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Population dynamical behaviors of stochastic logistic system with jumps

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Abstract: This paper is concerned with a stochastic logistic model driven by martingales with jumps. In the model, generalized noise and jump noise are taken into account. This model is new and more feasible. The explicit global positive solution of the system is presented, and then sufficient conditions for extinction and persistence are established. The critical value of extinction, nonpersistence in the mean, and weak persistence in the mean are obtained. The pathwise and moment properties are also investigated. Finally, some simulation figures are introduced to illustrate the main results.

Key words: Logistic equation, martingale, extinction, persistence

1. Introduction

The classical nonautonomous logistic equation is

$$\mathrm{d}x(t) = x(t)\big[a(t) - b(t)x(t)\big]\mathrm{d}t \tag{1.1}$$

for $t \geq 0$ with initial value $x(0) > 0$. In this model, $x(t)$ denotes the population size at time t, $a(t)$ is the intrinsic growth rate, and $a(t)/b(t)$ is the carrying capacity at time t. Both $a(t)$ and $b(t)$ are positive continuous functions. System (1.1) models the population density of a single species whose members compete among themselves for limited resources such as food or living space. For the detailed model construction, readers can refer to [24].

Because of its importance in theory and practice, many authors have studied model (1.1) and its generalization. Many good results on the dynamical behavior of solutions have been reported; see, e.g., Freedman and Wu [5], Lisena [16], Golpalsamy [8], Kuang [12], and the references therein. Among them, the books [8] and [12] are good references in this area.

However, in the real world, population systems are inevitably subject to much stochastic environmental noise, which is present in the ecosystem (see, e.g., Gard [6, 7]). In model (1.1), the parameters are all deterministic and independent of the environmental fluctuations; therefore, they have limitations in applications and it is difficult to fit data and predict the future accurately [2].

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According to the well-known central limit theorem, the sum of all small stochastic environmental noise follows a normal distribution, usually called the white noise and denoted by $\dot{B}(t)$. If we impose the perturbation on $a(t)$, then the deterministic model (1.1) changes into an Itô equation:

$$\mathrm{d}x(t) = x(t)\Big[\big(a(t) - b(t)x(t)\big)\mathrm{d}t + \sigma(t)\mathrm{d}B(t)\Big], \tag{1.2}$$

where $B(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and $\sigma^2(t)$ denotes the intensity of the noise. The noise has important effects on the model: Mao et al. [23] showed that the environmental Brownian noise suppresses explosion in population dynamics. There are many other papers in the literature on models with white noise; readers can refer to [11, 17, 13, 18, 14, 19, 20] and the references cited therein.

In addition, the population may suffer from sudden environmental shocks, e.g., red tides, tsunamis, earthquakes, floods, or epidemics. These events are so strong that they break the continuity of the solution. Thus, models with white noise cannot capture these phenomena. Introducing jump noise into the model may be a reasonable way to accommodate such phenomena; see [3, 4]. Models with jumps have received considerable attention in recent years, but, so far, there are very few papers on jump noise. For background on processes with jumps, readers can refer to [1, 25].

As we know, martingales constitute an important class of stochastic processes and generalize Brownian motion. The Brownian motion $B(t)$ is a special martingale. Therefore, we attempt to generalize the driving process.

Motivated by the above discussions, we propose the following logistic equation driven by martingales with jumps:

$$\mathrm{d}x(t) = x(t^-)\Big[\big(a(t) - b(t)x(t^-)\big)\mathrm{d}t + \sigma(t)\mathrm{d}M(t) + \int_{\mathbb{Z}} c(t,z)\tilde{N}(\mathrm{d}t, \mathrm{d}z)\Big]. \tag{1.3}$$

In this model, $x(t^-)$ is the left limit of $x(t)$, N is a Poisson counting measure with characteristic measure π on a measurable subset $\mathbb{Z}$ of $(0,\infty)$ with $\pi(\mathbb{Z}) < \infty$, and $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \pi(\mathrm{d}z)\mathrm{d}t$ is the corresponding martingale measure. $M(t)$ is a square integrable martingale with $M(0) = 0$. It is worth noting that a martingale usually does not share the good properties of Brownian motion, so there are many difficulties when we replace the Brownian motion in the stochastic integral with a martingale, and maybe this is one of the reasons why biological models driven by martingales have not been widely studied. For detailed information on martingales, readers can refer to [25, 10]. Throughout this paper, we assume that M is independent of N.

The main aim of this paper is to investigate the asymptotic behaviors of model (1.3). The rest of this paper is organized as follows. In Section 2, we present the explicit positive solution and give several useful lemmas. In Section 3, we consider the extinction and persistence of a solution to equation (1.3). The stochastically ultimate boundedness is investigated in Section 4. We give 2 numerical simulations for extinction and persistence in Section 5. Finally, we complete the paper with conclusions in Section 6.

2. Positive solutions and useful lemmas

Throughout this paper, we assume that $a(t), b(t)$, and $\sigma(t)$ are continuous bounded functions on $\mathbb{R}_+ = [0,\infty)$, and $\inf_{t\in\mathbb{R}_+} b(t) > 0$. $M(t)$ denotes a square integrable martingale, and $\langle M\rangle(t)$ is the unique integrable increasing process such that $M^2(t) - \langle M\rangle(t)$ is a martingale (see [25]).

