

1-1-2014

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### Recommended Citation

DEĞİRMENÇİ, NEDİM and BULUT, ŞENAY (2014) "Seiberg–Witten-like equations on 5-dimensional contact metric manifolds," *Turkish Journal of Mathematics*: Vol. 38: No. 5, Article 2. <https://doi.org/10.3906/mat-1303-34>

Available at: <https://journals.tubitak.gov.tr/math/vol38/iss5/2>

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## Seiberg–Witten-like equations on 5-dimensional contact metric manifolds\*\*

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Received: 18.03.2013 • Accepted: 26.02.2014 • Published Online: 01.07.2014 • Printed: 31.07.2014

**Abstract:** In this paper, we write Seiberg–Witten-like equations on contact metric manifolds of dimension 5. Since any contact metric manifold has a  $\text{Spin}^c$ -structure, we use the generalized Tanaka–Webster connection on a  $\text{Spin}^c$  spinor bundle of a contact metric manifold to define the Dirac-type operators and write the Dirac equation. The self-duality of 2-forms needed for the curvature equation is defined by using the contact structure. These equations admit a nontrivial solution on 5-dimensional strictly pseudoconvex CR manifolds whose contact distribution has a negative constant scalar curvature.

**Key words:** Seiberg–Witten equations, spinor, Dirac operator, contact metric manifold, self-duality

### 1. Introduction

Seiberg–Witten equations were defined on 4-dimensional Riemannian manifolds by Witten in [14]. The solution space of these equations gives differential topological invariants for 4-manifolds [1, 11]. Some generalizations were given later on higher dimensional manifolds [4, 7, 10].

Seiberg–Witten equations consist of 2 equations. The first is the Dirac equation, which is meaningful for the manifolds having  $\text{Spin}^c$ -structure. The second is the curvature equation, which couples the self-dual part of a connection 2-form with a spinor field. In order to be able to write down the curvature equation, the notion of the self-duality of a 2-form is needed. This notion is meaningful for 4-dimensional Riemannian manifolds. On the other hand, there are similar self-duality notions for some higher dimensional manifolds [5, 13]. In the present paper, we propose Seiberg–Witten-like equations for 5-dimensional contact metric manifolds by using the  $\text{Spin}^c$ -structure and the notion of self-duality given in [12] and [3], respectively.

The paper is organized as follows. We begin with a section introducing some basic facts concerning contact metric manifolds. In the following section, we study self-dual 2-forms on 5-dimensional contact metric manifolds. In Section 4, we discuss the  $\text{Spin}^c$ -structures and Dirac-type operators associated to the generalized Tanaka–Webster connection. In the final section we propose the Dirac and curvature equations and hence write Seiberg–Witten-like equations on contact metric manifolds of dimension 5. Finally, we obtain a special solution for these equations on the 5-dimensional strictly pseudoconvex CR manifolds whose contact distribution has a negative constant scalar curvature.

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2010 AMS Mathematics Subject Classification: 15A66, 58Jxx.

\*\* This paper was supported by Project 1105F099 of the Scientific Research Foundation of Anadolu University.

## 2. Contact metric manifolds

A contact form on a smooth manifold  $M$  of dimension  $(2n+1)$  is a 1-form  $\eta$  such that  $\eta \wedge (d\eta)^n \neq 0$  everywhere on  $M$ . The contact form  $\eta$  induces a hyperplane subbundle  $H$  of the tangent bundle  $TM$  given by  $H = Ker \eta$ . The Reeb vector field associated to  $\eta$  is the vector field  $\xi$  uniquely determined by  $\eta(\xi) = 1$  and  $d\eta(\xi, \cdot) = 0$ . Then  $(M, \eta)$  is called a contact manifold.

Note that given  $H = Ker \eta$  and  $\xi$  such that  $\eta(\xi) = 1$ , we can split the tangent bundle into  $TM = H \oplus \mathbb{R}\xi$ . If  $X$  is any vector field on  $M$ , then  $X$  decomposes as  $X = X_H + f\xi$  for any  $f \in C^\infty(M, \mathbb{R})$ .  $X_H$  is called the horizontal part of  $X$ .

