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$\xi^\perp$-submanifolds of para-Sasakian manifolds

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Abstract: Almost semiinvariant $\xi^\perp$-submanifolds of an almost paracontact metric manifold are defined and studied. Some characterizations of almost semiinvariant $\xi^\perp$-submanifolds and semiinvariant $\xi^\perp$-submanifolds are presented. A para-*CR*-structure is defined and it is proven that an almost semiinvariant $\xi^\perp$-submanifold of a normal almost paracontact metric (and hence para-Sasakian) manifold with the proper invariant distribution always possesses a para-*CR*-structure. A counter example is also given. Integrability conditions for certain natural distributions arising on almost semiinvariant $\xi^\perp$-submanifolds are obtained. Finally, certain parallel operators on submanifolds are investigated.

Key words: Almost paracontact metric manifold, para-Sasakian manifold, $\xi^\perp$-submanifold, almost semiinvariant $\xi^\perp$-submanifolds, para-*CR*-structure.

1. Introduction

The theory of almost paracontact structures on Riemannian manifolds was introduced by Sato [11, 12]. Since then, many authors contributed to the study of almost paracontact metric manifolds and their submanifolds. Specifically, several authors studied antiinvariant, semiinvariant, and almost semiinvariant submanifolds of para-Sasakian manifolds [3, 4, 5, 6, 8, 9]. However, it is known that [18] in a submanifold of a para-Sasakian manifold, if the structure vector field of the ambient manifold is tangent to the submanifold, then the submanifold cannot admit an antiinvariant distribution orthogonal to the structure vector field (see also [13]). Knowing the fact that in these submanifolds, the structure vector field of the ambient manifold is taken to be tangent to submanifolds, in this paper we study $\xi^\perp$-submanifolds of para-Sasakian manifolds, where the $\xi^\perp$-submanifolds are perpendicular to the structure vector field of the ambient manifold.

The paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3, some fundamental formulas concerning $\xi^\perp$-submanifolds of almost paracontact metric manifolds and para-Sasakian manifolds have been presented. In Section 4, we give the definition of the almost semiinvariant $\xi^\perp$-submanifold of an almost paracontact metric manifold along with some examples. Section 5 contains some characterizations of almost semiinvariant $\xi^\perp$-submanifolds and semiinvariant $\xi^\perp$-submanifolds. In Section 6, we define a para-*CR*-structure and prove that an almost semiinvariant $\xi^\perp$-submanifold of a normal almost paracontact metric (and

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hence also para-Sasakian) manifold with proper invariant distribution always possesses a para-CR-structure. A counterexample is also given. In Section 7, integrability conditions for certain natural distributions on almost semiinvariant $\xi^{\perp}$-submanifolds are obtained. In Section 8, we investigate certain parallel operators on submanifolds.

2. Preliminaries

Let $\widetilde{M}$ be an almost paracontact metric manifold [11] equipped with an almost paracontact metric structure (φ, ξ, η, g); that is, φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form, and g is an associated Riemannian metric such that

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$\Phi(X, Y) \equiv g(X, \varphi Y) = \Phi(Y, X), \quad g(X, \xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in T\widetilde{M}$. An almost paracontact metric structure is known to be a para-Sasakian structure if [10, 12]

$$(\widetilde{\nabla}_X \varphi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.4}$$

where $\widetilde{\nabla}$ is the Riemannian connection on $\widetilde{M}$, and we say that $\widetilde{M}$ is a para-Sasakian manifold.

Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ with a Riemannian metric g. Then Gauss and Weingarten formulae are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \qquad X, Y \in TM, \tag{2.5}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \qquad N \in T^{\perp}M, \tag{2.6}$$

where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian, and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M, respectively, and σ is the second fundamental form related to A by

$$g(\sigma(X, Y), N) = g(A_N X, Y). \tag{2.7}$$

Let M be a submanifold of an almost paracontact metric manifold $\widetilde{M}$. Let $X, Y \in TM$, $N \in T^{\perp}M$. We put

$$\varphi X \equiv PX + FX, \qquad PX \in TM, \; FX \in T^{\perp}M, \tag{2.8}$$

$$\varphi N \equiv tN + fN, \qquad tN \in TM, \; fN \in T^{\perp}M, \tag{2.9}$$

and then

$$\begin{aligned}(\widetilde{\nabla}_X \varphi)Y \;&=\; ((\nabla_X P)Y - A_{FY}X - t\sigma(X, Y)) \\ &\quad + ((\nabla_X F)Y + \sigma(X, PY) - f\sigma(X, Y)),\end{aligned} \tag{2.10}$$

$$\begin{aligned}(\widetilde{\nabla}_X \varphi)N \;&=\; ((\nabla_X t)N - A_{fN}X + PA_N X)) \\ &\quad + ((\nabla_X f)N + \sigma(X, tN) + FA_N X)),\end{aligned} \tag{2.11}$$

where

$$(\nabla_X P)\,Y \equiv \nabla_X PY - P\nabla_X Y, \quad (\nabla_X F)\,Y \equiv \nabla^{\perp}_X FY - F\nabla_X Y,$$
$$(\nabla_X t)\,N \equiv \nabla_X tN - t\nabla^{\perp}_X N, \quad (\nabla_X f)\,N \equiv \nabla^{\perp}_X fN - f\nabla^{\perp}_X N.$$

If $\nabla_X Q = 0$, $Q \in \{P, F, t, f\}$, then Q is said to be *parallel*.

3. Some properties of $\xi^{\perp}$-submanifolds

Definition 3.1 *A submanifold of an almost paracontact metric manifold such that ξ is normal to M is said to be an $\xi^{\perp}$-submanifold.*

From now on, all submanifolds of almost paracontact metric manifolds are assumed to be $\xi^{\perp}$-submanifolds, unless specifically stated otherwise. In this case, $\eta(X) = 0$, for all $X \in TM$.