In the sequel, for convenience and simplicity, we adopt the following notations. Define

$$\hat{h} = \inf_{t\in\mathbb{R}_+} h(t), \quad \check{h} = \sup_{t\in\mathbb{R}_+} h(t),$$

$$\overline{f(t)} = t^{-1}\int_0^t f(s)\mathrm{d}s, \quad f^* = \limsup_{t\to\infty} f(t), \quad f_* = \liminf_{t\to\infty} f(t).$$

For the jump-diffusion coefficient, we assume that

$$c(t,z) > -1, \quad t \geq 0, \ z \in \mathbb{Z}.$$

Since $c(t,z)$ is the coefficient of the effect of jump noise on the population, $c(t,z) > 0$ implies that jump noise is advantageous for the ecosystem, and $c(t,z) < 0$ implies that jump noise is disadvantageous for the ecosystem. Therefore, $c(t,z) > -1$ is needed; otherwise, the population will be extinct. See Remark 4 in [27].

If we attempt to study the properties of the solution, first we should guarantee the existence of the global solution. Here we present the explicit global positive solution by the variation-of-constants formula [3].

Theorem 2.1 *For any initial value $x(0) > 0$, Eq. (1.3) admits a unique global positive solution $x(t)$ on $t \geq 0$ almost surely (a.s.), which is prescribed by*

$$x(t) = \frac{\phi^{-1}(t)}{1/x(0) + \int_0^t \phi^{-1}(s)b(s)\mathrm{d}s}$$

where

$$\begin{aligned}\phi(t) &= \exp\Big[\int_0^t \big(-a(s) + \int_{\mathbb{Z}}(c(s,z) - \ln(1+c(s,z)))\pi(\mathrm{d}z)\big)\mathrm{d}s - \sigma(s)\mathrm{d}M(s) \\ &\quad + \frac{1}{2}\sigma^2(s)\mathrm{d}\langle M\rangle(s) - \int_0^t\int_{\mathbb{Z}} \ln(1+c(s,z))\tilde{N}(\mathrm{d}s,\mathrm{d}z)\Big].\end{aligned}$$

Proof Since the coefficients of the equation are locally Lipschitz continuous, for any initial condition $x(0) > 0$, Eq. (1.3) has a unique maximal local solution $x(t)$ on $[0,\tau_e)$, where τ_e is the explosion time [9]. To show that this solution is global, we will derive the solution. Letting $y(t) = 1/x(t)$, Itô's formula for semimartingales with jumps leads to

$$\begin{aligned}\mathrm{d}y(t) &= -y\Big[\big(a(t) - b(t)\frac{1}{y}\big)\mathrm{d}t + \sigma(t)\mathrm{d}M(t)\Big] + y\sigma^2(t)\mathrm{d}\langle M\rangle(t) \\ &\quad + y\int_{\mathbb{Z}}\big[\frac{1}{1+c(t,z)} - 1\big]\tilde{N}(\mathrm{d}t,\mathrm{d}z) + y\int_{\mathbb{Z}}\frac{c^2(t,z)}{1+c(t,z)}\pi(\mathrm{d}z)\mathrm{d}t \\ &= y\Big[\big(-a(t) + \int_{\mathbb{Z}}\frac{c^2(t,z)}{1+c(t,z)}\pi(\mathrm{d}z)\big)\mathrm{d}t - \sigma(t)\mathrm{d}M(t) + \sigma^2(t)\mathrm{d}\langle M\rangle(t) \\ &\quad + \int_{\mathbb{Z}}\big[\frac{1}{1+c(t,z)} - 1\big]\tilde{N}(\mathrm{d}t,\mathrm{d}z)\Big] + b(t)\mathrm{d}t, \end{aligned} \tag{2.1}$$

where, for simplicity, we omit t^- in $y(t^-)$. Clearly, Eq. (2.1) is a linear equation.

$$\begin{aligned} \mathrm{d}\phi(t) &= \phi(t)\Big[\big(-a(t)+\int_{\mathbb{Z}}\frac{c^2(t,z)}{1+c(t,z)}\pi(\mathrm{d}z)\big)\mathrm{d}t-\sigma(t)\mathrm{d}M(t)+\sigma^2(t)\mathrm{d}\langle M\rangle(t) \\ &\quad +\int_{\mathbb{Z}}\big[\frac{1}{1+c(t,z)}-1\big]\tilde{N}(\mathrm{d}t,\mathrm{d}z)\Big] \end{aligned} \tag{2.2}$$

is the corresponding homogeneous linear equation of (2.1). Note that

$$\begin{aligned} \phi(t) &= \exp\Big[\int_0^t\big(-a(s)+\int_{\mathbb{Z}}\big(c(s,z)-\ln(1+c(s,z))\big)\pi(\mathrm{d}z)\big)\mathrm{d}s-\sigma(s)\mathrm{d}M(s) \\ &\quad +\frac{1}{2}\sigma^2(s)\mathrm{d}\langle M\rangle(s)-\int_0^t\int_{\mathbb{Z}}\ln(1+c(s,z))\tilde{N}(\mathrm{d}s,\mathrm{d}z)\Big] \end{aligned}$$

is a fundamental solution of Eq. (2.2). It is then easy to see by Itô's formula that

$$y(t)=\phi(t)\left(y(0)+\int_0^t\phi^{-1}(s)b(s)\mathrm{d}s\right)$$

solves (2.1). Therefore,

$$x(t)=\frac{1}{y(t)}=\frac{\phi^{-1}(t)}{1/x(0)+\int_0^t\phi^{-1}(s)b(s)\mathrm{d}s}$$

is a positive solution of Eq. (1.3) and global on $t\in[0,\infty)$ (i.e. $\tau_e=\infty$). This completes the proof. □

Remark 2.2 *If $b(t)<0$, for $x(0)>0$, Eq. (1.3) has only the local solution*

$$x(t)=\frac{\phi^{-1}(t)}{1/x(0)-\int_0^t\phi^{-1}(s)|b(s)|\mathrm{d}s},\quad 0\le t<\tau_e,$$

where τ_e is the exploding time:

$$\tau_e=\inf\big\{t\ge 0,\quad \frac{1}{x(0)}=\int_0^t\phi^{-1}(s)|b(s)|\mathrm{d}s\big\}.$$

This means that in this case the general noise and the jump noise cannot suppress the potential explosion.