If  $(M, \eta)$  is a contact manifold, then the pair  $(H, d\eta|_H)$  is a symplectic vector bundle. We fix an almost complex structure  $J_H$  on  $H$  compatible with  $d\eta|_H$ , i.e.  $d\eta|_H(J_H(X), J_H(Y)) = d\eta|_H(X, Y)$ . We can extend  $J_H$  to an endomorphism  $J$  of the tangent bundle  $TM$  by setting  $J\xi = 0$ . The relation  $J^2 = -Id + \eta \otimes \xi$  then holds. With this in mind,  $g_\eta$ , given by

$$g_\eta(X, Y) = d\eta(X, JY) + \eta(X)\eta(Y),$$

defines a Riemannian metric on  $TM$ . The metric  $g_\eta$  is called a Webster metric and is said to be associated to  $\eta$ . Moreover, the following relations hold:

$$g_\eta(\xi, X) = \eta(X), \quad g_\eta(JX, Y) = d\eta(X, Y), \quad g_\eta(JX, JY) = g_\eta(X, Y) - \eta(X)\eta(Y)$$

for any  $X, Y \in \Gamma(TM)$ . We call  $(M, g_\eta, \eta, \xi, J)$  a contact metric manifold. For detailed information, see [2, 12].

The generalized Tanaka–Webster connection  $\nabla$  is a well-known connection on the contact metric manifold  $(M, g_\eta, \eta, \xi, J)$ . This connection satisfies the conditions  $\nabla\eta = 0$  and  $\nabla g_\eta = 0$ . Moreover, if  $J$  is integrable, i.e.  $\nabla J = 0$ , then the contact metric manifold  $(M, g_\eta, \eta, \xi, J)$  is called a strictly pseudoconvex CR manifold [12].

## 3. Self-dual 2-forms on 5-dimensional contact metric manifolds

Let  $(M, g_\eta, \eta, \xi, J)$  be a 5-dimensional contact metric manifold. The  $p$ -form  $\alpha$  is called a horizontal  $p$ -form if  $i(\xi)\alpha = 0$  where  $i$  is contraction operator. For any 2-form  $\alpha \in \Omega^2(M)$  we have the splitting  $\alpha = \alpha_H + \alpha_\xi$  where  $\alpha_H = \alpha \circ \Pi$ ,  $\Pi : TM \rightarrow H$  is the canonical projection and  $\alpha_\xi = \eta \wedge i(\xi)\alpha$ . The decomposition of  $\Omega^2(M)$  is then given by

$$\Omega^2(M) = \Omega_H^2(M) \oplus \eta \wedge \Omega_H^1(M), \tag{1}$$

where  $\Omega_H^2(M)$  and  $\Omega_H^1(M)$  are the bundles of horizontal forms. Moreover, any horizontal 2-form can be split into its self-dual and anti-self dual parts as follows.

Let  $\star$  be the Hodge-star operator acting on the cotangent bundle  $T^*M$ . We can define the operator

$$\star : \Omega^2(M) \rightarrow \Omega^2(M), \quad \star(\beta) := \star(\eta \wedge \beta).$$

We can restrict the operator  $\star$  to the space of horizontal 2-forms  $\Omega_H^2(M)$ :

$$\star_H : \Omega_H^2(M) \rightarrow \Omega_H^2(M), \quad \star_H(\beta) := \star(\eta \wedge \beta).$$

This operator satisfies  $\star_H^2 = id$ . Then we have the following orthogonal decomposition:

$$\Omega_H^2(M) = \Omega_H^2(M)^+ \oplus \Omega_H^2(M)^-, \tag{2}$$

where  $\Omega_H^2(M)^\pm$  is the eigenspace associated to eigenvalue  $\pm 1$  of the operator  $\star_H$ . The eigenspace  $\Omega_H^2(M)^+$  is called as the space of self-dual 2-forms. In a similar way, the eigenspace  $\Omega_H^2(M)^-$  is called the space of anti-self-dual 2-forms (see [3, 8]). From equalities (1) and (2), we have

$$\Omega^2(M) = \Omega_H^2(M)^+ \oplus \Omega_H^2(M)^- \oplus \eta \wedge \Omega_H^1(M).$$

Hence, any 2-form  $\alpha$  can be written as  $\alpha = \alpha_H^+ + \alpha_H^- + \eta \wedge \beta$  where  $\beta$  is a 1-form on  $H$ . The self-dual part of  $\alpha$  is defined as the self-dual part of  $\alpha_H$ , i.e.  $\alpha^+ := \alpha_H^+$ .

