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PETRE BABILUA

ELIZBAR NADARAYA

GRIGOL SOKHADZE

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Functionals of Gasser–Muller estimators

Petre BABILUA[1,*], **Elizbar NADARAYA**[1], **Grigol SOKHADZE**[1,2]
[1]Department of Mathematics, Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University, Tbilisi, Georgia

[2]I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia

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Abstract: The asymptotic properties of a general functional of the Gasser–Muller estimator are investigated in the Sobolev space. The convergence rate, consistency, and central limit theorem are established.

Key words: Nonparametric regression, Gasser–Muller estimator, functionals

1. Introduction

Many researchers show interest in the study of functionals of probability distribution densities or functionals of regression functions. They consider mostly functionals of the integral type. For instance, integral functionals of a probability density function and its derivatives were studied in [9, 2, 8], whereas the same problems were investigated for a regression function in [3, 6]. A special mention should be made of [5], a work by Goldstein and Messer where the general type functional of a probability density function and the functional of the Nadaraya–Watson regression function were considered. Additionally, in [5], the problem of optimality was studied for a plug-in estimator in a functional space.

In the present paper we consider a general functional of the Gasser–Muller regression function. We are concerned with the consistency issues and the conditions under which the central limit theorem is fulfilled. We determine convergence orders and deal with some related problems. An analogous topic was studied in [1] for integral functionals.

Let us consider a regression model of the form

$$Y(t) = a(t) + \varepsilon(t) \tag{1}$$

where $t \in [0,1]$, $\varepsilon(\cdot)$ is noise with $E\varepsilon(t) = 0$, $E\varepsilon^2(t) = \sigma^2 < \infty$, $Y(t)$ is a random function, and $a(t)$ is an unknown function. Suppose we have n numbers

$$0 \le t_1 \le t_2 \le \cdots \le t_n \le 1,$$

where each $t_k, k = 1, 2, \ldots, n$, is dependent on n.

The estimator of an unknown regression function $a(t)$ was introduced by Gasser and Muller and defined

*Correspondence: petre.babilua@tsu.ge
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by the expression

$$\widehat{a}_n(t)=\frac{1}{h_n}\sum_{i=1}^{n}\int\limits_{s_{i-1}}^{s_i} W\Big(\frac{t-u}{h_n}\Big)\,du\cdot Y(t_i), \tag{2}$$

where $0=s_0\le s_1\le s_2\le\cdots\le s_n=1$, $t_i\le s_i\le t_{i+1}$, $i=1,2,\dots,n-1$ and

$$\max_i |s_i-s_{i-1}|=O\Big(\frac{1}{n}\Big);$$

$\{h_n, n=1,2,\dots\}$ is the sequence of positive numbers that monotonically tend to zero, and $W(u)$ is the function with probability density properties.

Gasser and Muller also defined the estimator of the kth derivative of the regression function $a^{(k)}(t)$ by the formula

$$\widehat{a}_n^{(k)}(t)=\frac{1}{h_n^{k+1}}\sum_{i=1}^{n}\int\limits_{s_{i-1}}^{s_i} W^{(k)}\Big(\frac{t-u}{h_n}\Big)\,du\cdot Y(t_i) \tag{3}$$

for all $k=0,1,\dots,m$. It was assumed that $\widehat{a}_n^{(0)}(t)\doteqdot\widehat{a}_n(t)$.

In the above-mentioned works, the consistency and asymptotic normality theorems for these estimators were obtained by imposing certain conditions.

For some functional $\mathfrak{A}$, here we investigate the asymptotic properties of the expression $\mathfrak{A}(\widehat{a}_n)$ as $n\to\infty$.

2. Representation theorem

Let us introduce the notation and conditions that will be used in our argumentation.