For later applications, we give several lemmas. First we give the strong law of large numbers for local martingales.

Lemma 2.3 *[15] Let $M(t)$, $t\ge 0$, be a local martingale with $M(0)=0$. Then*

$$\lim_{t\to\infty}\frac{M(t)}{t}=0 \text{ a.s. provided that } \lim_{t\to\infty}\rho_M(t)<\infty \text{ a.s.}$$

where

$$\rho_M(t)=\int_0^t\frac{\mathrm{d}\langle M\rangle(s)}{(1+s)^2},\ t\ge 0.$$

Lemma 2.4 *[21] Suppose that $x(t) \in C[\Omega \times \mathbb{R}_+, \mathbb{R}_+]$ and $\lim_{t\to\infty} \tau(t) = 0$. If there exist 2 positive constants T and λ_0, for all $t \geq T$ such that*

$$\ln x(t) \leq \lambda t + \tau(t)t - \lambda_0 \int_0^t x(s)\mathrm{d}s,$$

then $\overline{x}^* \leq \lambda/\lambda_0$, when $\lambda \geq 0$, and $\lim_{t\to\infty} x(t) = 0$, when $\lambda < 0$.

Lemma 2.5 *Let $f: [0,\infty) \to \mathbb{R}$ and $h: [0,\infty) \times \mathbb{Z} \to \mathbb{R}$ be both predictable $\{\mathcal{F}_t\}$-adapted processes such that for any $T > 0$*

$$\int_0^T |f(t)|^2 \mathrm{d}\langle M\rangle(t) < \infty \ a.s. \ and \ \int_0^T \int_{\mathbb{Z}} |h(t,z)|^2 \pi(\mathrm{d}z)\mathrm{d}t < \infty \quad a.s.$$

Then, for any constants $\alpha, \beta > 0$,

$$\mathbb{P}\Big\{ \sup_{0\leq t\leq T} \ \Big[\int_0^t f(s)\mathrm{d}M(s) - \frac{\alpha}{2}\int_0^t |f(s)|^2 \mathrm{d}\langle M\rangle(s) + \int_0^t \int_{\mathbb{Z}} h(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)$$
$$-\frac{1}{\alpha}\int_0^t \int_{\mathbb{Z}} [e^{\alpha h(s,z)} - 1 - \alpha h(s,z)]\pi(\mathrm{d}z)\mathrm{d}s\Big] > \beta\Big\} \leq e^{-\alpha\beta}.$$

Proof This part is motivated by [1]. For every integer $k \geq 1$, define the stopping time

$$\tau_k = \inf \ \Big\{t \geq 0: \frac{\alpha}{2}\int_0^t \Big|f(s)\Big|^2 \mathrm{d}\langle M\rangle(s) + \Big|\int_0^t \int_{\mathbb{Z}} h(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\Big|$$
$$+\Big|\int_0^t f(s)\mathrm{d}M(s)\Big| + \frac{1}{\alpha}\Big|\int_0^t \int_{\mathbb{Z}} [e^{\alpha h(s,z)} - 1 - \alpha h(s,z)]\pi(\mathrm{d}z)\mathrm{d}s\Big| \geq k\Big\}$$

and the Itô process

$$\begin{aligned} x_k(t) &= \alpha \int_0^{t\wedge\tau_k} f(s)\mathrm{d}M(s) - \frac{\alpha^2}{2}\int_0^{t\wedge\tau_k} |f(s)|^2 \mathrm{d}\langle M\rangle(s) \\ &\quad +\alpha \int_0^{t\wedge\tau_k} \int_{\mathbb{Z}} h(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &\quad -\int_0^{t\wedge\tau_k} \int_{\mathbb{Z}} \big[e^{\alpha h(s,z)} - 1 - \alpha h(s,z)\big]\pi(\mathrm{d}z)\mathrm{d}s. \end{aligned}$$

From the definition of τ_k, $x_k(t)$ is bounded and $\tau_k \uparrow \infty$ a.s. when $k \to \infty$. Applying Itô's formula to $\exp[x_k(t)]$, we see that

$$\begin{aligned} e^{x_k(t)} &= 1 + \alpha \int_0^{t\wedge\tau_k} e^{x_k(s)} f(s)\mathrm{d}M(s) \\ &\quad + \int_0^{t\wedge\tau_k} \int_{\mathbb{Z}} e^{x_k(s)}(e^{\alpha h(s,z)} - 1)\widetilde{N}(\mathrm{d}s,\mathrm{d}z). \end{aligned}$$

This equality implies that $\exp[x_k(t)]$ is a local martingale with $\mathbb{E}(\exp[x_k(t)]) = 1$, so $\exp[x_k(t)]$ is a martingale (see Theorem 5.2.4 in [1]). By virtue of Doob's martingale inequality [22], we have

$$\mathbb{P}\left(\sup_{0\leq t\leq T} \exp[x_k(t)] > e^{\alpha\beta}\right) \leq e^{-\alpha\beta}\mathbb{E}(\exp[x_k(T)]) = e^{-\alpha\beta}.$$