Locally, we can specify the self-dual and anti-self-dual 2-forms. For this, choose a local orthonormal frame field  $\{e_1, e_2 = J(e_1), e_3, e_4 = J(e_3), \xi\}$  and denote by  $\{e^1, e^2, e^3, e^4, \eta\}$  the dual basis. From (2), the 2-form  $d\eta$  has the form  $d\eta = e^1 \wedge e^2 + e^3 \wedge e^4$ . The forms  $e^1 \wedge e^2 + e^3 \wedge e^4$ ,  $e^1 \wedge e^3 - e^2 \wedge e^4$  and  $e^1 \wedge e^4 + e^2 \wedge e^3$  are an orthonormal basis for  $\Omega_H^2(M)^+$ . An orthonormal basis for  $\Omega_H^2(M)^-$  is given by the forms  $e^1 \wedge e^2 - e^3 \wedge e^4$ ,  $e^1 \wedge e^3 + e^2 \wedge e^4$ , and  $e^1 \wedge e^4 - e^2 \wedge e^3$ .

#### 4. Dirac operators on contact metric manifolds

In this section we will describe Dirac operators on contact metric manifolds. For this, we need a  $Spin^c$ -structure. Any contact metric manifold admits a canonical  $Spin^c$ -structure. Then we have a  $Spin^c$ -bundle  $P_{Spin^c(2n)}$ , an  $S^1$ -bundle  $P_{S^1}$ , and the canonical line bundle  $\mathcal{L}$ . The spinor bundle  $S$  can be identified with the bundle  $\wedge_H^{0,*}M$  of the  $(0, *)$  forms. For the definitions and more details about these notions, we refer to [12]. For our purpose, we use the following representation of the complex Clifford algebra  $\mathbb{C}l_5$ :

$$\begin{aligned} \kappa(e_1) &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \kappa(e_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ \kappa(e_3) &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \kappa(e_4) = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\ \kappa(e_5) &= \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \kappa(d\eta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}. \end{aligned}$$

Let  $(M, g_\eta, \eta, \xi, J)$  be a contact metric manifold equipped with a  $Spin^c$ -structure. Each unitary connection  $A$  on  $\mathcal{L}$  induces a spinorial connection  $\nabla^A$  on  $S$  with the generalized Tanaka–Webster connection  $\nabla$ . The Kohn–Dirac operator  $D_H^A$  is defined as follows:

$$D_H^A = \sum_{i=1}^{2n} \kappa(e_i)(\nabla_{e_i}^A),$$

where  $\{e_i\}$  is a local orthonormal frame of  $H$ . The Dirac operator  $D_A$  is defined by

$$D_A = D_H^A + \xi \cdot \nabla_\xi^A$$

(see also [12]).

**5. Seiberg–Witten-like equations on 5-dimensional contact metric manifolds**

In [6], Seiberg–Witten-like equations on 5-dimensional Euclidean space  $\mathbb{R}^5$  were written. In this section, we will write Seiberg–Witten-like equations on 5-dimensional contact metric manifolds and give a solution to these equations on strictly pseudoconvex CR manifolds.

For a spinor  $\psi$  we define a 2-form  $\sigma(\psi)$  by the following formula:

$$\sigma(\psi)(X, Y) = \langle X \cdot Y \cdot \psi, \psi \rangle + g_\eta(X, Y)|\psi|^2,$$

where  $X, Y \in \Gamma(TM)$  and  $\langle, \rangle$  is the Hermitian inner product on the spinor space  $S$ . Note that  $\sigma(\psi)$  is an imaginary valued 2-form. The restriction of  $\sigma(\psi)$  to  $H$  is denoted by  $\sigma_H(\psi)$ .

**Definition 1** *Let  $(M, g_\eta, \eta, \xi, J)$  be a contact metric 5-manifold. Fix a  $Spin^c$ -structure and a connection  $A$  in the  $U(1)$ -principal bundle associated with the  $Spin^c$ -structure. For any  $\psi \in \Gamma(S)$  Seiberg–Witten equations are defined by*

$$\begin{aligned} D_A(\psi) &= 0, \\ F_A^+ &= -\frac{1}{4}\sigma(\psi)^+, \end{aligned} \tag{3}$$

where  $F_A^+$  is the self-dual part of the curvature  $F_A$  and  $\sigma(\psi)^+$  is the self-dual part of the 2-form  $\sigma(\psi)$ .

