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Generalized hypercenter of a finite group

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Abstract: Let G be a finite group. In this paper, we introduce the concept of super generalized supersolvably embedded subgroup of a group G and give a new characterization of the generalized hypercenter of G.

Key words: S-quasinormally embedded, generalized supersolvably embedded, super generalized supersolvably embedded, supersolvable

1. Introduction

All groups considered in this paper will be finite and G always means a finite group. We use conventional notions and notations, as in Doerk and Hawkes [4].

Two subgroups H and K of a group G are said to permute if $HK = KH$. Thus, we have that H and K permute if and only if HK is a subgroup of G. We say, following Kegel [5], that a subgroup of a group G is S-quasinormal in G if it permutes with every Sylow subgroup of G. In 1998, Ballester-Bolinches and Pedraza-Aguilera [3] introduced the following definition: a subgroup H of a group G is said to be S-quasinormally embedded in G if each Sylow subgroup of H is a Sylow subgroup of some S-quasinormal subgroup of G. Obviously, every S-quasinormal subgroup is S-quasinormally embedded. The converse does not hold in general. The Sylow 2-subgroups of S_3, the symmetric group of degree 3, are S-quasinormally embedded but not S-quasinormal subgroups of S_3.

Agrawal [1] defined the generalized center $genz(G)$ of a group G to be the subgroup

$$< g \in G : \ < g > \text{ is } S\text{-quasinormal in } G >.$$

The generalized hypercenter $genz_\infty(G)$, is the largest term of the chain

$$1 = genz_0(G) \leq genz_1(G) = genz(G) \leq genz_2(G) \leq \ldots,$$

where $genz_{i+1}(G)/genz_i(G) = genz(G/genz_i(G))$ for all $i > 0$. Asaad and Ezzat Mohamed [2] gave a new characterization of $genz_\infty(G)$ by introducing the following definition: a normal subgroup H of a group G is a generalized supersolvably embedded (GSE) in G if there exists a chain

$$1 = H_0 \leq H_1 \leq H_2 \leq \ldots \leq H_n = H$$

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such that H_i is S -quasinormal in G and $|H_{i+1} : H_i|$ is a prime, where $0 \leq i \leq n-1$. It is easily verified that if H and K are normal GSE subgroups of G, then HK is GSE in G. From this, a group G has a unique maximal generalized supersolvably embedded subgroup and it is denoted by $GSE(G)$. The generalized hypercenter $genz_\infty(G)$ is the maximal generalized supersolvably embedded $GSE(G)$ (see [2; p. 2243]). It was proven in [5; Theorem 1, p. 209] that an S- quasinormal subgroup of a finite group is subnormal. Therefore, by the definition of a GSE subgroup, it is clear that H_i is normal in H_{i+1}. As a development, we now replace the S-quasinormality by S -quasinormally embedded and introduce the following new concept:

Definition 1.1 *A normal subgroup H of a group G is super generalized supersolvably embedded ($SGSE$) in G if there exists a chain*

$$1 = H_0 \lhd H_1 \lhd H_2 \lhd ... \lhd H_n = H$$

such that H_i is S -quasinormally embedded in G and $|H_{i+1} : H_i|$ is a prime for every $0 \leq i \leq n-1$.

In this paper, we give a new characterization of generalized hypercenter and generalized supersolvably embedded.

2. Preliminary results

The following known results will be useful later.

Lemma 2.1 *Suppose that U is S-quasinormally embedded in a group G, H is a subgroup of G, and K is a normal subgroup of G. Then:*

(a) If $U \leq H$, then U is S-quasinormally embedded in H.

(b) UK is S-quasinormally embedded in G and UK/K is S-quasinormally embedded in G/K.

(c) If $K \leq H$ and H/K is S-quasinormally embedded in G/K, then H is S-quasinormally embedded in G.