Conditions on a***:***

(a1) The function $a=a(t)$ is well defined and continuous on $[0,1]$ and takes its values in the interval $[-\mathbb{k};\mathbb{k}]$;

(a2) The function $a(t)$ has continuous derivatives up to order m inclusive;

(a3) For any $i=0,1,\dots,m$, $a^{(i)}(t)$ takes its values in $[-\mathbb{k};\mathbb{k}]$ and $a^{(i)}(\,\cdot\,)\in L_1([0,1])$.

Conditions on ε_k***:***

(ε1) Random values $\varepsilon_k=\varepsilon(t_k)$, $k=1,2,\dots$, are independent and equally distributed;

(ε2) $E\varepsilon_k=0$, $E\varepsilon_k^2=\sigma^2<\infty$.

Conditions on W***:***

(w1) $\int\limits_{-\infty}^{\infty} W(t)\,dt=1$;

(w2) Functions $W^{(i)}(t)$, $i=0,1,\dots,m$ have the compact support $[-\tau,\tau]$,

$$W^{(i)}(-\tau)=W^{(i)}(\tau)=0;$$

(w3) The function $W(t)$ has continuous derivatives up to order m, $m \geq 1$;

(w4) There exists a constant $C_W > 0$, for which

$$\sup_{t\in R} |W^{(i)}(t)| \leq C_W < \infty, \quad i = 0, 1, \ldots, m;$$

(w5) For any $i = 0, 1, \ldots, m$, $W^{(i)} \in L_1([-\tau, \tau])$.

Denote by $a_n(t)$ the mathematical expectation $\widehat{a}_n(t)$:

$$a_n(t) = E\widehat{a}_n(t) = E\,\frac{1}{h_n}\sum_{i=1}^{n}\int\limits_{s_{i-1}}^{s_i} W\Big(\frac{t-u}{h_n}\Big)\,du\cdot Y(t_i) = \frac{1}{h_n}\sum_{i=1}^{n}\int\limits_{s_{i-1}}^{s_i} W\Big(\frac{t-u}{h_n}\Big)\,du\cdot a(t_i).$$

Then we obtain

$$a_n^{(k)}(t) = E\widehat{a}_n^{(k)}(t) = \frac{1}{h_n^{i+1}}\sum_{i=1}^{n}\int\limits_{s_{i-1}}^{s_i} W^{(k)}\Big(\frac{t-u}{h_n}\Big)\,du\cdot a(t_i).$$

Let $C^m[0,1]$ denote the space of bounded real functions that are defined and continuous on $[0,1]$, having continuous derivatives of at least mth order. In this space we introduce the norm

$$\|f\|_m = \Bigg(\sum_{k=0}^{m}\int\limits_0^1 \Big(\frac{d^k f}{dt^k}\Big)^2\,dt\Bigg)^{\frac{1}{2}}, \quad f \in C^m[0,1].$$

The closure of $C^m[0,1]$ in this norm is denoted by W_m^2 and called the Sobolev space. This is a complete separable Hilbert space with the scalar product

$$\langle f, g\rangle_m = \sum_{k=0}^{m}\int\limits_0^1 \frac{d^k f}{dt^k}\,\frac{d^k g}{dt^k}\,dt, \quad f, g \in W_m^2.$$

Conditions on $\mathfrak{A}$:

($\mathfrak{A}$1) The functional $\mathfrak{A} : W_m^2 \to R$ is considered in the space W_m^2. It is assumed that this functional is smooth in a strong sense. This means that there exists a bounded linear functional $T_{\mathfrak{A}}$ such that for any 2 elements from W_m^2, $f, g \in W_m^2$, we have

$$\mathfrak{A}(f) - \mathfrak{A}(g) = T_{\mathfrak{A}}(f-g) + O\big(\|f-g\|_m^2\big).$$

By the Riesz theorem there is an element $t_{\mathfrak{A}}$ of the space W_m^2 such that

$$T_{\mathfrak{A}} w = \langle t_{\mathfrak{A}}, w\rangle_m.$$

The formulation of our problem reads as follows: Consider the Gasser–Muller scheme, where the components of (1), (2), and (3) satisfy conditions (a1)–(a3), (ε1)–(ε3), (w1)–(w5), and ($\mathfrak{A}$1). Construct the estimator of the variable $\mathfrak{A}(a)$ using observations $\{(t_1, Y(t_1)), \ldots, (t_n, Y(t_n))\}$.