Now we give a solution for Seiberg–Witten equations in dimension 5. To do this, we follow the method given in [9]. From now on we suppose that  $(M, g_\eta, \eta, \xi, J)$  is a strictly pseudoconvex CR manifold.

Let  $(M, g_\eta)$  be a contact metric manifold endowed with  $Spin^c$ -structure. The spinor bundle is then  $S = \wedge_H^{0,*}(M)$ . Namely,

$$S = \wedge_H^{0,2}(M) \oplus \wedge_H^{0,1}(M) \oplus \wedge_H^{0,0}(M),$$

where  $\wedge_H^{0,2}(M)$  is the eigenspace corresponding to the eigenvalue  $2i$  of the mapping  $\kappa(d\eta) : S \rightarrow S$  and has dimension 1,  $\wedge_H^{0,1}(M)$  is the eigenspace corresponding to the eigenvalue 0 of the mapping  $\kappa(d\eta) : S \rightarrow S$  and has dimension 2, and  $\wedge_H^{0,0}(M)$  is the eigenspace corresponding to the eigenvalue  $-2i$  of the mapping  $\kappa(d\eta) : S \rightarrow S$  and has dimension 1.

If  $\psi_0 \in \wedge_H^{0,0}(M)$ , then  $\psi_0$  denotes the spinor corresponding to the constant function 1 in the chosen coordinates

$$\psi_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Moreover, we have  $d\eta \cdot \psi_0 = -2i\psi_0$ . By using the expression of  $\sigma_H(\psi)$  in the local coordinates, we obtain the following identity:

$$\sigma_H(\psi_0) = -id\eta. \tag{4}$$

**5.1. Some identities**

In this part, we collect some identities needed for the special solution of Seiberg–Witten equations.

When  $M$  is a strictly pseudoconvex CR manifold,  $M$  also has a complex CR structure [2]. Let  $\{Z_1, \dots, Z_n\}$  be a local unitary frame of  $T^{1,0}$  over  $U \subset M$  where  $Z_\alpha = \frac{1}{\sqrt{2}}(e_\alpha - \sqrt{-1}Je_\alpha)$ ,  $1 \leq \alpha \leq n$ . Let us denote by  $\omega := (\omega_{\alpha\beta})$  the matrix of the connection form of  $\nabla$  with respect to the frame. Then we can write the following:

$$\nabla Z_\alpha = \sum_{\beta} \omega_{\alpha\beta} Z_\beta.$$

$\{Z_1, \dots, Z_n, \bar{Z}_1, \dots, \bar{Z}_n, \xi\}$  is a local frame of the complexified tangent bundle  $TM^{\mathbb{C}}$  over  $U$ .

Let  $\{\theta^1, \dots, \theta^n, \bar{\theta}^1, \dots, \bar{\theta}^n, \eta\}$  be the corresponding dual basis. Thus,

$$\zeta = \bar{\theta}^1 \wedge \dots \wedge \bar{\theta}^n : U \rightarrow \Lambda_H^{0,n}(M)$$

is a local section in determinant line bundle  $\Lambda_H^{0,n}(M)$ . The Webster connection  $\nabla$  defines a covariant derivative in the canonical line bundle  $\Lambda_H^{0,n}(M)$  such that

$$\nabla(\bar{\theta}^1 \wedge \dots \wedge \bar{\theta}^n) = -Tr(\omega)\bar{\theta}^1 \wedge \dots \wedge \bar{\theta}^n.$$

Since  $\nabla$  is a metric with respect to  $g_\eta$ , the trace  $Tr(\omega)$  is purely imaginary. Therefore, this connection  $\nabla$  in  $\Lambda_H^{0,n}(M)$  induces a connection on the associated  $S^1$ -principal bundle  $P_{S^1}$ . Let us denote this connection by  $A$ . Then,

$$\zeta^* A = -Tr \bar{\omega} = Tr \omega$$

is a local connection form on  $S^1$ -bundle  $P_{S^1}$ . Let  $F_A$  be the curvature form of the connection  $A$ . The curvature form  $F_A$  is a 2-form on  $M$  with values in  $i\mathbb{R}$ . Over  $U \subset M$  we have

$$F_A = dA = Tr d\omega. \tag{5}$$

Moreover,

$$Ric(X, Y) = Tr(d\omega) - Tr(\omega \wedge \omega) = Tr d\omega. \tag{6}$$

From (5) and (6) it follows that

$$F_A = Ric. \tag{7}$$

Here we follow the similar procedures given in [2].