Proof See [3; Lemma 1, p. 114]. □

Lemma 2.2 *(a) If K is a normal subgroup of a group G and H is S-quasinormal in G, then $H \cap K$ is S-quasinormal in G.*

(b) If K is a normal subgroup of a group G and H is S-quasinormally embedded in G, then $H \cap K$ is S-quasinormally embedded in G.

Proof (a) Since H is S-quasinormal in G, it follows by [5; Theorem 1, p. 209] that H is subnormal in G, and so HK and $H \cap K$ are subnormal in G. Let P be an arbitrary Sylow subgroup of G. Clearly $H \cap P \in Syl(H)$, $K \cap P \in Syl(K)$, $HK \cap P \in Syl(HK)$, and $H \cap K \cap P \in Syl(H \cap K)$. Then $|(H \cap P)(K \cap P)| = \frac{|H \cap P||K \cap P|}{|H \cap K \cap P|} = |HK \cap P|$. Since $(H \cap P)(K \cap P) \leq HK \cap P$, it follows that $(H \cap P)(K \cap P) = HK \cap P$. Hence, by [4; Lemma 1.2, p. 2], $(H \cap K)P = HP \cap KP$ and so $H \cap K$ is S-quasinormal in G.

(b) Let P be an arbitrary Sylow subgroup of $H \cap K$. Since K is a normal subgroup of G, it follows that $P = S \cap K$, where $S \in Syl(H)$. Since H is S-quasinormally embedded in G, it follows that S is a Sylow subgroup of an S-quasinormal subgroup L of G. By (a), $L \cap K$ is S-quasinormal in G. Clearly $P = S \cap K$ is a Sylow subgroup of $L \cap K$. Then every Sylow subgroup of $H \cap K$ is a Sylow subgroup of S-quasinormal subgroup of G. Hence, $H \cap K$ is S-quasinormally embedded in G. □

Lemma 2.3 *(a) Let H be a normal subgroup of G such that $H \leq genz_\infty(G)$, then $genz_\infty(G)/H = genz_\infty(G/H)$.*

(b) $genz_\infty(G)$ is supersolvable.

(c) G is supersolvable if and only if $G/genz_\infty(G)$ is supersolvable.

(d) If K is a supersolvable subgroup of G, then $Kgenz_\infty(G)$ is supersolvable.

Proof See [1; Proposition 2.3, p. 27 and Theorems 2.7, 2.8(ii), and 2.9, pp. 18–19]. □

3. Results

Lemma 3.1 *Let H and K be normal subgroups of G.*

(a) If $K \leq H$ and H is $SGSE$ in G, then H/K is $SGSE$ in G/K.

(b) If $K \leq H$ and H is $SGSE$ in G, then K is $SGSE$ in G.

(c) If K and H are $SGSE$ in G, then HK is $SGSE$ in G.

Proof (a) Suppose that H is $SGSE$ in G. Then there exists a chain $1 = H_0 \lhd H_1 \lhd H_2 \lhd ... \lhd H_n = H$, such that H_i is S-quasinormally embedded in G and $|H_{i+1} : H_i|$ is a prime, for every $0 \leq i \leq n-1$. Since H_i is S-quasinormally embedded in G and K is a normal subgroup of G, it follows by Lemma 2.1(b) that H_iK/K is S-quasinormally embedded in G/K. Since $H_i \lhd H_{i+1}$, it follows that

$$(H_{i+1}K/K)/(H_iK/K) \cong H_{i+1}K/H_iK = H_{i+1}(H_iK)/H_iK \cong H_{i+1}/(H_{i+1} \cap H_iK) = H_{i+1}/H_i(H_{i+1} \cap K),$$

where the equality $(H_{i+1} \cap H_iK) = H_i(H_{i+1} \cap K)$ follows from the modular law. So, as $|H_{i+1} : H_i|$ is a prime, $|H_{i+1}K/K : H_iK/K|$ is prime or 1. Hence, there is a chain

$$1 = K/K = L_0/K \lhd L_1/K \lhd L_2/K \lhd ... \lhd L_m/K = H/K,$$

such that L_i/K is S-quasinormally embedded in G/K and $|L_{i+1}/K : L_i/K|$ is a prime, for every $0 \leq i \leq m-1$. Therefore, H/K is $SGSE$ in G/K.