As an estimator of the expression $\mathfrak{A}(a)$ we will use the so-called plug-in estimator $\mathfrak{A}(\widehat{a}_n)$. To prove the asymptotic properties we use a simple fact: the smoothness representation of the considered functional.

Let us consider the expression $\mathfrak{A}(\widehat{a}_n)$. We can write the following difference:

$$\mathfrak{A}(\widehat{a}_n)-\mathfrak{A}(a_n)=T_{\mathfrak{A}}(\widehat{a}_n-a_n)+O\big(\|\widehat{a}_n-a_n\|_m^2\big). \tag{4}$$

We call this expression the representation and will use it to obtain the desired results. To begin with, let us estimate the remainder:

$$R_n=O\big(\|\widehat{a}_n-a_n\|_m^2\big).$$

We have

$$\|\widehat{a}_n-a_n\|_m^2=\int_0^1\sum_{i=0}^m\left(\widehat{a}_n^{(i)}(t)-a_n^{(i)}(t)\right)^2dt.$$

Denote

$$U_k=U_k(t)=\frac{1}{h_n}\int_{s_{k-1}}^{s_k}W\Big(\frac{t-u}{h_n}\Big)\,du\big[Y(t_k)-a(t_k)\big],\quad k=1,2,\dots,n,$$

where $a(t_k)=EY(t_k)$. Then

$$\sum_{k=1}^nU_k=\frac{1}{h_n}\sum_{k=1}^n\int_{s_{k-1}}^{s_k}W\Big(\frac{t-u}{h_n}\Big)\,du\big[Y(t_k)-a(t_k)\big]=\widehat{a}_n(t)-a_n(t).$$

Therefore,

$$\|\widehat{a}_n-a_n\|_m^2=\Big\|\sum_{k=1}^nU_k\Big\|_m^2. \tag{5}$$

For each $k=1,2,\dots,n$ estimate the norm $\|\cdot\|_m$ of one summand in U_k (5). We have

$$\|U_k\|_m=\left(\sum_{i=0}^m\int_0^1\left|\frac{1}{h_n^{i+1}}\int_{s_{k-1}}^{s_k}W^{(i)}\Big(\frac{t-u}{h_n}\Big)\,du\big[Y(t_k)-a(t_k)\big]\right|^2dt\right)^{\frac12}=$$

$$=\left(\sum_{i=0}^m\frac{1}{h_n^{2i+2}}\int_0^1\left|h_n\int_{\frac{t-s_k}{h_n}}^{\frac{t-s_{k-1}}{h_n}}W^{(i)}\Big(\frac{t-u}{h_n}\Big)\,d\Big(\frac{t-u}{h_n}\Big)\right|^2\big|Y(t_k)-a(t_k)\big|^2dt\right)^{\frac12}\le$$

$$\le 2|\varepsilon_k|C_W\left(\sum_{i=0}^m\frac{1}{h_n^{2i}}\int_0^1\left|\frac{t-s_{k-1}}{h_n}-\frac{t-s_k}{h_n}\right|^2dt\right)^{\frac12}=$$

$$=2|\varepsilon_k|C_W\,\frac{|s_k-s_{k-1}|\sqrt{1-h_n^{2m+2}}}{h_n^{m+1}\sqrt{1-h_n^2}}\le L\,\frac{1}{nh_n^{m+1}}=M_m\sim O\Big(\frac{1}{nh_n^{m+1}}\Big)\ \text{ for sufficiently large } L>0. \tag{6}$$

To estimate $\|\widehat{a}_n-a_n\|_m^2$ we use McDiarmid's inequality, which is given here for convenience (for more information see [4]).

