip.Taking s as an arc length parameter, one may take an orthonormal base $\left\{v_0^j, v_{11}^j\right\}$ on M_1. Take $\alpha(s_1) = P$ and $\beta(s_1) = Q$ and the vector $\overrightarrow{PQ} = v(s_1)$.Then $v(s_1)$ satisfies the following equation:

$$v(s_1) = \cos\varphi \; v_0^j + \sin\varphi \; v_{11}^j. \tag{3.4}$$

On the other hand,

$$\beta(s_1) = \alpha(s_1) + \lambda(s_1)v(s_1). \tag{3.5}$$

Taking s_2 as an arclength parameter of β and differentiating both sides of (3.5), we obtain

$$D_{s_2}\beta \; \frac{ds_2}{ds_1} = D\alpha(s_1) + D\lambda(s_1)v(s_1) + \lambda(s_1)Dv(s_1)$$

or

$$v_{02}^j \;=\; \frac{ds_1}{ds_2}\left(\begin{array}{c} v_0^j \; + \lambda(s_1) \\ \left(-D\varphi\sin\varphi \; v_0^j + \cos\varphi \; Dv_0^j + D\varphi\cos\varphi \; v_{11}^j + \sin\varphi \; Dv_{11}^j\right) \end{array}\right)$$

where $v_{02}^j = D_{s_2}\beta$.

Using Dv_0^j and Dv_{11}^j, assuming $a_1 = 0$ in the last equation, we get

$$\begin{aligned} v_{02}^j \quad &= \quad \frac{ds_1}{ds_2}((v_0^j + \;\lambda(s_1)) - D\varphi\sin\varphi \; v_0^j \\ &\quad + \cos\varphi\left(P_j^i\left(c_1 v_{11}^j - b_1 v_{21}^j\right) + Q_a^b D\delta_b^j v_0^q, v_{22}^j\right) \end{aligned}$$

$$+D\varphi\cos\varphi\ v^j_{11}+\sin\varphi P^i_j\left(-c_1 v^j_0+a_1 v^j_{21}\right))$$

or

$$v^j_{02} = \frac{ds_1}{ds_2}((1-\lambda(s_1))\left(D\varphi+P^i_j c_1\right)\sin\varphi v^j_0 + \lambda(s_1)\left(\left(D\varphi+P^i_j c_1\right)\cos\varphi\right)\ v^j_{11}-\lambda(s_1)\left(P^j_i b_1\cos\varphi\right)v^j_{21}). \tag{3.6}$$

Along the curves α and β, the tangent planes of M_1 and M_2 are common, and then we have

$$v^j_{21}\ (s_1)=v^j_{22}\ (s_2)\,.$$

Differentiating both parts with respect to s_1 , we get

$$D_{s_2}v^j_{22}(s_2)\ \frac{ds_2}{ds_1}=Dv^j_{21}(s_1)=\ P^i_j b_1 v^j_0$$

or

$$D_{s_2}v^j_{22}(s_2)\ =\frac{ds_1}{ds_2}\ P^i_j b_1 v^j_0. \tag{3.7}$$

From the derivation equation of (β, M_2), one may have

$$D_{s_2}v^j_{22}\ =\ P^i_j\left(b_2 v^j_0-a_2 v^j_{12}\right)+Q^b_a D\delta^j_b v^q_{22}$$

or

$$g_{ij}\left(D_{s_2}v^j_{22},v^j_{12}\right)\ =\ -P^i_j a_2+g_{ij}\left(Q^b_a D\delta^j_b v^q_{22},v^j_{12}\right)=0.$$