(b) Suppose that H is $SGSE$ in G. Then there exists a chain $1 = H_0 \lhd H_1 \lhd H_2 \lhd ... \lhd H_n = H$, such that H_i is S-quasinormally embedded in G and $|H_{i+1} : H_i|$ is a prime, for every $0 \leq i \leq n-1$. Since H_i is S-quasinormally embedded in G and K is a normal subgroup of G, it follows by Lemma 2.2(b) that $H_i \cap K$ is S-quasinormally embedded in G. Since $H_i \lhd H_{i+1}$, it follows that $H_i \cap K \lhd H_{i+1} \cap K$. Then $(H_{i+1} \cap K)/(H_i \cap K) \cong (H_{i+1} \cap K)H_i/H_i \leq H_{i+1}/H_i$. Then $|H_{i+1} \cap K : H_i \cap K|$ is prime or 1, as $|H_{i+1} : H_i|$ is a prime. Thus, we have a chain $1 = H_0 \cap K \lhd H_1 \cap K \lhd H_2 \cap K \lhd ... \lhd H_n \cap K = H \cap K = K$ such that $H_i \cap K$ is S-quasinormally embedded and $|H_{i+1} \cap K : H_i \cap K|$ is prime or 1, for every $0 \leq i \leq n-1$. Hence, we can choose a chain $1 = K_0 \lhd K_1 \lhd K_2 \lhd ... \lhd K_m = K$, such that K_i is S-quasinormally embedded in G and $|K_{i+1} : K_i|$ is a prime, where $0 \leq i \leq m-1$. Therefore, K is $SGSE$ in G.

(c) Suppose that H and K are $SGSE$ in G. Then there exist 2 chains $1 = H_0 \lhd H_1 \lhd H_2 \lhd ... \lhd H_n = H$ and $1 = K_0 \lhd K_1 \lhd K_2 \lhd ... \lhd K_m = K$ such that H_i, K_j are S-quasinormally embedded in G and $|H_{i+1} : H_i|$, $|K_{j+1} : K_j|$ are primes, where $0 \leq i \leq n-1$, $0 \leq j \leq m-1$. Since K_j is S-quasinormally embedded in G and $H \lhd G$, it follows by Lemma 2.1(b) that K_jH is S-quasinormally embedded in G. Then $HK_{j+1}/HK_j = K_{j+1}(HK_j)/HK_j \cong K_{j+1}/(K_{j+1} \cap HK_j) = K_{j+1}/K_j(K_{j+1} \cap H)$. Since $|K_{j+1} : K_j|$ is a prime, it follows that $|HK_{j+1} : HK_j|$ is prime or 1. Thus, we have a chain $1 = H_0 \lhd H_1 \lhd H_2 \lhd ... \lhd H_n =$

$H = HK_0 \lhd HK_1 \lhd HK_2 \lhd ... \lhd HK_m = HK$ such that H_i , HK_j are S-quasinormally embedded in G and $|H_{i+1} : H_i|$ is a prime, and $|HK_{j+1} : HK_j|$ is prime or 1, for every $0 \leq i \leq n-1$, $0 \leq j \leq m-1$. Hence, we can choose a chain $1 = L_0 \lhd L_1 \lhd L_2 \lhd ... \lhd L_r = HK$, such that L_i is S-quasinormally embedded in G and $|L_{i+1} : L_i|$ is a prime, for every $0 \leq i \leq r-1$. Therefore, HK is $SGSE$ in G. □

From this last property, we can see that a group G has a unique maximal ($SGSE$), namely the product of all $SGSE$ normal subgroups of G, and it will be denoted by $SGSE(G)$.