McDiarmid's inequality. Let $H(t_1,\dots,t_k)$ be a real function such that for each $i=1,\dots,k$ and some c_i, the supremum in $t_1,\dots,t_k,t$, of the difference:

$$\Big|H(t_1,\dots,t_{i-1},t_i,t_{i+1},\dots,t_k)-H(t_1,\dots,t_{i-1},t,t_{i+1},\dots,t_k)\Big|\le c_i.$$

If $X_1,\dots,X_k$ are independent random variables taking values in the domain of the function $H(t_1,\dots,t_k)$, then for every $\varepsilon>0$,

$$P\Big\{\big|H(X_1,\dots,X_k)-EH(X_1,\dots,X_k)\big|>\varepsilon\Big\}\le 2\exp\left(-\frac{2\varepsilon^2}{\sum\limits_{i=1}^{k}c_i^2}\right).$$

We use McDiarmid's inequality for the functions

$$H(U_1,\dots,U_m)=\Big\|\sum_{k=1}^{n}U_k\Big\|_m.$$

As c_k we take $c_k\equiv 2M_m, k=1,\dots,n$. For any $\delta>0$ we obtain

$$P\left\{\bigg|\Big\|\sum_{k=1}^{n}U_k\Big\|_m-E\Big\|\sum_{k=1}^{n}U_k\Big\|_m\bigg|\ge\delta\right\}\le 2\exp\left\{-\frac{\delta^2 n h_n^{2m+2}}{2L^2}\right\}.$$

We substitute here

$$\delta=\frac{2L\sqrt{\log n}}{\sqrt{n}\,h_n^{m+1}}$$

and, by the Borel–Cantelli lemma, write

$$\Big\|\sum_{k=1}^{n}U_k\Big\|_m=E\Big\|\sum_{k=1}^{n}U_k\Big\|_m+O\Big(\frac{\sqrt{\log n}}{\sqrt{n}\,h_n^{m+1}}\Big). \tag{7}$$

After applying Jensen's inequality,

$$E\Big\|\sum_{k=1}^{n}U_k\Big\|_m^2\le 2\sum_{k=1}^{n}E\|U_k\|_m^2=2\sum_{k=1}^{n}\sum_{i=0}^{m}\int_0^1 E\bigg|\frac{1}{h_n^{i+1}}\int_{s_{k-1}}^{s_k}W^{(i)}\Big(\frac{t-u}{h_n}\Big)\,du\big[Y(t_k)-a(t_k)\big]\bigg|^2dt\le$$

$$\le 2C_W^2\sum_{k=1}^{n}\sum_{i=0}^{m}\int_0^1\frac{1}{h_n^{2i+2}}\bigg|\int_{s_{k-1}}^{s_k}du\bigg|^2E\big[Y(t_k)-a(t_k)\big]^2dt=$$

$$=2C_W^2\sigma^2\frac{(1-h_n^{2m+2})}{(1-h_n^2)h_n^{2m+2}}\sum_{k=1}^{n}(s_k-s_{k-1})^2\le K\cdot\frac{1}{nh_n^{2m+2}}, \tag{8}$$

from (6), (7), and (8) we conclude that

$$R_n=O\Big(\frac{\log n}{nh_n^{2m+2}}\Big).$$

Thus the following statement is true.

Theorem 1 *Assume that the conditions* (a1)–(a3), (ε1)–(ε3), (w1)–(w5), *and* ($\mathfrak{A}$1) *are fulfilled. Then a representation formula holds with the remainder of order*

$$R_n = O\Big(\frac{\log n}{nh_n^{2m+2}}\Big).$$

3. Consistency

In this section of the paper we use Theorem 1 to prove the strict consistency of the estimator $\mathfrak{A}(\widehat{a}_n)$.

Theorem 2 *Let the conditions of Theorem 1 be fulfilled. If the positive sequence* h_n, $n = 1, 2, \ldots$, $0 < h_n < 1$, *is chosen so that*

$$\frac{\log n}{nh_n^{2m+2}} \longrightarrow 0, \tag{9}$$

then with probability 1 *we have*

$$\mathfrak{A}(\widehat{a}_n) \longrightarrow \mathfrak{A}(a)$$

as $n \to \infty$.