Proposition 3.2 *Let M be an $\xi^{\perp}$-submanifold of an almost paracontact metric manifold. Then*

$$P^2 + tF = I, \tag{3.1}$$

$$FP + fF = 0, \tag{3.2}$$

$$f^2 + Ft = I - \eta \otimes \xi, \tag{3.3}$$

$$Pt + tf = 0. \tag{3.4}$$

Consequently,

$$\ker P = \ker\left(P^2\right) = \ker\left(tF - I\right), \tag{3.5}$$

$$\ker F = \ker\left(tF\right) = \ker\left(P^2 - I\right), \tag{3.6}$$

$$\ker t = \ker\left(Ft\right) = \ker\left(f^2 - I + \eta \otimes \xi\right), \tag{3.7}$$

$$\ker f = \ker\left(f^2\right) = \ker\left(Ft - I + \eta \otimes \xi\right). \tag{3.8}$$

Proof For $X \in TM$, in $\varphi^2 X = X$, using (2.8) and (2.9), we get

$$\left(P^2 + tF\right)X + \left(FP + fF\right)X = X,$$

from which we get (3.1) and (3.2). Similarly, using (2.8) and (2.9), in $\varphi^2 N = N - \eta\,(N)\,\xi$ for $N \in T^{\perp}M$, we get

$$\left(Pt + tf\right)N + \left(f^2 + Ft\right)N = N - \eta\,(N)\,\xi,$$

which implies (3.3) and (3.4). The remaining part is straightforward. □

Proposition 3.3 *If M is an $\xi^{\perp}$-submanifold of a para-Sasakian manifold, then*

$$(\nabla_X P)Y - A_{FY}X - t\sigma(X, Y) = 0, \tag{3.9}$$

$$(\nabla_X F)Y + \sigma(X, PY) - f\sigma(X, Y) + g(X, Y)\xi = 0, \tag{3.10}$$

$$(\nabla_X t)N - A_{fN}X + PA_N X + \eta(N)X = 0, \tag{3.11}$$

$$(\nabla_X f)N + \sigma(X, tN) + FA_N X = 0, \tag{3.12}$$

$$P[X,Y] = \nabla_X PY - \nabla_Y PX + A_{FX}Y - A_{FY}X, \tag{3.13}$$

$$F[X,Y] = \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX + \sigma(X, PY) - \sigma(PX, Y) \tag{3.14}$$

for all $X, Y \in TM$ *and* $N \in T^{\perp}M$.

Proof Using (2.4), (2.8), and $\eta(Y) = 0$ in (2.10) and equating tangential and normal parts in the resulting equation, we get (3.9) and (3.10), respectively. Similarly, using (2.4) and (2.9) in (2.11) and equating tangential and normal parts, we get (3.11) and (3.12), respectively. Lastly, (3.13) and (3.14) follow from (3.9) and (3.10), respectively. □

Proposition 3.4 *For an* $\xi^{\perp}$*-submanifold* M *of a para-Sasakian manifold* $\widetilde{M}$, *it follows that*

$$-A_\xi X = PX, \tag{3.15}$$

$$\nabla^{\perp}_X \xi = FX, \tag{3.16}$$

$$\eta(\sigma(X,Y)) = g(X, PY), \tag{3.17}$$

$$\eta(H) = -\frac{1}{n}\operatorname{trace}(P) \tag{3.18}$$

for any $X, Y \in TM$, *where* H *is the mean curvature vector.*

Proof From (2.4) it follows that

$$\widetilde{\nabla}_X \xi = \varphi X. \tag{3.19}$$

Using (3.19), (2.8), and $\eta(X) = 0$ in (2.5), we get

$$-A_\xi X + \nabla^{\perp}_X \xi = \varphi X = PX + FX.$$

Equating tangential and normal parts in the above equation we get (3.15) and (3.16), respectively. In view of $(2.3)_2$, (2.7), and (3.15), it follows that

$$\eta(\sigma(X,Y)) = g(\sigma(X,Y), \xi) = g(A_\xi X, Y) = -g(PX, Y),$$

which gives (3.17). If $\{e_1, \ldots, e_n\}$, $n = \dim M$, is a local orthonormal frame field, then in view of (3.17) one gets

$$\eta(H) = \frac{1}{n}\eta\left(\sum_{i=1}^{n}\sigma(e_i, e_i)\right) = -\frac{1}{n}\left(\sum_{i=1}^{m} g(Pe_i, e_i)\right),$$

which gives (3.18). □

In view of (3.18), we have the following:

Corollary 3.5 *Let* M *be an* $\xi^{\perp}$*-submanifold of a para-Sasakian manifold. If* $\operatorname{trace}(P) \neq 0$, *then* M *can not be minimal.*

In view of (3.15), we have the following:

Theorem 3.6 *Let* M *be an* $\xi^{\perp}$*-submanifold of a para-Sasakian manifold. Then* M *is antiinvariant if and only if* $A_\xi = 0$.

4. Almost semiinvariant $\xi^\perp$-submanifolds

Let M be an $\xi^\perp$-submanifold of an almost paracontact metric manifold $\widetilde{M}$. Since $g(X, PY) = g(PX, Y)$, it therefore follows that $(P^2)_x$ is symmetric on T_xM. Hence, its eigenvalues are real and it is diagonalizable. If $X \in T_xM$ is an eigenvector corresponding to an eigenvalue $\mu(x)$ of $(P^2)_x$, then

$$\mu(x) \|X\|^2 = \mu(x) g(X, X) = g\left(P^2X, X\right) = g(PX, PX) = \|PX\|^2,$$

which implies that $\mu(x) \geq 0$. On the other hand, from (2.2) for all $Z \in T\widetilde{M}$, we get $\|\varphi Z\| \leq \|Z\|$ and therefore

$$\mu(x) \|\varphi X\|^2 \leq \mu(x) \|X\|^2 = \|PX\|^2.$$

Since decomposition of φX given by (2.8) is orthogonal, $\mu(x)$ is bounded by 0 and 1. At every point $x \in M$, we may set