This is equivalent to

$$\mathbb{P}\Big\{\sup_{0\le t\le T}\quad \Big[\int_0^{t\wedge\tau_k} f(s)\mathrm{d}M(s)-\frac{\alpha}{2}\int_0^{t\wedge\tau_k}|f(s)|^2\mathrm{d}\langle M\rangle(s)$$
$$-\frac{1}{\alpha}\int_0^{t\wedge\tau_k}\int_{\mathbb{Z}}[e^{\alpha h(s,z)}-1-\alpha h(s,z)]\pi(\mathrm{d}z)\mathrm{d}s$$
$$+\int_0^{t\wedge\tau_k}\int_{\mathbb{Z}} h(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\Big]>\beta\Big\}.\le e^{-\alpha\beta}$$

Letting $k\to\infty$, we arrive at our desired result. This completes the proof. □

3. Persistence and extinction

Theorem 2.1 shows that Eq. (1.3) has a unique global positive solution. However, from the biological point of view, the nonexplosion property and positivity in a population dynamical system are often not good enough. In this section, we will discuss in detail how the positive solutions of Eq. (1.3) vary on $\mathbb{R}_+$. First, we give several definitions, and then we try to derive sufficient conditions for them.

Definition 3.1 *[17] Let $x(t)$ be a solution of Eq. (1.3):*

a. *if $\lim_{t\to+\infty} x(t)=0$, the species modeled by (1.3) is said to be extinct.*

b. *if $\lim_{t\to+\infty}\overline{x(t)}=0$, the species modeled by (1.3) is said to be nonpersistent in the mean.*

c. *if $\overline{x(t)}^*>0$, the species modeled by (1.3) is said to be weakly persistent in the mean.*

d. *if $\overline{x(t)}_*>0$, the species modeled by (1.3) is said to be strongly persistent in the mean.*

From these definitions we can see that extinction implies nonpersistence in the mean; strong persistence in the mean implies weak persistence in the mean. We will discuss these one by one. In the sequel, let $K>0$ be a generic constant whose value may vary for its different appearances.

For later discussions, we make the following assumptions.

Assumption 3.2 $\lim_{t\to\infty}\int_0^t \frac{\mathrm{d}\langle M\rangle(s)}{(1+s)^2}<\infty$.

Assumption 3.3 *There is a constant $K>0$ such that*

$$\int_{\mathbb{Z}}[\ln(1+c(t,z))]^2\pi(\mathrm{d}z)\le K,\quad for\ \ all\ \ t\ge 0.$$

Theorem 3.4 *Let Assumptions 3.2 and 3.3 hold. Then the solution $x(t)$ with $x(0)>0$ of system (1.3) satisfies*

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t}\le h^*.$$

In particular, if $h^*<0$, then species $x(t)$ modeled by (1.3) will go to extinction a.s., where

$$h(t)=\frac{\int_0^t r(s)\mathrm{d}s-\frac{1}{2}\int_0^t\sigma^2(s)\mathrm{d}\langle M\rangle(s)}{t},\tag{3.1}$$

$$r(t) = a(t) - \int_{\mathbb{Z}} [c(t,z) - \ln(1 + c(t,z))]\pi(\mathrm{d}z). \tag{3.2}$$

Proof Applying Itô's formula to $\ln x(t)$, we deduce that

$$\begin{aligned}
\mathrm{d}\ln x(t) &= \big[a(t) - b(t)x(t)\big]\mathrm{d}t + \sigma(t)\mathrm{d}M(t) - \frac{1}{2}\sigma^2(t)\mathrm{d}\langle M\rangle(t) \\
&\quad + \int_{\mathbb{Z}} \big[\ln(x(t) + x(t)c(t,z)) - \ln x(t)\big]\widetilde{N}(\mathrm{d}t, \mathrm{d}z) \\
&\quad + \int_{\mathbb{Z}} \big[\ln(x(t) + x(t)c(t,z)) - \ln x(t) - c(t,z)\big]\pi(\mathrm{d}z)\mathrm{d}t \\
&= \big[r(t) - b(t)x(t)\big]\mathrm{d}t + \sigma(t)\mathrm{d}M(t) - \frac{1}{2}\sigma^2(t)\mathrm{d}\langle M\rangle(t) \\
&\quad + \int_{\mathbb{Z}} \ln(1 + c(t,z))\widetilde{N}(\mathrm{d}t, \mathrm{d}z).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\ln x(t) &= \ln x(0) + \int_0^t r(s)\mathrm{d}s - \frac{1}{2}\int_0^t \sigma^2(s)\mathrm{d}\langle M\rangle(s) + \int_0^t \sigma(s)\mathrm{d}M(s) \\
&\quad - \int_0^t b(s)x(s)\mathrm{d}s + \int_0^t \int_{\mathbb{Z}} \ln(1 + c(s,z))\widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&\leq \ln x(0) + \int_0^t r(s)\mathrm{d}s - \frac{1}{2}\int_0^t \sigma^2(s)\mathrm{d}\langle M\rangle(s) + \int_0^t \sigma(s)\mathrm{d}M(s) \\
&\quad + \int_0^t \int_{\mathbb{Z}} \ln(1 + c(s,z))\widetilde{N}(\mathrm{d}s, \mathrm{d}z) \qquad (3.3) \\
&:= \ln x(0) + \int_0^t r(s)\mathrm{d}s - \frac{1}{2}\int_0^t \sigma^2(s)\mathrm{d}\langle M\rangle(s) + w_1(t) + w_2(t),
\end{aligned}$$