Proof By Theorem 1 and formula (4)

$$\mathfrak{A}(\widehat{a}_n) - \mathfrak{A}(a_n) = T_{\mathfrak{A}}(\widehat{a}_n - a_n) + R_n, \tag{10}$$

where $R_n = O(\|\widehat{a}_n - a_n\|_m^2) = o(1)$ a.e. and

$$T_{\mathfrak{A}}(\widehat{a}_n - a_n) = \big\langle t_{\mathfrak{A}}, \widehat{a}_n - a_n \big\rangle_m.$$

According to conditions (a1)–(a3),

$$\Big\{\big(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}\big) : \ t \in [0,1]\Big\} \subset [0,1] \times [-\Bbbk; \Bbbk]^{m+1}.$$

Keeping this in mind and by condition ($\mathfrak{A}$1), we can write

$$\begin{aligned} \big|T_{\mathfrak{A}}(\widehat{a}_n - a_n)\big| &\le \|T_{\mathfrak{A}}\| \sum_{i=0}^{m} \int_0^1 \frac{1}{h_n^{i+1}} \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} \Big|W^{(i)}\Big(\frac{t-u}{h_n}\Big)\Big|\, du \cdot \big[Y(t_k) - a(t_k)\big]\, dt \le \\ &\le 2C_W \|T_{\mathfrak{A}}\| \sum_{i=0}^{m} \int_0^1 \frac{1}{h_n^{i+1}} \sum_{k=1}^{n} |\varepsilon_k|\, |s_k - s_{k-1}|\, dt \sim \ \text{(by (w4))} \sim M\, \frac{1}{nh_n^{m+1}} \ \text{(for some } M\text{)}. \end{aligned} \tag{11}$$

Let us apply McDiarmid's inequality for

$$Y = \sum_{k=1}^{n} X_k, \tag{12}$$

where

$$X_k = \sum_{i=0}^{m} \int_0^1 \frac{1}{h_n^{i+1}} \int_{s_{k-1}}^{s_k} \Big|W^{(i)}\Big(\frac{t-u}{h_n}\Big)\Big|\, du \cdot \big[Y(t_k) - a(t_k)\big]\, dt.$$

Analogously to (11), it can be shown that X_k takes its values in the interval $[-M\,\frac{1}{n^2h_n^{m+1}}, M\,\frac{1}{n^2h_n^{m+1}}]$. Therefore,

$$\sum_{i=1}^{n} c_i^2 = \frac{2M}{n^3 h_n^{2m+2}}\,.$$

Furthermore, we take

$$t = \frac{2\sqrt{M\log n}}{n^{3/2}\,h_n^{m+1}}\,.$$

Then we obtain

$$P\bigg\{\big|T_{\mathfrak{A}}(\widehat{a}_n - a_n)\big| > \frac{2\sqrt{M\log n}}{n^{3/2}h_n^{m+1}},\bigg\} \le 2\exp\big\{-2\log n\big\}$$

by means of which, using the Borelli–Cantelli lemma, we can conclude that

$$T_{\mathfrak{A}}(\widehat{a}_n - a_n) = O\Big(\frac{\sqrt{\log n}}{n^{3/2}h_n^{m+1}}\Big).$$

It is obvious that, for condition (9), $\frac{\sqrt{\log n}}{n^{3/2}h_n^{m+1}}$, too, tends to zero. Thus we conclude that

$$T_{\mathfrak{A}}(\widehat{a}_n - a_n) \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

By formula (6) from [10] we can write

$$Ea_n^{(k)}(t) = \int_{-\tau}^{\tau} W(u)a^{(k)}(t - uh_n)\,du + O\Big(\frac{1}{nh_n^k}\Big).$$

Hence, we make the following conclusions:

(i) for condition (9), $\frac{1}{nh_n^k}$, too, tends to zero for any $k = 0, 1, \ldots, m$;

(ii) $Ea_n^{(k)}(t) \to a^{(k)}(t)$ as $n \to \infty$.