$$\mathcal{D}_x^\lambda = \ker(P^2 - \lambda^2(x)I)_x,$$

where $\lambda(x) \in [0, 1]$ is such that $\lambda^2(x)$ is an eigenvalue of $(P^2)_x$. Since $(P^2)_x$ is symmetric and diagonalizable, there is some integer q such that $\lambda_1^2(x), \ldots, \lambda_q^2(x)$ are distinct eigenvalues of $(P^2)_x$ and T_xM can be decomposed as the direct sum

$$T_xM = \mathcal{D}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_x^{\lambda_q}$$

of the mutually orthogonal P-invariant eigenspaces. Note that

$$\mathcal{D}_x^1 = \ker(F_x) = \{X \in T_xM : \|X\| = \|PX\|\},$$

$$\mathcal{D}_x^0 = \ker(P_x) = \{X \in T_xM : \|X\| = \|FX\|\}.$$

Thus, $\mathcal{D}_x^1$ and $\mathcal{D}_x^0$ are the maximal φ-invariant and the maximal φ-anti-invariant subspaces of T_xM, respectively.

Now we define an almost semiinvariant $\xi^\perp$-submanifold of an almost paracontact metric manifold, which is analogous to the definition of an almost semiinvariant $\xi^\perp$-submanifold of an almost contact metric manifold [16].

Definition 4.1 *An $\xi^\perp$-submanifold M of an almost paracontact metric manifold $\widetilde{M}$ is said to be an almost semiinvariant $\xi^\perp$-submanifold of $\widetilde{M}$ if there are k functions $\lambda_1, \ldots, \lambda_k$ defined on M with values in the open interval $(0, 1)$ such that*

(1) *$\lambda_1^2(x), \ldots, \lambda_k^2(x)$ are distinct eigenvalues of P^2 at each $x \in M$ with*

$$T_xM = \mathcal{D}_x^1 \oplus \mathcal{D}_x^0 \oplus \mathcal{D}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_x^{\lambda_k},$$

(2) *the dimensions of $\mathcal{D}_x^1$, $\mathcal{D}_x^0$, $\mathcal{D}_x^{\lambda_1}, \ldots, \mathcal{D}_x^{\lambda_k}$ are independent of $x \in M$.*

In view of condition **(2)** in Definition 4.1 we can define P-invariant mutually orthogonal distributions

$$\mathcal{D}^\lambda = \bigcup_{x \in M} \mathcal{D}_x^\lambda, \qquad \lambda \in \{0, \lambda_1, \ldots, \lambda_k, 1\},$$

on M such that

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \cdots \oplus \mathcal{D}^{\lambda_k}.$$

Moreover, in view of [7] these distributions are differentiable.

If $k = 0$ in Definition 4.1, then it follows that P is an $f(3,-1)$-structure [14] on M and hence $\dim(\mathcal{D}_x^1) = \mathrm{rank}(P_x)$ is independent of $x \in M$ [15]; therefore, $\dim(\mathcal{D}_x^0)$ also does not depend on $x \in M$. Thus, in the special case of $k = 0$, **(1)** implies **(2)** and M is called a *semiinvariant* $\xi^\perp$*-submanifold*. If $k = 0$ and $\mathcal{D}_x^1 = \{0\}$ (resp. $\mathcal{D}_x^0 = \{0\}$), then M becomes an *antiinvariant* (resp. *invariant*) $\xi^\perp$*-submanifold*. If $\mathcal{D}_x^1 = \{0\} = \mathcal{D}_x^0$ and $k = 1$ with and $\lambda_1^2(x)$ is constant, then M may be said to be a θ*-slant submanifold* with the slant angle $\cos\theta = \lambda_1$.

Example 4.2 *We consider the Euclidean space $\mathbb{R}^9$ and denote its points by $x = (x^i)$. Let (e_j), $j = 1, \ldots, 9$, be the natural basis defined by $e^j = \partial/\partial x^j$. We define a vector field ξ by $\xi = \partial/\partial x^9$, a 1-form η by $\eta = dx^9$, and a $(1,1)$ tensor field φ by*

$$\begin{aligned}
\varphi e_1 &= e_2,\ \varphi e_2 = e_1,\ \varphi e_3 = e_8,\ \varphi e_8 = e_3,\\
\varphi e_4 &= \cos\nu(x)e_5 - \sin\nu(x)e_6,\\
\varphi e_5 &= \cos\nu(x)e_4 + \sin\nu(x)e_7,\\
\varphi e_6 &= -\sin\nu(x)e_4 + \cos\nu(x)e_7,\\
\varphi e_7 &= \sin\nu(x)e_5 + \cos\nu(x)e_6,\ \varphi e_9 = 0,
\end{aligned}$$

where $\nu : \mathbb{R}^9 \to (0, \pi/2)$ is a smooth function. Then it is easy to verify that $\mathbb{R}^9$ is an almost paracontact metric manifold with almost paracontact structure (φ, ξ, η) and the canonical associated metric g given by $g(e_i, e_j) = \delta_{ij}$. The submanifold

$$M = \left\{ \left(x^1, \ldots, x^9\right) \in \mathbb{R}^9 \mid x^6, x^7, x^8, x^9 = 0 \right\}$$

of $\mathbb{R}^9$ is an almost semiinvariant $\xi^\perp$-submanifold with

$$\mathcal{D}^1 = \mathrm{Span}\{e_1, e_2\},\ \mathcal{D}^0 = \mathrm{Span}\{e_3\},\ \mathcal{D}^\lambda = \mathrm{Span}\{e_4, e_5\},$$

where for $x \in M$ one has $\lambda(x) = \cos\nu(x)$.