From $v^j_{12}\ =\ v^j_{22}\wedge v^j_{02}$, we obtain

$$-P^i_j a_2+g_{ij}\left(Q^b_a D\delta^j_b v^q_{22},v^j_{12}\right)=\det\left(D_{s_2}v^j_{22},v^j_{22},v^j_{02}\right)$$

$$P^i_j a_2+g_{ij}\left(Q^b_a D\delta^j_b v^q_{22},v^j_{12}\right)=\det\left(v^j_{22},D_{s_2}v^j_{21},v^j_{02}\right)$$

$$=\left(\frac{ds_1}{ds_2}\right)^2$$

$$P^i_j\begin{vmatrix} 0 & 0 & 1\\ b_1 & 0 & 0\\ (1-\lambda(s_1))\left(\frac{D\varphi}{P^i_j}+c_1\sin\varphi\right) & \frac{\lambda(s_1)}{P^i_j}D\varphi+c_1\cos\varphi & -\lambda(s_1)b_1\cos\varphi\end{vmatrix} \tag{3.8}$$

or

$$P^i_j a_2+g_{ij}\left(Q^b_a D\delta^j_b v^q_{22},v^j_{12}\right)=\left(\frac{ds_1}{ds_2}\right)^2 P^i_j\left(\frac{\lambda(s_1)b_1}{P^i_j}D\varphi+c_1\right)\cos\varphi. \tag{3.9}$$

Since $\left\{v^j_{02}\ ,v^j_{12},v^j_{22}\right\}$ is an orthonormal frame, from (3.9) we get

$$\lambda(s_1)b_1\cos\varphi=0.$$

Thus, $a_2 = 0$, and this means that $(\beta\ ,\ M_2)$ is a curvature strip. $(ii, iii)\ \Rightarrow (i)$ Let us show that $(\alpha\ ,\ M_1)$ and $(\beta\ ,\ M_2)$ is a curvature strip, and $\lambda(s_1)$ is constant.

Considering equation (3.5), equation (3.4) can be rewritten as

$$\beta(s_1) = \alpha(s_1) + \lambda(s_1)\left(\cos\varphi\ v_0^j + \sin\varphi\ v_{11}^j\right).$$

Differentiating both parts, we get

$$\begin{aligned} D_{s_2}\beta(s_1)\frac{ds_2}{ds_1} &= v_0^j + D\lambda(s_1)\left(\cos\varphi\ v_0^j + \sin\varphi\ v_{11}^j\right) \\ &\quad +\lambda(s_1)(-D\varphi\sin\varphi\ v_0^j + \cos\varphi D v_0^j + D\varphi\cos\varphi\ v_{11}^j + \sin\varphi\ D v_{11}^j). \end{aligned}$$

From the derivation equation, we may write $a_1 = 0$, and v_{02}^j can be calculated as

$$\begin{aligned} v_{02}^j &= \frac{ds_1}{ds_2}\{v_0^j + D\lambda(s_1)\left(\cos\varphi\ v_0^j + \sin\varphi\ v_{11}^j\right) + \\ &\quad \lambda(s_1)(-D\varphi\sin\varphi\ v_0^j + \\ &\quad \cos\varphi\left(P_j^i\left(-c_1 v_{11}^j - b_1 v_{21}^j\right) + Q_a^b D\delta_b^j v_0^q\right) + \\ &\quad D\varphi\cos\varphi\ v_{11}^j + \sin\varphi\ \left(P_j^i\left(-c_1 v_0^j + a_1 v_{21}^j\right) + Q_b^a D\delta_b^j v_{11}^q\right))\} \end{aligned}$$

or

$$\begin{aligned} v_{02}^j &= \frac{ds_1}{ds_2}\{\left[1 + D\lambda(s_1)\cos\varphi - \lambda(s_1)\left(D\varphi + c_1 P_j^i\right)\sin\varphi\right]\ v_0^j \\ &\quad + \left[D\lambda(s_1)\sin\varphi + \lambda(s_1)\left(D\varphi + c_1 P_j^i\right)\cos\varphi\ \right] v_{11}^j \\ &\quad - \left[\lambda(s_1) b_1 P_j^i\cos\varphi\right] v_{21}^j\} \\ &\quad + \cos\varphi Q_a^b D\delta_b^j v_0^q + \sin\varphi Q_b^a D\delta_b^j v_{11}^q. \end{aligned} \tag{3.10}$$