The following 2 results are easily verified by using Lemmas 2.1 and 3.1:

Lemma 3.2 *If H is a subgroup of G and $SGSE(G) \leq H$, then $SGSE(G) \leq SGSE(H)$. In particular, $SGSE(SGSE(G)) = SGSE(G)$.*

Lemma 3.3 *Let N be a normal subgroup of G. Then $SGSE(G)N/N \leq SGSE(G/N)$.*

The inclusion here can be proper, for example, the alternating group of degree 4.

As a further Lemma, we have:

Lemma 3.4 *Let H be a normal subgroup of G such that $H \leq SGSE(G)$. Then $SGSE(G)/H = SGSE(G/H)$.*

Proof Put $SGSE(G/H) = L/H$. Consider a chain

$$1 = H/H = H_0/H \lhd H_1/H \lhd ... \lhd H_n/H = SGSE(G/H) = L/H,$$

such that H_i/H is S-quasinormally embedded in G/H and $|H_{i+1}/H : H_i/H|$ is a prime, for every $0 \leq i \leq n-1$. Since H_i/H is S-quasinormally embedded in G/H, it follows by Lemma 2.1(c) that H_i is S-quasinormally embedded in G. Hence, we have a chain $H = H_0 \lhd H_1 \lhd H_2 \lhd ... \lhd H_n = L$, such that H_i is S-quasinormally embedded in G and $|H_{i+1} : H_i| = |H_{i+1}/H : H_i/H|$ is a prime, for every $0 \leq i \leq n-1$. Since H is a normal subgroup of G and $H \leq SGSE(G)$, it follows by Lemma 3.1(b) that H is $SGSE$ in G. Then H_n is $SGSE$ in G and so $SGSE(G/H) = L/H \leq SGSE(G)/H$. On the other hand, $SGSE(G)/H \leq SGSE(G/H)$, by Lemma 3.3. Therefore, $SGSE(G)/H = SGSE(G/H)$. □

Now we can prove the following result:

Theorem 3.5 *Let G be a group. Then $SGSE(G)$ is supersolvable.*

Proof We prove the result by the induction on the order of G. By Lemma 3.2, we may assume that $SGSE(G) = G$. Then we have:

(1) $1 < GSE(G) = genz_{\infty}(G)$.

If $GSE(G) = genz_{\infty}(G) = 1$, then $genz(G) = 1$. As $SGSE(G) = G$, it is clear by the definition of $SGSE(G)$ that G is solvable. Then $F(G) \neq 1$. So $O_p(G) \neq 1$ for some prime p dividing the order of G. By Lemma 3.1(b), $O_p(G)$ is $SGSE$ in G. Then $O_p(G)$ contains a subgroup H of order p such that H is S-quasinormally embedded in G. Thus, there exists an S-quasinormal subgroup K of G such that H is a Sylow p-subgroup of K. Clearly $H = O_p(G) \cap K$. Since $O_p(G) \lhd G$ and K is S-quasinormal in G, it follows by Lemma 2.2(b) that $H = O_p(G) \cap K$ is S-quasinormal in G. Then $1 \neq H \leq genz(G)$, a contradiction. Thus, $1 < GSE(G) = genz_{\infty}(G)$.

(2) G contains a normal Sylow r -subgroup R, where r is the largest prime dividing the order of G.