Example 4.3 *Let $\widetilde{M} = \{(x^1, \ldots, x^5, y^1, \ldots, y^5, t) \in \mathbb{R}^{11} : x^i, y^j, t \in \mathbb{R}, i, j = 1, ..., 5\}$ be an 11-dimensional manifold with the usual Euclidean metric g, where $(x^1, \ldots, x^5, y^1, \ldots, y^5, t)$ are standard coordinates of $\mathbb{R}^{11}$. Define a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η on $\widetilde{M}$ by*

$$\varphi\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i}, \qquad \varphi\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial y^j}, \qquad \varphi\left(\frac{\partial}{\partial t}\right) = 0,$$

$$\xi = \frac{\partial}{\partial t}, \qquad \eta = dt,$$

where $i,j = 1,2,3,4,5$. It is easy to see that $\widetilde{M}$ is an almost paracontact metric manifold with the almost paracontact metric structure (φ,ξ,η,g). Now let us consider the submanifold M of the almost paracontact metric manifold $\widetilde{M}$ given by

$$\begin{aligned}\iota(u^1,\dots,u^6) &= \big(u^1+u^3, ku^1+u^4, ku^2+k\sin u^5, u^2-k\cos u^6,\\ &\quad -k\cos u^5+k\sin u^6, u^5, u^6, u^3, 0, u^4, 0\big),\end{aligned}$$

where u^i, $i\in\{1,\dots,6\}$, is a real parameter with $u^i\neq 0,\frac{\pi}{2}$ and $k\in\mathbb{R}-\{-1,0,1\}$ is a constant. It is easy to see that

$$e_1 = \frac{\partial}{\partial x^1}+k\,\frac{\partial}{\partial x_2}, \qquad e_2 = k\,\frac{\partial}{\partial x^3}+\frac{\partial}{\partial x^4},$$

$$e_3 = \frac{\partial}{\partial x^1}+\frac{\partial}{\partial y^3}, \qquad e_4 = \frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^5},$$

$$e_5 = k\cos u^5\,\frac{\partial}{\partial x^3}+k\sin u^5\,\frac{\partial}{\partial x^4}+\frac{\partial}{\partial y^1}, \quad e_6 = k\sin u^6\,\frac{\partial}{\partial x^3}+k\cos u^6\,\frac{\partial}{\partial x^4}+\frac{\partial}{\partial y^2}$$

form a local orthogonal basis for TM. The submanifold M of $\widetilde{M}$ is an almost semiinvariant $\xi^{\perp}$-submanifold with

$$TM=\mathcal{D}^1\oplus\mathcal{D}^0\oplus\mathcal{D}^{\lambda}, \quad \mathcal{D}^1=\mathrm{Span}\{e_1,e_2\}, \quad \mathcal{D}^0=\mathrm{Span}\{e_3,e_4\}, \quad \mathcal{D}^{\lambda}=\mathrm{Span}\{e_5,e_6\},$$

where $\lambda=\left(k^2-1\right)/\left(k^2+1\right)$ and $\theta=\cos^{-1}\lambda$ is the slant angle of the distribution $\mathcal{D}^{\lambda}$.

5. Some characterizations

Like the operator P^2, the operators tF, Ft, and f^2 are also symmetric and their eigenvalues are bounded by 0 and 1. Let $\lambda^2(x)$, $0\leq\lambda(x)\leq 1$, be an eigenvalue of $f^2|_{\{\xi\}^{\perp}}$ at $x\in M$ and $\bar{\mathcal{D}}_x^{\lambda}$ denote the corresponding eigenspace; that is,

$$\bar{\mathcal{D}}_x^{\lambda}\equiv\ker(f^2|_{\{\xi\}^{\perp}}-\lambda^2(x)I)_x.$$

For $\lambda\neq 1$, we have $F\mathcal{D}_x^{\lambda}=\bar{\mathcal{D}}_x^{\lambda}$ and $t\bar{\mathcal{D}}_x^{\lambda}=\mathcal{D}_x^{\lambda}$. Equivalently, at $x\in M$, X_x (resp. N_x) is an eigenvector of P^2 (resp. $f^2|_{\{\xi\}^{\perp}}$) corresponding to an eigenvalue $\lambda^2(x)$ if and only if FX_x (resp. tN_x) is an eigenvector of $f^2|_{\{\xi\}^{\perp}}$ (resp. P^2) corresponding to the same eigenvalue $\lambda^2(x)$. Consequently, $\dim(\mathcal{D}_x^{\lambda})=\dim(\bar{\mathcal{D}}_x^{\lambda})$. Thus, for an $\xi^{\perp}$-submanifold M of an almost paracontact metric manifold $\widetilde{M}$, the following 2 statements are equivalent:

(1) $T_xM=\mathcal{D}_x^1\oplus\mathcal{D}_x^0\oplus\mathcal{D}_x^{\lambda_1}\oplus\cdots\oplus\mathcal{D}_x^{\lambda_k}$,

(2) $T_x^{\perp}M=\bar{\mathcal{D}}_x^1\oplus\bar{\mathcal{D}}_x^0\oplus\bar{\mathcal{D}}_x^{\lambda_1}\oplus\cdots\oplus\bar{\mathcal{D}}_x^{\lambda_k}\oplus\{\xi\}_x$.

In view of the above discussion we immediately have the following:

Proposition 5.1 *M is an almost semiinvariant $\xi^{\perp}$-submanifold of an almost paracontact metric manifold $\widetilde{M}$ if and only if there are k functions $\lambda_1,\dots,\lambda_k$, defined on M with values in the open interval $(0,1)$ such that*

(1) $\lambda_1^2(x), \dots, \lambda_k^2(x)$ *are distinct eigenvalues of* $f^2|_{\{\xi\}^\perp}$ *with*

$$T_x^\perp M = \bar{\mathcal{D}}_x^1 \oplus \bar{\mathcal{D}}_x^0 \oplus \bar{\mathcal{D}}_x^{\lambda_1} \oplus \cdots \oplus \bar{\mathcal{D}}_x^{\lambda_k} \oplus \{\xi\}_x, \qquad x \in M,$$

(2) *the dimensions of* $\bar{\mathcal{D}}_x^1$, $\bar{\mathcal{D}}_x^0$, $\bar{\mathcal{D}}_x^{\lambda_1}, \dots, \bar{\mathcal{D}}_x^{\lambda_k}$ *are independent of* $x \in M$.