where $w_1(t) = \int_0^t \sigma(s)\mathrm{d}M(s)$ and $w_2(t) = \int_0^t \int_{\mathbb{Z}} \ln(1 + c(s,z))\widetilde{N}(\mathrm{d}s, \mathrm{d}z)$ are local martingales. Their quadratic variations are $\langle w_1\rangle(t) = \int_0^t \sigma^2(s)\mathrm{d}\langle M\rangle(s)$ and $\langle w_2\rangle(t) = \int_0^t \int_{\mathbb{Z}} \ln^2(1 + c(s,z))\pi(\mathrm{d}z)\mathrm{d}s$, respectively. Note that, by Assumption 3.2,

$$\lim_{t\to+\infty} \rho_{w_1}(t) = \lim_{t\to+\infty} \int_0^t \frac{\sigma^2(s)\mathrm{d}\langle M\rangle(s)}{(1+s)^2} \leq (\check{\sigma})^2 \lim_{t\to+\infty} \int_0^t \frac{d\langle M\rangle(s)}{(1+s)^2} < \infty.$$

By Lemma 2.3, we obtain

$$\lim_{t\to+\infty} \frac{w_1(t)}{t} = 0, \quad a.s. \tag{3.4}$$

On the other hand, by Assumption 3.3,

$$\mathrm{d}\langle w_2\rangle(t) \leq \int_{\mathbb{Z}} \big[\ln(1 + c(t,z))\big]^2 \pi(\mathrm{d}z)\mathrm{d}t \leq K\mathrm{d}t.$$

Making use of Lemma 2.3, we get

$$\lim_{t\to\infty} \frac{w_2(t)}{t} = 0, \quad a.s. \tag{3.5}$$

Taking the superior limit for (3.3) and using (3.4) and (3.5), we find

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \le h^* \quad \text{a.s.},$$

which is our desired assertion. This completes the proof. □

Remark 3.5 *Theorem 3.4 presents sufficient conditions for the extinction of system (1.3). It reveals the fact that the jump noise and the general noise can make the population extinct. In particular, the effect of the jump noise on the model is based on the concavity of the logarithm function.*

Remark 3.6 *It is evident that $x(t) \equiv 0$ is the trivial solution of system (1.3). Theorem 3.4 also demonstrates that the trivial solution is almost surely exponentially stable if $h^* < 0$.*

Theorem 3.7 *Under Assumptions (3.2) and (3.3), if $h^* = 0$, then species modeled by (1.3) will be nonpersistent in the mean a.s.*

Proof By the limit properties, for any $\varepsilon > 0$, there exists a constant $T > 0$ such that

$$w_1(t)/t < \varepsilon/3, \quad w_2(t)/t < \varepsilon/3, \quad h(t) < h^* + \varepsilon/3 = \varepsilon/3,$$

for $t > T$. Substituting above inequalities into (3.3), we obtain

$$\ln x(t) \le \ln x(0) + \varepsilon t - \int_0^t b(s)x(s)\mathrm{d}s \le \ln x(0) + \varepsilon t - \hat{b}\int_0^t x(s)\mathrm{d}s, \quad t > T.$$

Making use of Lemma 2.4, we follow that $\overline{x}^* \le \varepsilon/\hat{b}$. By the arbitrariness of ε, we get our result. □

In order to get sufficient conditions for weak persistence in the mean, we need the following theorem, which shows that the total population of the ecosystem cannot grow too fast.

Theorem 3.8 *Let Assumptions (3.2) and (3.3) hold. For any initial value $x(0) > 0$, the solution $x(t)$, $t \ge 0$, of Eq.(1.3) has the property*

$$\limsup_{t\to\infty} \frac{\ln x(t)}{\ln t} \le 1.$$

Proof Our proof is motivated by Zhu and Yin [26]. Applying Itô's formula to $[e^t \ln x(t)]$ leads to

$$\begin{aligned} e^t \ln x(t) \;&=\; \ln x(0) + \int_0^t e^s \ln x(s)\mathrm{d}s + \int_0^t e^s(r(s) - b(s)x(s))\mathrm{d}s + \int_0^t e^s\sigma(s)\mathrm{d}M(s) \\ &\quad -\frac{1}{2}\int_0^t e^s\sigma^2(s)\mathrm{d}\langle M\rangle(s) + \int_0^t\int_{\mathbb{Z}} e^s \ln(1 + c(s,z))\widetilde{N}(\mathrm{d}s,\mathrm{d}z). \end{aligned} \tag{3.6}$$

Let $w_3(t) = \int_0^t e^s\sigma(s)\mathrm{d}M(s)$, $w_4(t) = \int_0^t\int_{\mathbb{Z}} e^s \ln(1 + c(s,z))\widetilde{N}(\mathrm{d}s,\mathrm{d}z)$, and then $\langle w_3\rangle(t) = \int_0^t e^{2s}\sigma^2(s)\mathrm{d}\langle M\rangle(s)$. Choose $T = k\gamma$, $\alpha = \varepsilon e^{-k\gamma}$, $\beta = (\theta e^{k\gamma}\ln k)/\varepsilon$, where $k \in N$, $0 < \varepsilon < 1$, $\theta > 1$ and $\gamma > 0$. By Lemma 2.5, we deduce that

$$\mathbb{P}\left\{\sup_{0\le t\le k\gamma}\left[w_3(t)-\frac{\varepsilon e^{-k\gamma}}{2}\langle w_3\rangle(t)+w_4(t)-\frac{e^{k\gamma}}{\varepsilon}\int_0^t\int_{\mathbb{Z}}[(1+c(s,z))^{\varepsilon e^{s-k\gamma}}-1\right.\right.$$
$$\left.\left.-\varepsilon e^{s-k\gamma}\ln(1+c(s,z))]\pi(\mathrm{d}z)\mathrm{d}s\right]>(\theta e^{k\gamma}\ln k)/\varepsilon\right\}\le k^{-\theta}. \tag{3.7}$$