In the following, the Ricci form  $\rho_H$  is defined by

$$\rho_H(X, Y) = Ric(X, J_H Y) = g_\eta(X, J_H Ric Y)$$

for any  $X, Y \in \Gamma(H)$ . In the case of a strictly pseudoconvex CR manifold, the almost complex structure  $J_H$  is complex. Therefore, we have the equation

$$Ric(X, Y) = i\rho_H(X, Y) \tag{8}$$

for any  $X, Y \in \Gamma(H)$ .

**Proposition 2** Let  $\rho_H$  be a Ricci form on  $H$  and  $s_H$  be a scalar curvature of the subbundle  $H$ . Then the following identity holds:

$$\rho_H^+ = -\frac{s_H}{4}d\eta, \tag{9}$$

where  $\rho_H^+$  is a the self-dual part of the Ricci form  $\rho_H$ .

**Proof** In local coordinates the almost complex structure  $J$  is given as follows.

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since  $J \circ Ric = Ric \circ J$ , we obtain the reduced form of the Ric as follows.

$$Ric = \begin{pmatrix} R_{11} & 0 & R_{13} & R_{14} & 0 \\ 0 & R_{11} & -R_{14} & R_{13} & 0 \\ R_{13} & -R_{14} & R_{33} & 0 & 0 \\ R_{14} & R_{13} & 0 & R_{33} & 0 \\ 0 & 0 & 0 & 0 & R_{55} \end{pmatrix}$$

The Ricci form  $\rho_H$  can be written in the following way:

$$\rho_H = -R_{11}e^1 \wedge e^2 - R_{33}e^3 \wedge e^4 - R_{13}(e^1 \wedge e^4 - e^2 \wedge e^3) + R_{14}(e^1 \wedge e^3 + e^2 \wedge e^4).$$

Since the 2-forms  $e^1 \wedge e^4 - e^2 \wedge e^3$  and  $e^1 \wedge e^3 + e^2 \wedge e^4$  are anti-self-dual 2-forms, the self-dual part of  $\rho_H$  is given by

$$\rho_H^+ = \frac{-R_{11} - R_{33}}{2}d\eta = -\frac{R_{11} + R_{22} + R_{33} + R_{44}}{4}d\eta = -\frac{s_H}{4}d\eta,$$

where  $s_H$  is the restricted scalar curvature to  $H$ . □

### 5.2. A special solution to 5-dimensional Seiberg–Witten equations

Let  $(M, g_\eta, \eta, \xi, J)$  be a strictly pseudoconvex contact manifold of dimension 5. Suppose that the scalar curvature  $s_H$  of the subbundle  $H$  is negative and constant. Then let  $\psi = \sqrt{-s_H}\psi_0$ . In this case,  $\psi \in \Lambda_H^{0,0}(M)$ . From (4) we have

$$\sigma_H(\psi) = is_H d\eta. \tag{10}$$

By using (7),(8), (9), and (10) we obtain

$$F_A^+ = Ric^+ = i\rho_H^+ = -i\frac{s_H}{4}d\eta = -\frac{1}{4}\sigma_H(\psi). \tag{11}$$

Note that since  $d\eta$  is a self-dual 2-form,  $\sigma_H(\psi)$  is also i.e.,  $\sigma_H(\psi)^+ = \sigma_H(\psi)$ . Because of  $\sigma(\psi)^+ = \sigma_H(\psi)^+$  and with identity (11), we get

$$F_A^+ = -\frac{1}{4}\sigma(\psi)^+.$$

One can show that  $\nabla_{e_i}^A \psi_0 = 0$ . Therefore, we deduce that

$$D_H^A \psi = 0.$$

Moreover,

$$D_A \psi = 0.$$

The pair  $(A, \psi = \sqrt{-s_H} \psi_0)$  is a solution of Seiberg–Witten-like equations in (3).

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