Let $1-\lambda^2(x)$, $0 \le \lambda(x) \le 1$, be an eigenvalue of tF (resp. $(Ft)|_{\{\xi\}^\perp}$) at $x \in M$ and $\mathcal{C}_x^\lambda$ (resp. $\bar{\mathcal{C}}_x^\lambda$) be denoted by

$$\mathcal{C}_x^\lambda = \mathrm{Ker}(tF + (\lambda^2(x)-1)I)_x \quad (\text{resp. } \bar{\mathcal{C}}_x^\lambda = \mathrm{Ker}(Ft|_{\{\xi\}^\perp} + (\lambda^2(x)-1)I)_x.$$

Then X (resp. N) is an eigenvector of P^2 (resp. $f^2|_{\{\xi\}^\perp}$) corresponding to an eigenvalue $\lambda^2(x)$ if and only if X (resp. N) is an eigenvector of tF (resp. $Ft|_{\{\xi\}^\perp}$) corresponding to the eigenvalue $1-\lambda^2(x)$. Consequently, $\mathcal{D}_x^\lambda = \mathcal{C}_x^\lambda$ and $\bar{\mathcal{D}}_x^\lambda = \bar{\mathcal{C}}_x^\lambda$ and hence we have the following:

Proposition 5.2 *M is an almost semiinvariant $\xi^\perp$-submanifold of an almost paracontact metric manifold $\widetilde{M}$ if and only if there are k functions $\lambda_1, \dots, \lambda_k$, defined on M with values in $(0,1)$ such that*

(1) $(1-\lambda_1^2(x)), \dots, (1-\lambda_k^2(x))$ *are distinct eigenvalues of tF (resp. $Ft|_{\{\xi\}^\perp}$) with*

$$T_xM = \mathcal{C}_x^1 \oplus \mathcal{C}_x^0 \oplus \mathcal{C}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{C}_x^{\lambda_k},$$

$$\left(\text{resp. } T_x^\perp M = \bar{\mathcal{C}}_x^1 \oplus \bar{\mathcal{C}}_x^0 \oplus \bar{\mathcal{C}}_x^{\lambda_1} \oplus \cdots \oplus \bar{\mathcal{C}}_x^{\lambda_k} \oplus \{\xi\}_x\right), \qquad x \in M,$$

(2) *the dimensions of* $\mathcal{C}_x^1, \mathcal{C}_x^0, \mathcal{C}_x^{\lambda_1}, \dots, \mathcal{C}_x^{\lambda_k}$ *(resp.* $\bar{\mathcal{C}}_x^1, \bar{\mathcal{C}}_x^0, \bar{\mathcal{C}}_x^{\lambda_1}, \dots, \bar{\mathcal{C}}_x^{\lambda_k}$*) are independent of* $x \in M$.

Now we give the following characterization of semiinvariant $\xi^\perp$-submanifolds.

Proposition 5.3 *M is a semiinvariant $\xi^\perp$-submanifold of an almost paracontact metric manifold if and only if one of the following equivalent conditions holds.*

(1)	$T_xM = \mathcal{D}_x^1 \oplus \mathcal{D}_x^0, \quad x \in M,$			(2)	$T_x^\perp M = \bar{\mathcal{D}}_x^1 \oplus \bar{\mathcal{D}}_x^0 \oplus \{\xi\}_x, \quad x \in M,$		
(3)	$FP = 0,$	(4)	$fF = 0,$	(5)	$tf = 0,$	(6)	$Pt = 0,$
(7)	$tFP = 0,$	(8)	$tfF = 0,$	(9)	$Ptf = 0,$	(10)	$P^3 - P = 0,$
(11)	$f^2F = 0,$	(12)	$tFP = 0,$	(13)	$FP^2 = 0,$	(14)	$FtF - F = 0,$
(15)	$Ftf = 0,$	(16)	$FPt = 0,$	(17)	$fFt = 0,$	(18)	$f^3 - f = 0,$
(19)	$P^2t = 0,$	(20)	$Ptf = 0,$	(21)	$tf^2 = 0,$	(22)	$tFt - t = 0.$

Proof The equivalence of statements (1) and (2) is obvious. The equivalence of statements (3)–(22) can also be easily verified. Now we show the equivalence of statements (1) and (3). Since $\ker(FP)_x = \mathcal{D}_x^1 \oplus \mathcal{D}_x^0$, statement (1) implies statement (3). Conversely, if statement (3) is true, then $\varphi(PX) = P^2X$ for $X \in T_xM$. Consequently, for $\mathcal{D}_x \equiv P(T_xM)$, we get $\varphi(\mathcal{D}_x) \subset \mathcal{D}_x$. In view of $\varphi X = PX$ for $X \in \mathcal{D}_x$, we get $X_x = \varphi^2 X_x = \varphi P(X_x)$; that is, $\mathcal{D}_x \subset \varphi(\mathcal{D}_x)$. Thus, $\varphi(\mathcal{D}_x) = P(\mathcal{D}_x) = \mathcal{D}_x$, which shows that $\mathcal{D}_x = \mathcal{D}_x^1$. Let $\mathcal{D}_x^\perp$ denote the orthogonal complement to $\mathcal{D}_x^1$ in T_xM. Now, for $X \in \mathcal{D}_x^\perp$ and $Y \in T_xM$, we have $g(\varphi X, Y) = g(X, \varphi Y) = g(X, PY) = 0$, which implies that $\mathcal{D}_x^\perp = \mathcal{D}_x^0$. Hence statement (3) implies statement (1).