Summarizing the above discussion, we ascertain that

$$\mathfrak{A}(a_n) \longrightarrow \mathfrak{A}(a)$$

as $n \to \infty$.

Since $\mathfrak{A}(\widehat{a}_n) - \mathfrak{A}(a_n) = o(1)$, we conclude that

$$\mathfrak{A}(\widehat{a}_n) - \mathfrak{A}(a) \longrightarrow 0 \ \text{ a.e.}$$

The theorem is proved. □

4. Central limit theorem

Using our representation theorem we can obtain the limit distribution property for a smooth functional $\mathfrak{A}$ of $\widehat{a}_n(t)$ in the space W_m^2.

Consider the difference

$$\mathfrak{A}(\widehat{a}_n) - \mathfrak{A}(a_n) = T_{\mathfrak{A}}(\widehat{a}_n - a_n) + O\big(\|\widehat{a}_n - a_n\|_m^2\big) \tag{13}$$

where, for any $h_n > 0$, $T_{\mathfrak{A}}(\widehat{a}_n - a_n)$ is the sum of independent random variables

$$T_{\mathfrak{A}}(\widehat{a}_n - a_n) = \langle t_{\mathfrak{A}}, \widehat{a}_n - a_n \rangle_m = \sum_{k=0}^{m} \int_0^1 t_{\mathfrak{A}}^{(k)}(t)\big(\widehat{a}_n^{(k)}(t) - a_n^{(k)}(t)\big)\, dt. \tag{14}$$

The remainder R_n has the form

$$R_n = O\big(\|\widehat{a}_n - a_n\|_m^2\big).$$

It is clear that

$$ET_{\mathfrak{A}}(\widehat{a}_n - a_n) = 0 \ \text{ and } \ ER_n \to 0 \ \text{ as } \ n \to \infty. \tag{15}$$

Moreover,

$$E\big(T_{\mathfrak{A}}(\widehat{a}_n - a_n)\big)^2 = \sigma^2 \sum_{k=0}^{m} \bigg(\int_0^1 t_{\mathfrak{A}}^{(k)}(t)\, dt\bigg)^2 \ \text{ and } \ \operatorname{Var} R_n \to 0 \ \text{ as } \ n \to \infty. \tag{16}$$

Using appropriate conditions, we have to prove that the value

$$\sqrt{n}\,\big(\mathfrak{A}(\widehat{a}_n) - \mathfrak{A}(a_n)\big)$$

is asymptotically normal and calculate the limiting variance. By the conditions of the theorem with formulas (8), (12), and (13), for this we must show the asymptotic normality of the value $\sqrt{n}\, T_{\mathfrak{A}}(\widehat{a}_n - a_n)$. As seen from (11), for this it suffices to study this property for the variables

$$d_k = Y(t_k) \sum_{i=0}^{m} \frac{1}{h_n^{i+1}} \int_0^1 \int_{s_{k-1}}^{s_k} W^{(i)}\Big(\frac{t-u}{h_n}\Big) t_{\mathfrak{A}}^{(k)}(t)\, dt\, du.$$

It is easy to see that

$$Ed_k = a(t_k) \sum_{i=0}^{m} \frac{1}{h_n^{i+1}} \int_0^1 \int_{s_{k-1}}^{s_k} W^{(i)}\Big(\frac{t-u}{h_n}\Big) t_{\mathfrak{A}}^{(k)}(t)\, dt\, du.$$

Thus, we consider the sequence of independent random variables

$$f_k(n) = \alpha(n,k)\big(Y(t_k) - a(t_k)\big) = \alpha(n,k)\varepsilon_k,$$

where

$$\alpha(n,k) = \sum_{i=0}^{m} \frac{1}{h_n^{i+1}} \int_0^1 \int_{s_{k-1}}^{s_k} W^{(i)}\Big(\frac{t-u}{h_n}\Big) t_{\mathfrak{A}}^{(k)}(t)\, dt\, du.$$