Since $genz_{\infty}(G) \lhd G$, it follows by Lemma 3.4 that

$$G/genz_{\infty}(G) = SGSE(G)/genz_{\infty}(G) = SGSE(G/genz_{\infty}(G)).$$

By (1) and the induction on the order of G, $G/genz_{\infty}(G)$ is supersolvable. If r divides the order of $G/genz_{\infty}(G)$, then $Rgenz_{\infty}(G)/genz_{\infty}(G)$ is a Sylow r-subgroup of $G/genz_{\infty}(G)$. Since $G/genz_{\infty}(G)$ is supersolvable and since r is the largest prime divisor of the order of $G/genz_{\infty}(G)$, it follows that $Rgenz_{\infty}(G)/genz_{\infty}(G) \lhd G/genz_{\infty}(G)$ and so $Rgenz_{\infty}(G) \lhd G$. By Lemma 2.3(d), $Rgenz_{\infty}(G)$ is supersolvable. Then R is characteristic in $Rgenz_{\infty}(G)$. Since $Rgenz_{\infty}(G) \lhd G$, it follows that $R \lhd G$. Thus, we may assume that r is not dividing the order of $G/genz_{\infty}(G)$. Hence, R is a Sylow r-subgroup of $genz_{\infty}(G)$. Since $genz_{\infty}(G)$ is supersolvable by Lemma 2.3(b), it follows that R is characteristic in $genz_{\infty}(G)$. Since $genz_{\infty}(G) \lhd G$, it follows that $R \lhd G$.

(3) G is supersolvable.

Since $R \lhd G = SGSE(G)$, it follows by Lemma 3.1(b) that R is $SGSE$ in G. Then there exists a chain $1 = R_0 \lhd R_1 \lhd ... \lhd R_n = R$ such that R_i is S-quasinormally embedded in G and $|R_{i+1} : R_i| = r$, for every $0 \leq i \leq n-1$. Thus, for every R_i there exists an S-quasinormal subgroup L_i in G such that R_i is a Sylow subgroup of L_i. Clearly $R_i = L_i \cap R$. Since $R \lhd G$ and L_i is S-quasinormal in G, it follows by Lemma 2.2(b) that $R_i = L_i \cap R$ is S-quasinormal in G. Then R is GSE in G, and so $R \leq genz_{\infty}(G)$ (because $GSE(G) = genz_{\infty}(G)$). Since G/R is supersolvable by the induction on the order of G and $R \leq genz_{\infty}(G)$, it follows by Lemma 2.3(c) that G is supersolvable. □

Finally, we prove the following:

Theorem 3.6 *Let G be a group. Then* $SGSE(G) = genz_{\infty}(G)$.

Proof We prove the result by the induction on the order of G. We consider the following 2 cases:

Case 1. $genz_{\infty}(G) \neq 1$.

By the induction on the order of G,

$$SGSE(G/genz_{\infty}(G)) = genz_{\infty}(G/genz_{\infty}(G)).$$

Since $genz_{\infty}(G) \leq SGSE(G)$, it follows by Lemma 3.4 that

$$SGSE(G/genz_{\infty}(G)) = SGSE(G)/genz_{\infty}(G).$$

Also by Lemma 2.3(a), $genz_{\infty}(G/genz_{\infty}(G)) = genz_{\infty}(G)/genz_{\infty}(G)$. Then $SGSE(G)/genz_{\infty}(G) = genz_{\infty}(G)/genz_{\infty}(G)$ and so $SGSE(G) = genz_{\infty}(G)$.

Case 2. $genz_{\infty}(G) = 1$.

We argue that $SGSE(G) = 1$. If not, then $SGSE(G) \neq 1$. By Theorem 3.5, $SGSE(G)$ is supersolvable. Then $SGSE(G)$ contains a characteristic Sylow subgroup R, and since $SGSE(G) \lhd G$, it follows that $R \lhd G$. By Lemma 3.1(b), R is $SGSE$ in G. Then there exists a chain $1 = R_0 \lhd R_1 \lhd ... \lhd R_n = R$ such that R_i is S-quasinormally embedded in G and $|R_{i+1} : R_i|$ is a prime, for every $0 \leq i \leq n-1$. Thus, for every R_i there exists an S-quasinormal subgroup L_i in G such that R_i is a Sylow subgroup of L_i. Clearly $R_i = L_i \cap R$. Since $R \lhd G$ and L_i is S-quasinormal in G, it follows by Lemma 2.2(a) that $R_i = L_i \cap R$ is S-quasinormal in G. Then R is GSE in G and so $1 < R \leq genz_{\infty}(G)$, a contradiction. Thus, $SGSE(G) = genz_{\infty}(G) = 1$. □

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