Finally, if M is a semiinvariant $\xi^{\perp}$-submanifold, then statement (1) is obvious by the definition. Conversely, if statement (1) is true then it implies statement (3), which is equivalent to statement (10). From statement (10) it follows that $\dim(\mathcal{D}^1_x) = \operatorname{rank}(P)$ is independent of $x \in M$ [15] (thus, P becomes an $f(3,-1)$-structure on M). Therefore, $\dim(\mathcal{D}^0_x) = \ker(P_x)$ also does not depend on $x \in M$. This completes the proof. □

6. Para-CR-structure

First we recall the notion of a *CR*-manifold. Let M be a differentiable manifold and $T^{\mathbb{C}}M$ be the complexified tangent bundle to M. A *CR-structure* [2] on M is a complex subbundle $\mathcal{H}$ of $T^{\mathbb{C}}M$ such that $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$ and $\mathcal{H}$ is involutive. A manifold endowed with a *CR*-structure is called a *CR-manifold*. It is known that a differentiable manifold M admits a *CR*-structure [1] if and only if there is a differentiable distribution $\mathcal{D}$ and a $(1,1)$ tensor field P on M such that for all $X, Y \in \mathcal{D}$

$$P^2X = -X,$$

$$[P,P](X,Y) \equiv [PX,PY] - [X,Y] - P[PX,Y] - P[X,PY] = 0,$$
$$[PX,PY] - [X,Y] \in \mathcal{D}.$$

Analogous to the definition of *CR*-structure, we now define a para-*CR*-structure.

Definition 6.1 *A differentiable manifold M is said to admit a para-CR-structure if there is a differentiable distribution $\mathcal{D}$ and a $(1,1)$ tensor field P on M such that for all $X, Y \in \mathcal{D}$*

$$P^2X = X, \tag{6.1}$$

$$[P,P](X,Y) \equiv [PX,PY] + [X,Y] - P[PX,Y] - P[X,PY] = 0, \tag{6.2}$$

$$[PX,PY] + [X,Y] \in \mathcal{D}. \tag{6.3}$$

A manifold equipped with a para-CR-structure is called a para-CR-manifold.

An almost paracontact structure (φ, ξ, η) is *normal* if the Nijenhuis tensor $[\varphi, \varphi]$ of φ satisfies [17]

$$[\varphi,\varphi] - 2d\eta \otimes \xi = 0. \tag{6.4}$$

Now we prove the following:

Theorem 6.2 *If M is an almost semiinvariant $\xi^{\perp}$-submanifold of a normal almost paracontact metric manifold $\widetilde{M}$ with nontrivial invariant distribution, then M possesses a para-CR-structure.*

Proof Since $\widetilde{M}$ is normal, for $X, Y \in \mathcal{D}^1$ we get $P^2X = X$ and, in view of $[\varphi,\varphi] = 2d\eta \otimes \xi$, we have

$$0 = [P,P](X,Y) - F([X,PY] + [PX,Y]),$$

from which it follows that

$$F([PX,Y] + [X,PY]) = 0;$$

that is, $[PX,Y]+[X,PY]\in\mathcal{D}^1$. Thus

$$[PX,PY]+[X,Y]=P\left([PX,Y]+[X,PY]\right)\in\mathcal{D}^1,$$

and hence $(\mathcal{D}^1,P)$ is a para-CR-structure on M. □

Theorem 6.3 *An almost semiinvariant $\xi^\perp$-submanifold of a para-Sasakian manifold with nontrivial invariant distribution is a para-CR-manifold.*

Proof Since every para-Sasakian manifold is normal [17], by Theorem 6.2, the proof is immediate. □

From Theorem 6.2, it is obvious that normality of $\widetilde{M}$ is a sufficient condition for an almost semiinvariant $\xi^\perp$-submanifold with nontrivial invariant distribution to carry a para-CR-structure. However, this is not necessary, and now we construct an example of a semiinvariant $\xi^\perp$-submanifold M of an almost paracontact metric manifold $\widetilde{M}$ such that M is a para-CR-manifold and $\widetilde{M}$ is not normal.

Example 6.4 *Consider the Euclidean space $\mathbb{R}^5$ and denote its points by $x=(x^1,\ldots,x^5)$. Let (e_j), $j=1,\ldots,5$, be the natural basis defined by $e_j\equiv\partial/\partial x^j$, and g the canonical metric defined by $g(e_i,e_j)=\delta_{ij}$, $i,j=1,\ldots,5$. For each $x\in\mathbb{R}^5$, the set (E_j) defined by*

$$E_1=e_1,\ E_2=\cos(x^1)e_2+\sin(x^1)e_3,\ E_3=-\sin(x^1)e_2+\cos(x^1)e_3,\ E_4=e_4, E_5=e_5$$

forms an orthonormal basis. As the point x varies in $\mathbb{R}^5$ the above set of equations defines 5 vector fields also denoted by (E_j). Now we define a vector field ξ by $\xi\equiv\partial/\partial x^5$, a 1-form η by $\eta\equiv dx^5$, and a $(1,1)$ tensor field φ by

$$\varphi(E_1)=E_2,\ \varphi(E_2)=E_1,\ \varphi(E_3)=E_4,\ \varphi(E_4)=E_3,\ \varphi(E_5)=0.$$

Then (φ,ξ,η,g) define an almost paracontact metric structure on $\mathbb{R}^5$. Since

$$[\varphi,\varphi](E_1,E_4)-2d\eta(E_1,E_4)\xi=E_1\neq 0,$$

the almost paracontact structure is not normal. The submanifold

$$M=\{x\in\mathbb{R}^5: x^4,x^5=0\}$$

is a semiinvariant $\xi^\perp$-submanifold of $\mathbb{R}^5$ with $\mathcal{D}^1=\mathrm{Span}\{E_1,E_2\}$ and $\mathcal{D}^0=\mathrm{Span}\{E_3\}$ such that $(\mathcal{D}^1,\varphi)$ is a para-CR-structure on M. Moreover, $\mathcal{D}^1$ is not integrable because $[E_1,E_2]=E_3$.