By the Borel–Cantelli lemma, for almost all $\omega\in\Omega$, there is an integer $k_0=k_0(\omega)$ such that for $k\ge k_0$

$$w_3(t)-\frac{\varepsilon e^{-k\gamma}}{2}\langle w_3\rangle(t)+w_4(t)-\frac{e^{k\gamma}}{\varepsilon}\int_0^t\int_{\mathbb{Z}}[(1+c(s,z))^{\varepsilon e^{s-k\gamma}}-1$$
$$-\varepsilon e^{s-k\gamma}\ln(1+c(s,z))]\pi(\mathrm{d}z)\mathrm{d}s\le(\theta e^{k\gamma}\ln k)/\varepsilon,\quad \mathit{for}\quad \mathit{every}\quad t\in[0,k\gamma].$$

Substituting the above inequality into (3.6), we see that

$$\begin{aligned}e^t\ln x(t)\quad\le\quad&\ln x(0)+\int_0^t e^s\ln x(s)\mathrm{d}s+\int_0^t e^s(r(s)-b(s)x(s))\mathrm{d}s\\&+\frac{\varepsilon e^{-k\gamma}}{2}\int_0^t e^{2s}\sigma^2(s)\mathrm{d}\langle M\rangle(s)-\frac{1}{2}\int_0^t e^s\sigma^2(s)\mathrm{d}\langle M\rangle(s)+\frac{\theta e^{k\gamma}\ln k}{\varepsilon}\\&+\frac{e^{k\gamma}}{\varepsilon}\int_0^t\int_{\mathbb{Z}}[(1+c(s,z))^{\varepsilon e^{s-k\gamma}}-1-\varepsilon e^{s-k\gamma}\ln(1+c(s,z))]\pi(\mathrm{d}z)\mathrm{d}s.\end{aligned}\tag{3.8}$$

Note that

$$\varepsilon e^{-k\gamma}e^{2s}\sigma^2(s)-e^s\sigma^2(s)=e^s\sigma^2(s)(\varepsilon e^{-k\gamma}e^s-1)\le e^s\sigma^2(s)(\varepsilon e^{-k\gamma}e^{k\gamma}-1)<0,$$

and

$$(1+c(s,z))^{\varepsilon e^{s-k\gamma}}-1-\varepsilon e^{s-k\gamma}\ln(1+c(s,z))\le\varepsilon e^{s-k\gamma}(c(s,z)-\ln(1+c(s,z))),$$

where in the second inequality, we use the inequality $x^r\le 1+r(x-1),\quad x\ge 0, 1\ge r\ge 0$. Therefore, (3.8) changes into

$$\begin{aligned}e^t\ln x(t)\quad\le\quad&\ln x(0)+\int_0^t e^s[\ln x(s)+r(s)-b(s)x(s)\\&+\int_{\mathbb{Z}}(c(s,z)-\ln(1+c(s,z)))\pi(\mathrm{d}z)]\mathrm{d}s+\frac{\theta e^{k\gamma}\ln k}{\varepsilon}.\end{aligned}$$

Since $\hat{b}>0$, we claim that for almost all $0\le s\le k\gamma$ and $x(s)>0$, there exists a constant K such that

$$\ln x(s)+r(s)-b(s)x(s)+\int_{\mathbb{Z}}(c(s,z)-\ln(1+c(s,z)))\pi(\mathrm{d}z)\le K.$$

Thus,

$$e^t\ln x(t)\le\ln x(0)+Ke^t+\frac{\theta e^{k\gamma}\ln k}{\varepsilon}.$$

So, for $(k-1)\gamma \le t \le k\gamma$, we have

$$\ln x(t) \quad \le \quad e^{-t}\ln x(0) + K + \frac{\theta e^{\gamma}\ln k}{\varepsilon}.$$

Furthermore, we obtain

$$\frac{\ln x(t)}{\ln t} \le \frac{\ln x(0)}{e^{t}\ln t} + \frac{K}{\ln t} + \frac{\theta e^{\gamma}\ln k}{\varepsilon \ln(k-1)\gamma}.$$

Letting $k \to \infty$ (and so $t \to \infty$), we deduce that

$$\limsup_{t\to\infty}\frac{\ln x(t)}{\ln t} \le \frac{\theta e^{\gamma}}{\varepsilon}.$$

Letting $\gamma \downarrow 0$, $\theta \downarrow 1$, and $\varepsilon \uparrow 1$, we can get our desired assertion. This completes the proof. □

By the fact that $\lim_{t\to\infty}(\ln t)/t = 0$, we can easily deduce from Lemma 3.8 that the sample Lyapunov exponent of Eq. (1.3) is less than or equal to zero, which is stated as the following corollary.