Let us consider the sum

$$T_{\mathfrak{A}}(\widehat{a}_n - a_n) = \sum_{k=1}^{n} \alpha(n,k)\varepsilon_k.$$

Let $F_{k,n}$ be a probability distribution function of a random variable $\alpha(n,k)\varepsilon_k$, and F_ε be a distribution function of a random variable ε_k. Lindeberg's condition has the form $\forall\,\delta > 0$, $\lim\limits_{n\to\infty} L_n(\delta) = 0$, where

$$L_n(\delta) =$$

$$= \Big(\sigma^2 \sum_{k=1}^{n} \alpha^2(n,k)\Big)^{-1} \sum_{j=1}^{n} \int x^2 J\bigg(|x| \geq \delta\sigma\Big(\sum_{k=1}^{n} \alpha^2(n,k)\Big)^{\frac{1}{2}}\bigg)\, dF_{k,n}(x),$$

where $J(A)$ is the indicator function of the set A.

We easily conclude that

$$L_n(\delta) \leq \frac{1}{\sigma^2} \max_{1\leq j\leq n} \int x^2 J\big(|x| \geq \delta\sigma v(n,j)\big)\, dF_\varepsilon,$$

where

$$v(n,j) = \frac{|\alpha(n,j)|}{\big(\sum\limits_{j=1}^{n} \alpha^2(n,j)\big)^{1/2}}\,.$$

It remains to show that $\max\limits_{1\leq j\leq n} v(n,j) \to 0$ as $n \to \infty$. We have

$$\max_{1\leq j\leq n} |\alpha(n,j)| \leq C_W |t_{\mathfrak{A}}\|_m |s_j - s_{j-1}| \sum_{i=1}^{n} \frac{1}{h_n^{m+1}} = O\Big(\frac{1}{n h_n^{m+1}}\Big).$$

On the other hand,

$$\alpha(n,k) = \sum_{i=0}^{m} \frac{1}{h_n^{i+1}} \int\limits_0^1 \int\limits_{t-uh_n}^{t+uh_n} W^{(i)}(x) t_{\mathfrak{A}}^{(k)}(x+uh_n)\, dx\, du =$$

$$= 2\sum_{i=0}^{m} \frac{1}{h_n^{i}} \int\limits_0^1 W^{(i)}(\widetilde{x}) t_{\mathfrak{A}}^{(k)}(\widetilde{x}+uh_n) u\, du \sim \quad \text{(for some } \widetilde{x}\text{)}$$

$$O\Big(\frac{1}{h_n^{m}}\Big).$$

Therefore,

$$\sum_{k=1}^{n} \alpha^2(n,k) \sim O\Big(\frac{n}{h_n^{2m}}\Big)$$

and

$$v(n,j) \sim \frac{h_n^m}{n\sqrt{n}\, h_n^{m+1}} = \frac{1}{n^{3/2} h_n} \to 0 \text{ as } n \to \infty.$$

Thus, Lindeberg's condition is fulfilled and we can state that the following theorem is valid.

Theorem 3 *Let the conditions of Theorem 1 be fulfilled. Then, if* $h_n \to 0$ *and* $nh_n^{m+1} \to \infty$ *as* $n \to \infty$*, we have*

$$\sqrt{n}\,\big(\mathfrak{A}(\widehat{a}_n) - \mathfrak{A}(a_n)\big) \xrightarrow{d} N(0, r^2),$$

where

$$r^2 = \sigma^2 \sum_{k=0}^{m} \Big(\int_0^1 t_{\mathfrak{A}}^{(k)}(t)\,dt \Big)^2 .$$