7. Integrability of distributions

Theorem 7.1 *If M is an almost semiinvariant $\xi^\perp$-submanifold of a para-Sasakian manifold, then $\mathcal{D}^0$ is integrable if and only if*

$$A_{FX}Y=0,\qquad X,Y\in\mathcal{D}^0, \tag{7.1}$$

or equivalently

$$0=g(\sigma(X,PY)-\sigma(PX,Y),FZ),\qquad X,Y\in\mathcal{D}^1,\ Z\in TM.$$

Proof First we note that

$$g(X, tN) = g(FX, N), \qquad X \in TM,\ N \in T^{\perp}M. \tag{7.2}$$

For $X, Y \in \mathcal{D}^0$, $Z \in TM$, in view of (2.7), (7.2), (3.9), and (3.5), we have

$$\begin{aligned} g(A_{FX}Y, Z) &= g(\sigma(Y,Z), FX) = g(t\sigma(Y,Z), X) \\ &= g(\nabla_Z PY - P\nabla_Z Y - A_{FY}Z, X) \\ &= -g(\nabla_Z Y, PX) - g(A_{FY}Z, X) = -g(A_{FY}X, Z), \end{aligned}$$

which implies

$$A_{FX}Y + A_{FY}X = 0, \qquad X, Y \in \mathcal{D}^0. \tag{7.3}$$

On the other hand, in view of $\ker P = \mathcal{D}^0$ and (3.13), the distribution $\mathcal{D}^0$ is integrable if and only if

$$A_{FX}Y - A_{FY}X = 0, \qquad X, Y \in \mathcal{D}^0,$$

which in view of (7.3) completes the proof. □

In the following theorem necessary and sufficient conditions for $\mathcal{D}^1$ to be integrable have been obtained.

Theorem 7.2 *If M is an almost semiinvariant $\xi^{\perp}$-submanifold of a para-Sasakian manifold, then $\mathcal{D}^1$ is integrable if and only if*

$$\sigma(X, PY) - \sigma(PX, Y) = 0, \qquad X, Y \in \mathcal{D}^1, \tag{7.4}$$

or equivalently,

$$g(\sigma(X, PY) - \sigma(PX, Y), FZ) = 0, \qquad X, Y \in \mathcal{D}^1,\ Z \in TM. \tag{7.5}$$

Proof In view of $\ker F = \mathcal{D}^1$ and (3.14), $\mathcal{D}^1$ is integrable if and only if (7.4) holds. Next, for $X \in \mathcal{D}^1$, $Y \in TM$, $N \in \bar{\mathcal{D}}^1$ in view of (3.10), (3.4), and (3.7), we obtain

$$\begin{aligned} g(\varphi\sigma(X,Y), N) &= g(f\sigma(X,Y), N) \\ &= g(\nabla^{\perp}_Y FX - F\nabla_Y X + \sigma(Y, PX) + g(X,Y)\xi, N) \\ &= g(\sigma(PX, Y) + g(X,Y)\xi, N), \end{aligned}$$

which gives

$$g(\varphi\sigma(X,Y), N) = g(\sigma(PX,Y), N) + g(X,Y)\,\eta(N) \tag{7.6}$$

for $X \in \mathcal{D}^1$, $Y \in TM$, $N \in \bar{\mathcal{D}}^1$. From (7.6) we get

$$(\sigma(X, PY) - \sigma(PX, Y)) \perp \bar{\mathcal{D}}^1, \qquad X, Y \in \mathcal{D}^1. \tag{7.7}$$

In view of $F(TM) = \bar{\mathcal{D}}^0 \oplus \bar{\mathcal{D}}^{\lambda_1} \oplus \cdots \oplus \bar{\mathcal{D}}^{\lambda_k}$ and (7.7), it follows that (7.4) and (7.5) are equivalent. □

Theorem 7.3 *In an almost semiinvariant $\xi^{\perp}$-submanifold of a para-Sasakian manifold, the distribution $\mathcal{D}^1 \oplus \mathcal{D}^0$ is integrable if and only if the following statements are true.*

(a) $\nabla_X PY - \nabla_Y PX \in \mathcal{D}^1, \qquad X, Y \in \mathcal{D}^1$,

(b) $A_{FX}Y \in \mathcal{D}^1, \qquad X, Y \in \mathcal{D}^0$,

(c) $\nabla_X PY + A_{FX}Y \in \mathcal{D}^1, \qquad Y \in \mathcal{D}^1,\ X \in \mathcal{D}^0$.

Proof Using equivalence of $Z \in \mathcal{D}^1 \oplus \mathcal{D}^0$ and $PZ \in \mathcal{D}^1$ in (3.13) and taking into account equation (7.3), the proof is complete. □

Theorem 7.4 *In an almost semiinvariant $\xi^\perp$-submanifold of a para-Sasakian manifold, the distribution $\mathcal{D}^1 \oplus \mathcal{D}^0$ is integrable if and only if the following statements are true.*

(a) $\sigma(X, PY) - \sigma(PX, Y) \in \bar{\mathcal{D}}^0, \qquad X, Y \in \mathcal{D}^1$.

(b) $\nabla^\perp_X FY - \nabla^\perp_Y FX \in \bar{\mathcal{D}}^0, \qquad X, Y \in \mathcal{D}^0$.

(c) $\nabla^\perp_X FY - \sigma(PX, Y) \in \bar{\mathcal{D}}^0, \qquad X \in \mathcal{D}^1, Y \in \mathcal{D}^0$.

Proof Using equivalence of $Z \in \mathcal{D}^1 \oplus \mathcal{D}^0$ and $FZ \in \bar{\mathcal{D}}^0$ in (3.14), the proof is complete. □

Theorem 7.5 *For direct sum $\mathcal{D}$ of a subfamily of $\{\mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}\}$ on an almost semiinvariant $\xi^\perp$-submanifold of a para-Sasakian manifold, the following statements are equivalent.*

(1) *$\mathcal{D}$ is integrable,*

(2) (a) $\nabla_X PY - \nabla_Y PX + A_{FX}Y - A_{FY}X \in \mathcal{D}, \quad X, Y \in \mathcal{D}$,

(2) (b) $\nabla^\perp_X FY - \nabla^\perp_Y FX + \sigma(X, PY) - \sigma(PX, Y) \in \bar{\mathcal{D}}, \quad X, Y \in \mathcal{D}$,

where $\bar{\mathcal{D}}$ is the direct sum of the corresponding subfamily $\{\bar{\mathcal{D}}^{\lambda_1}, \dots, \bar{\mathcal{D}}^{\lambda_k}\}$.