Corollary 3.9 *Under conditions of Theorem 3.8, the solution $x(t)$ of Eq. (1.3) obeys*

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t} \le 0 \quad \text{a.s.} \tag{3.9}$$

Theorem 3.10 *Let Assumptions 3.2 and 3.3 hold. If $h^* > 0$, then species $x(t)$ modeled by (1.3) is weakly persistent in the mean a.s.*

Proof Suppose that $\overline{x}^* > 0$ is not true; then $\mathbb{P}(E) > 0$, where $E = \{\overline{x}^* = 0\}$. By (3.3), we have

$$t^{-1}[\ln x(t) - \ln x(0)] \quad = \quad h(t) - \overline{b(t)x(t)} + w_1(t)/t + w_2(t)/t.$$

Note that for $\omega \in E$, $\overline{b(t)x(t,\omega)}^* = 0$, combining (3.4) and (3.5), we conclude that $[t^{-1}\ln x(t)]^* = h^* > 0$. So $\mathbb{P}\{[t^{-1}\ln x(t)]^* > 0\} > 0$, which contradicts (3.9). □

We conclude this section with a stronger property: strong persistence in the mean.

Theorem 3.11 *Let Assumptions 3.2 and 3.3 hold. If $h_* > 0$, then species $x(t)$ modeled in (1.3) is strongly persistent in the mean a.s., where $h(t)$ is defined by (3.1).*

Proof By (3.3), we have

$$\frac{1}{t}\int_0^t b(s)x(s)\mathrm{d}s \quad = \quad -\frac{\ln x(t) - \ln x(0)}{t} + \frac{w_1(t)}{t} + \frac{w_2(t)}{t} + h(t).$$

Making use of (3.4), (3.5), and (3.9), we see that

$$\check{b}\overline{x}_* \ge -[t^{-1}\ln x(t)]^* + h_* \ge h_*,$$

which is our desired assertion. □

Remark 3.12 *Theorems 3.4, 3.7, 3.10, and 3.11 have an interesting biological interpretation. It is easy to see that the extinction and persistence of species $x(t)$ modeled by (1.3) depend only on the values of h^* and h_*. In other words, the general noise and the jump noise can adjust and control the population size.*

4. Stochastically ultimate boundedness

In this section, we will consider stochastically ultimate boundedness, which is interesting from the biological point of view. We first give its definition.

Definition 4.1 *[13] A solution $x(t)$ of Eq. (1.3) is said to be stochastically ultimate bounded if $\forall\epsilon \in (0,1)$, $\exists H = H_\epsilon > 0$ such that*

$$\limsup_{t\to+\infty} \mathbb{P}[|x(t)| > H] < \epsilon,$$

for any initial value $x(0) > 0$.

To discuss stochastically ultimate boundedness, we first examine the pth moment boundedness.

Theorem 4.2 *Let Assumptions 3.2 and 3.3 hold. For any $p \in (0,1)$, there is a constant $K > 0$ such that*

$$\limsup_{t\to\infty} \mathbb{E}|x(t)|^p \le K.$$

Proof For $p \in (0,1)$, $x \in \mathbb{R}_+$, define a C^2-function $V(x) = x^p$. Applying Itô's formula leads to

$$\begin{aligned} \mathrm{d}V(x(t)) &= px^p\big(a(t)-b(t)x\big)\mathrm{d}t + px^p\sigma(t)\mathrm{d}M(t) + \frac{1}{2}p(p-1)x^p\sigma^2(t)\mathrm{d}\langle M\rangle(t) \\ &\quad + x^p\int_{\mathbb{Z}}\big[(1+c(t,z))^p - 1\big]\widetilde{N}(\mathrm{d}t,\mathrm{d}z) \\ &\quad + x^p\int_{\mathbb{Z}}\big[(1+c(t,z))^p - 1 - pc(t,z)\big]\pi(\mathrm{d}z)\mathrm{d}t. \end{aligned}$$

Further, for $p \in (0,1)$, we have

$$\mathbb{E}\big[e^tV(x(t))\big] \le x^p(0) + \mathbb{E}\int_0^t e^s x^p(s)L(x(s),s)\mathrm{d}s$$

where

$$\begin{aligned} L(x,t) &= 1 + p(a(t)-b(t)x) + \int_{\mathbb{Z}}\big[(1+c(t,z))^p - 1 - pc(t,z)\big]\pi(\mathrm{d}z) \\ &\le 1 + p(a(t)-b(t)x) \le 1 + p(\check{a} - \hat{b}x). \end{aligned}$$

Note that $x^pL(x(t),t) \le x^p[1+p(\check{a}-\hat{b}x)]$. Since the leading coefficient of $x^p[1+p(\check{a}-\hat{b}x)]$ is less than zero, there exists a constant $K > 0$ such that

$$x^pL(x,t) \le K.$$

Therefore,

$$\mathbb{E}\big[e^tV(x(t))\big] \le x^p(0) + \int_0^t Ke^s\mathrm{d}s = x^p(0) + K(e^t - 1).$$

Dividing both sides by e^t and taking the limit supremum, we get

$$\limsup_{t\to\infty} \mathbb{E}|x(t)|^p \le K.$$

This completes the proof. □

As an application of Theorem 4.2, together with Chebyshev's inequality, we can deduce the stochastically ultimate boundedness of $x(t)$. We state it as a theorem below.

Theorem 4.3 *Under conditions of Theorem 4.2, the solution $x(t)$ of Eq. (1.3) is stochastically ultimately bounded.*

Proof For any $\epsilon \in (0,1)$, let $H = (K/\epsilon)^{1/p}$. By Chebyshev's inequality, we have

$$\mathbb{P}[|x(t)| > H] \leq \frac{\mathbb{E}|x(t)|^p}{H^p}.$$

Hence

$$\limsup_{t\to+\infty} \mathbb{P}[|x(t)| > H] \leq \epsilon,$$

as required. □

5. Examples and numerical simulations

In this section, we give 2 examples and numerical simulations to illustrate our results.