5. Three examples

(a) Let the functional $\mathfrak{A}a = \int_0^1 [a'^2(t) + a^2(t)]\,dt$ be considered in $W_1^2[0,1]$. Then

$$(T_{\mathfrak{A}}a)(g) = \int_0^1 2a'(s)g'(s)\,ds + \int_0^1 2a(s)g(s)\,ds.$$

If $h_n \to 0$ and $\frac{\log n}{nh_n^4} \to 0$ as $n \to 0$, the consistency theorem is fulfilled. When $nb_n^2 \to \infty$ as $n \to 0$, we have

$$\sqrt{n}\,\big[\mathfrak{A}(\widehat{a}_n) - \mathfrak{A}(a)\big] \xrightarrow{d} N(0, r^2),$$

where

$$r^2 = 4a^2 \bigg\{ \Big[\int_0^1 a(t)\,dt \Big]^2 + \big[a(1) - a(0)\big]^2 \bigg\}.$$

(b) Consider the integral functional

$$I(a) = \int_{-\infty}^{\infty} \varphi\big(t, a(t), a'(t), \dots, a^{(m)}(t)\big)\,dt$$

in $W_m^2[0,1]$.

Assume that the following conditions are fulfilled:

($\varphi1$) The function $\varphi : R^{m+2} \to R$ is continuous, bounded, and integrable and has bounded continuous derivatives up to the second order, inclusive, in some open convex domain A, which contains the domain $R \times [-\Bbbk; \Bbbk]^{m+1}$.

($\varphi2$) All first and second derivatives of the function φ are uniformly bounded in the domain A by a constant $C_\varphi > 0$.

According to conditions ($\varphi1$) and ($\varphi2$), for the function φ we have

$$\sup \Big\{ |\varphi_{(ij)}|(s, s_0, s_1, \dots, s_m) : \ (s, s_0, s_1, \dots, s_m) \in A \Big\} \le C_\varphi$$

for each $i, j = 0, 1, \dots, m$.

Then if $h_n \to 0$ and $\sqrt{n}\, h_n^{2m+2} \to \infty$ as $n \to \infty$, we have

$$\sqrt{n}\,(\mathfrak{A}(\widehat{a}_n) - \mathfrak{A}(a)) \xrightarrow{d} N(0, r^2),$$

where

$$r^2 = \sigma^2 \sum_{i=0}^{m} \Big(\int_0^1 \varphi_{(i)}\big(t, a(t), a'(t), \dots, a^{(m)}(t)\big)\, dt \Big)^2 .$$

(c) Consider the functional

$$\mathfrak{A} f = \big(f^{(m)}(t_0)\big)^2 .$$

Assume that the conditions (a1)–(a3), $(\varepsilon 1)$, $(\varepsilon 2)$, (w1)–(w5), $(\varphi 1)$, and $(\varphi 2)$ are fulfilled. If $h_n \to 0$ and $\frac{\log n}{n h_n^{2m+2}} \to 0$ as $n \to 0$, the consistency theorem is fulfilled. When $n h_n^{m+1} \to \infty$ as $n \to 0$, we have that the central limit theorem is true.

6. Iterated logarithm law

Applying the well-known iterated logarithm law from Kuelbs' paper [7], we ascertain that the following theorem is valid.

Theorem 4 *If the sequence h_n is chosen so that*

$$R_n = o\Big(\sqrt{\frac{\log\log n}{n}}\Big),$$

then

$$\limsup_{n\to\infty} \pm \frac{\sqrt{n}\,(\mathfrak{A}(\widehat{a}_n) - \mathfrak{A}(a_n))}{\sqrt{2\log\log n}} = r,$$

where

$$r^2 = \sigma^2 \sum_{k=0}^{m} \Big(\int_0^1 t_{\mathfrak{A}}^{(k)}(t)\, dt \Big)^2 .$$

Indeed, as is easily seen,

$$\limsup_{n\to\infty} \pm \frac{\sqrt{n}\,[I(\widehat{a}_n) - I(a_n)]}{\sqrt{2\log\log n}} = \limsup_{n\to\infty} \pm \frac{\sqrt{n}\,\big[\alpha(n,k)Y(t_k) - \alpha(n,k)a(t_k)\big]}{\sqrt{2\log\log n}} = r.$$

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