Proof The proof follows from (3.13), (3.14), and the equivalence of $Z \in \mathcal{D}^{\lambda_i}$ to $PZ \in \mathcal{D}^{\lambda_i}$ along with $FZ \in \bar{\mathcal{D}}^{\lambda_i}$. □

Theorem 7.6 *For an almost semiinvariant $\xi^\perp$-submanifold of a para-Sasakian manifold, the following statements are equivalent:*

(1) *$\mathcal{D}^1 \oplus \mathcal{D}$ is integrable,*

(2) $\nabla^\perp_X FY - \nabla^\perp_Y FX + \sigma(X, PY) - \sigma(PX, Y) \in \bar{\mathcal{D}}, \qquad X, Y \in \mathcal{D}^1 \oplus \mathcal{D}$,

where $\mathcal{D}$ is the direct sum of a subfamily of $\{\mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}\}$ and $\bar{\mathcal{D}}$ is the direct sum of the corresponding subfamily of $\{\bar{\mathcal{D}}^{\lambda_1}, \dots, \bar{\mathcal{D}}^{\lambda_k}\}$.

Proof In view of (3.14) and the equivalence of $Z \in \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_i}$ and $FZ \in \bar{\mathcal{D}}^{\lambda_i}$, the proof becomes obvious. □

Theorem 7.7 *For direct sum $\mathcal{D}$ of a subfamily of $\{\mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}\}$ on an almost semiinvariant $\xi^\perp$-submanifold of a para-Sasakian manifold, the following statements are equivalent:*

(1) $\mathcal{D}^0 \oplus \mathcal{D}$ *is integrable,*

(2) (a) $(\nabla_X PY + A_{FX}Y - A_{FY}X) \in \mathcal{D}, \qquad X \in \mathcal{D}^0,\ Y \in \mathcal{D},$

(2) (b) $(\nabla_X PY - \nabla_Y PX + A_{FX}Y - A_{FY}X) \in \mathcal{D}, \qquad X, Y \in \mathcal{D}.$

(2) (c) $A_{FX}Y \in \mathcal{D}, \qquad X, Y \in \mathcal{D}^0.$

Proof Using the equivalence of $Z \in \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_i}$ and $PZ \in \mathcal{D}^{\lambda_i}$ and (7.3) in (3.13), we get **(1)** $\Leftrightarrow$ **(2)**. □

8. Certain parallel operators

In this section we investigate certain parallel operators on $\xi^\perp$-submanifolds of almost paracontact metric manifolds and para-Sasakian manifolds.

Analogous to Definition 8.1 of [16], first we give the following definition:

Definition 8.1 *An $\xi^\perp$-submanifold M of an almost paracontact metric manifold is said to satisfy*

(1) *the condition* **(A)** *if M is an almost semiinvariant $\xi^\perp$-submanifold such that each λ_i is constant and each of the distributions $\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}$ is parallel, and*

(2) *the condition* **(B)** *if M is an almost semiinvariant $\xi^\perp$-submanifold such that each λ_i is constant and each of the subbundles $\bar{\mathcal{D}}^1, \bar{\mathcal{D}}^0, \bar{\mathcal{D}}^{\lambda_1}, \dots, \bar{\mathcal{D}}^{\lambda_k}$ and $\{\xi\}$ are parallel with respect to $\nabla^\perp$.*

We note that if M satisfies condition **(A)**, then it is locally product of leaves of $\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}$.

Analogous to Theorem 7.3 of [19], we may state the following:

Theorem 8.2 *For an $\xi^\perp$-submanifold M of an almost paracontact metric manifold, we have the following flow diagram.*

$$\begin{array}{ccccccc} \nabla P = 0 & \Rightarrow & \nabla(P^2) = 0 & \Leftrightarrow & (\mathbf{A}) & \Leftarrow & \nabla t = 0 \\ & & & & & & \Updownarrow \\ \nabla f = 0 & \Rightarrow & \nabla(f^2) = 0 & \Leftrightarrow & (\mathbf{B}) & \Leftarrow & \nabla F = 0. \end{array}$$

Theorem 8.3 *An $\xi^\perp$-submanifold M of a para-Sasakian manifold is an invariant $\xi^\perp$-submanifold if and only if ξ is parallel with respect to the normal connection.*

Proof The proof follows from (3.16). □

Theorem 8.4 *Let M be an $\xi^\perp$-submanifold of a para-Sasakian manifold. If f^2 is parallel, then M is an invariant $\xi^\perp$-submanifold.*

Proof If $\nabla(f^2) = 0$, in view of Theorem 8.2, it follows that M satisfies condition **(B)**. Consequently, ξ is parallel with respect to the normal connection, which in view of Theorem 8.3 provides the proof. □

Similarly, we can prove the following:

Theorem 8.5 *For an $\xi^{\perp}$-submanifold M of a para-Sasakian manifold, if F is parallel (or equivalently t is parallel), then M is an invariant $\xi^{\perp}$-submanifold.*

In a smooth manifold M, an almost product Riemannian structure consists of a $(1,1)$ tensor P and an associated Riemannian metric g such that $P^2 = I$ and $g(PX, PY) = g(X, Y)$ for all vector fields X and Y on M. Moreover, if P is covariantly constant with respect to the Levi-Civita connection, then (M, P) is said to be a locally Riemannian product manifold [20].

Theorem 8.6 *If M is an invariant $\xi^{\perp}$-submanifold of a para-Sasakian manifold, then it is a locally Riemannian product manifold. Moreover, f is parallel.*

Proof Since F is zero for an invariant $\xi^{\perp}$-submanifold, from (3.9) it follows that $\nabla P = 0$. Thus, P provides a locally Riemannian product structure on M. The remaining part follows from (3.12). □

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