Obviously $a_2 = 0$; then, taking into account (3.7), (3.9), and $a_2 = \det\left(v_{22}^j, D_{s_2}v_{21}^j, v_{02}^j\right)$, we get the simplified form of $g_{ij}\left(Q_a^b D\delta_b^j v_{22}^q, v_{12}^j\right)$ as follows:

$$\left(\frac{ds_1}{ds_2}\right)^2 (b_1\lambda(s_1)P_j^i\cos\varphi\left(D\varphi + c_1 P_j^i\right) - b_1 P_j^i D\lambda(s_1)\sin\varphi) = 0 \tag{3.11}$$

Since $g_{ij}\left(v_{02}^j, v_{22}^j\right) = 0$, if we write (3.9) and (3.7) in this equation after routine calculations, we get:

$$b_1\lambda(s_1)P_j^i\cos\varphi = 0.$$

Taking into account the last equation in (3.10),

$$\left(\frac{ds_1}{ds_2}\right)^2 D\lambda(s_1) b_1 P_j^i\sin\varphi) = 0. \tag{3.12}$$

From the regularity of the curves α and β,

$$\left(\frac{ds_1}{ds_2}\right) \neq 0.$$

On the other hand, we know that the curve α is nonplanar so that $b_1 \neq 0$. Then we get

$$D\lambda(s_1)P_j^i \sin\varphi = 0, \qquad P_j^i \neq 0$$

and $b_1\lambda(s_1)P_j^i \cos\varphi = 0$, and so we get $\cos\varphi = 0$, which means that $\varphi = \frac{\pi}{2}$ and $\varphi = \frac{3\pi}{2}$. Hence, we get $\sin\varphi = \mp 1$. Then we obtain $D\lambda(s_1) = 0$ or $\lambda(s_1) = c\ e^t$.

$(i, iii) \Rightarrow (ii)$ The proof is ommitted because of the similarity to the above part. □

Theorem 9 *A closed, simple, and regular curve $c : I \to R - O_3$, $\left|c^{'}(s)\right| = 1$ of length L and torsion $k_2(s)$ is the union of principal curvature lines and umbilic points of an oriented surface (see [12]) iff $\theta^{'}(s) + k_2(s) = 0$, $\int_0^L k_2(s)ds = 2k\pi$, $k \in N$, and $Q_b^a D\delta_b^j = 0$.*

Proof Consider the Frenet frame $\left\{v_0^j, v_1^j, v_2^j\right\}$ associated to c. Letting

$$N(s) = cos\theta(s)v_1^j + \sin\theta(s)v_2^j$$

be a unit normal vector to c, it follows that

$$N^{'}(s) = -\sin\theta(s)v_1^j + \cos\theta(s)Dv_1^j + \cos\theta(s)v_2^j + \sin\theta(s)Dv_2^j$$

$$\begin{aligned} N^{'}(s) &= \cos\theta(s)\left(P_j^i\left(-k_1 v_0^i + k_2 v_2^i\right) + Q_b^a D\delta_j^b v_1^q\right) \\ &\quad + \sin\theta(s)\left(P_j^i\left(-k_2 v_1^i\right) + Q_b^a D\delta_j^b v_2^q\right) \\ &\quad - \sin\theta(s)v_1^j + \cos\theta(s)v_2^j \end{aligned}$$

or

$$\begin{aligned} N^{'}(s) &= \left(\theta^{'}(s) + k_2(s)\right)\left(P_j^i\left(-\sin\theta(s)v_1^j + \cos\theta(s)v_2^j\right)\right) \\ &\quad -P_j^i \cos\theta(s)k_1 v_0^i + \cos\theta(s)Q_b^a D\delta_j^b v_1^q \\ &\quad + \left(\sin\theta(s)Q_b^a D\delta_j^b v_2^q\right). \end{aligned}$$

Therefore, $P_j^i \neq 0$, c is a principal line (union of maximal and minimal principal lines and umbilic points (see [12])) iff $\theta^{'}(s) + k_2(s) = 0$, which means that

$$\theta(s) = \int_0^L k_2(s)ds$$

$$\theta(L) - \theta(0) = \int k_2(s)ds$$

$$N(L) - N(0) \quad \text{iff} \int_0^L k_2(s)ds = 2k\pi.$$

□

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