Example 1 *Consider the following autonomous equation:*

$$\mathrm{d}x(t) = x(t^-)\Big[\big(a - bx(t^-)\big)\mathrm{d}t + \sigma \mathrm{d}M(t) + \int_{\mathbb{Z}} c(z)\tilde{N}(\mathrm{d}t, \mathrm{d}z)\Big].$$

Choose the initial datum $x(0) = 0.3$. The parameters are chosen as follows: $a = 0.05$, $b = 0.05$, $\sigma = 0.5$, $c(z) = 0.5$, $\mathbb{Z} = (0,\infty)$, $\pi(\mathbb{Z}) = 1$, $M_t = B_t^2 - t$. By computation, we have

$$\begin{aligned} h(t) &= \frac{\int_0^t r(s)\mathrm{d}s - \frac{1}{2}\int_0^t \sigma^2(s)\mathrm{d}\langle M\rangle(s)}{t} \\ &\leq \frac{\int_0^t [a(s) - \int_{\mathbb{Z}}(c(s,z) - \ln(1 + c(s,z)))\pi(\mathrm{d}z)]\mathrm{d}s}{t} \cong -0.04. \end{aligned}$$

Therefore, $h^* < 0$, by Theorem 3.4, and species $x(t)$ will go into extinction. Figure 1 confirms this.

Example 2 *Now consider the following equation:*

$$\mathrm{d}x(t) = x(t^-)\Big[\big(a(t) - b(t)x(t^-)\big)\mathrm{d}t + \sigma(t)\mathrm{d}M_t + \int_{\mathbb{Z}} c(t,z)\tilde{N}(\mathrm{d}t, \mathrm{d}z)\Big].$$

Choose the initial datum $x(0) = 0.5$ with the following choice of parameters: $a(t) = 0.5 + 0.1\sin t$, $b(t) = 0.5 + 0.2\sin t$, $\sigma^2(t) = 0.9 + 0.8\sin t$, $c(t,z) = -0.2$, $\mathbb{Z} = (0,\infty)$, $\pi(\mathbb{Z}) = 1$, $M_t = B_t$. We can see that $h(t) = 0.03 > 0$, by Theorem 3.11, and species $x(t)$ is strongly persistent in the mean. Figure 2 illustrates this.

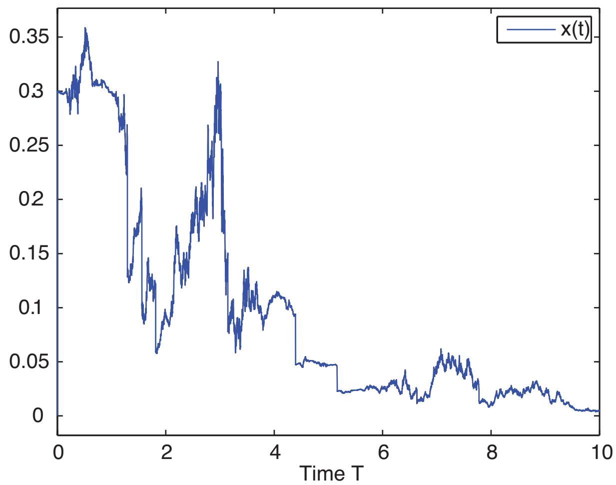

Figure 1.

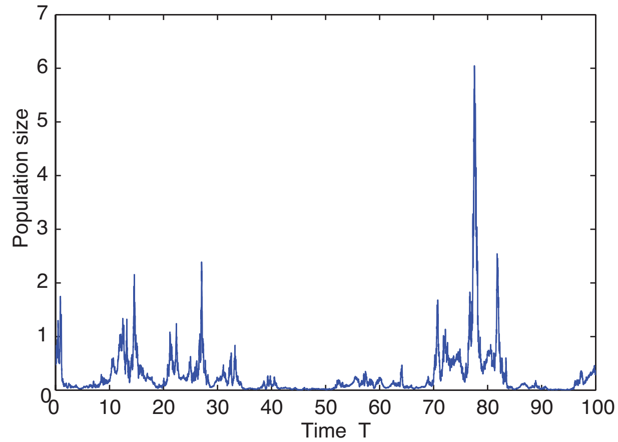

Figure 2.

6. Conclusions and further directions

This paper is concerned with a stochastic logistic system driven by martingales with jumps. The asymptotic properties of positive solutions are examined. Our key contributions are the following.

(a) In the model, the martingale and the jump noise are introduced at the same time. This is a new research subject. The effects of the martingale and jump noise on the model are analyzed.

(b) The critical value of extinction, nonpersistence in the mean, and weak persistence in the mean are obtained. Our results show that the extinction and persistence of the species depend only on h^* and h_*. In other words, the martingale and jump noise affect the extinction and persistence of the population.

(c) The asymptotic path-wise estimation (Theorem 3.8 and Corollary 3.9) and the stochastically ultimate boundedness (Theorem 4.3) are considered.

Some interesting topics deserve further investigation. One may investigate the stochastic permanence, which is a more important and difficult subject, and this is also a problem that we have tried and will continue to study. Moreover, one may consider n-species population systems and talk about their dynamics. In the study of dynamics, the terms with quadratic variation and joint quadratic variation for high-dimensional systems are more difficult to manage, and this demonstrates that the stochastic models with martingales are more difficult to study than with white noise.

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