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## On the equivariant cohomology algebra for solenoidal actions

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**Abstract:** We prove, under certain conditions, that if a solenoidal group (i.e. 1-dimensional compact connected abelian group) acts effectively on a compact space then the fixed point set is nonempty and  $H_G^*(X, \mathbb{Q})$  has a presentation similar to the presentation of  $H^*(X, \mathbb{Q})$  as proven by Chang in the case of a circle group.

**Key words:** TNHZ, solenoid,  $c$ -symplectic, algebra presentation

### 1. Introduction

In the cohomology theory of transformation groups (based on the Borel construction), most of the results concern Lie group, especially abelian Lie group actions. Results for non-Lie group actions are fewer. The main reason for this is the complexity of determining the cohomology ring of classifying space for non-Lie groups and equivariant cohomology algebra of the space on which the non-Lie group acts. However, there is considerable information about compact non-Lie transformation groups in [10].

It is well known that locally compact groups can be “approximated” by Lie groups. This means if  $G$  is a locally compact group with finitely many components then  $G$  has arbitrarily small compact normal subgroup  $N$  such that  $G/N$  is a Lie group. This was proven by Yamabe [20]; see also the work of Montgomery and Zippin [17].

We say that  $G$  is an  $n$ -dimensional compact connected abelian group if  $G$  is the projective limit of  $n$ -dimensional tori and write  $\dim G = n$ . One can say that if  $G$  is an  $n$ -dimensional compact connected abelian group then  $G$  has a totally disconnected closed subgroup  $N$  such that  $G/N \simeq T^n$ , an  $n$  torus. For details see [10], 8.17–8.24. We say  $G$  is a finite-dimensional compact connected abelian group if  $\dim G = n$  for some  $n \in \mathbb{N}$ . If  $\dim G = 1$ , then  $G$  is called solenoid.

As a well-known example for a solenoid, let us choose a prime number  $p$ . Let set  $G_n$  be the circle group  $T = \{z \in \mathbb{C} : |z| = 1\}$  and define  $f_n^{n+1} : G_{n+1} \rightarrow G_n$ ,  $f_n^{n+1}(z) = z^p$  for all  $n \in \mathbb{N}$  and  $z \in T$ . The projective limit of the projective system  $\{G_n, f_n^{n+1}\}$  is called the  $p$ -adic solenoid  $T_p$ . This projective limit would have the  $p$ -adic integers,  $Z_p$ , as a totally disconnected closed subgroup such that  $T_p/Z_p \simeq T$ . Solenoids are one of the prototypes of compact abelian groups that are connected, but not arc-wise connected.

If an  $n$ -dimensional compact connected abelian group  $G$  acts effectively on a Hausdorff space  $X$  (all actions are assumed to be continuous), then there is an induced, almost effective action of the  $n$  torus  $G/N$  on

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the orbit space  $X/N$  and  $X \rightarrow X/N$  induces a homeomorphism  $X^G \approx (X/N)^{G/N}$ . (Here  $X^G$  denotes fixed point set of action  $G$ .)

The orbit space of the action of  $G/N$  on  $X/N$  is homeomorphic to the orbit space  $X/G$ . Moreover, the orbit space  $X/N$  inherits global and local cohomological properties from the space  $X$ . Thus, many questions about the cohomological properties of orbit spaces and fixed point set of actions of finite-dimensional compact connected abelian groups are reduced to questions about torus actions by comparing actions of  $G$  on  $X$  and  $G/N$  on  $X/N$ .

The study of such actions is motivated by a classic unresolved problem of topological transformation groups, known as the generalized Hilbert–Smith conjecture, which states that a locally compact effective transformation group on manifold is a Lie group. A well-known fact (see [11]) states that the Hilbert–Smith conjecture is equivalent the following conjecture:

**Conjecture 1.1** *A  $p$ -adic group cannot act effectively on a connected finite dimensional manifold.*

The construction of effective  $p$ -adic spaces plays an important role in the study of the Hilbert–Smith conjecture. One way to obtain a compact space where a  $p$ -adic group acts effectively is to take the inverse limit of inverse systems of effective  $T$ -spaces with bonding maps that satisfy certain equivariance properties. This is because, if  $\{X_\alpha, f_\alpha^\beta\}$  is an inverse system of topological spaces and  $\{G_\alpha, \varphi_\alpha^\beta\}$  is an inverse system of topological groups, where each  $X_\alpha$  is a  $G_\alpha$ -space and each bonding map  $f_\alpha^\beta$  is  $\varphi_\alpha^\beta$ -equivariant, then  $\varprojlim X_\alpha$  is a  $\varprojlim G_\alpha$ -space with the action given by

$$(g_\alpha)(x_\alpha) = (g_\alpha x_\alpha).$$

In this paper, under certain conditions, we try to determine the structure of equivariant cohomology algebra with rational coefficients for solenoidal actions on compact spaces.

## 2. Preliminaries

Throughout this paper  $X$  will be a compact space and we shall use sheaf cohomology with coefficients in a field  $k$  of characteristic 0.

We need to recall definitions on the notion of effectiveness.

**Definition 2.1** (1) *Let  $G$  be a topological group and  $X$  a  $G$  space. If the ineffective kernel,  $\bigcap_{x \in X} G_x$ , is finite, then this action is called almost effective.*

(2) *Let  $G$  be a compact connected Lie group and let  $X$  be a  $G$  space. The action of  $G$  on  $X$  is said to be cohomologically effective (with coefficients in  $k$ ) if the restriction homomorphism*

$$H^*(X, k) \rightarrow H^*(X^K, k)$$

*is not a monomorphism for any subcircle  $K \subseteq G$ .*

**Remark 2.2** *If  $G$  is a compact connected Lie group and  $X$  is a closed orientable manifold, then an action of  $G$  on  $X$  is cohomologically effective if and only if it is almost effective. More generally, this holds if  $X$  is a compact orientable cohomology manifold over  $\mathbb{Q}$ . (See [5], Chapter 1, Corollary 4.6, and Chapter 5, Theorem 3.2.)*

For any topological group  $G$ , by introducing a suitable topology in the  $(m+1)$ -join  $G^{(m+1)} = G * \dots * G$  and letting  $G$  act on it naturally, we obtain an  $m$ -universal  $G$ -bundle  $(G^{(m+1)}, p, G^{(m+1)}/G, G)$  and a contractible space  $E_G = \lim_m G^{(m+1)}$  by taking the direct limit, on which  $G$  acts freely and properly. We denote the quotient space by  $B_G$ , which is called a classifying space of  $G$  [14, 15]. Thus, we have principal  $G$ -bundle  $E_G \rightarrow B_G$ , called the universal  $G$ -bundle. Let  $X$  be a  $G$ -space. A technique for studying  $G$ -actions is the construction of the so-called Borel space  $E_G \times_G X = (E_G \times X)/G$  associated to the  $G$ -space  $X$ . (On  $E_G \times X$ , there is the diagonal action given by  $g(e, x) = (ge, gx)$ .) This leads to the following commutative diagram:

$$\begin{array}{ccccc}
 X & \longleftarrow & E_G \times X & \longrightarrow & E_G \\
 \downarrow & & \downarrow & & \downarrow \\
 & & i_G & & \\
 X/G & \xleftarrow{\pi_2} & X_G & \xrightarrow{\pi_1} & B_G
 \end{array}$$

where  $\pi_1$  is a fiber bundle mapping with fiber  $X$  and structure group  $G/K$  where  $K$  is the ineffective kernel of the  $G$  action on  $X$ ,  $\pi_2$  is a mapping such that  $\pi_2^{-1}(x^*) = B_{G_{x^*}}$ , where  $x^* \in X/G$ , and  $x \in x^*$ . The equivariant graded cohomology algebra of  $X$  with coefficient  $k$  is then defined by  $H_G^*(X; k) = H^*(X_G; k)$ .

$X$  is said to be totally nonhomologous to zero (TNHZ) in  $X_G \rightarrow B_G$  with respect to  $H^*(-, k)$  if

$$i_G^* : H_G^*(X; k) \rightarrow H^*(X, k)$$

is surjective.

**Definition 2.3** Let  $X$  be a Poincare duality space of formal dimension  $fd(X) = 2n$ . (i.e.  $H^i(X, k) = 0$  for  $i > 2n$ ,  $H^{2n}(X, k) \cong k$ ,  $\dim_k H^i(X, k) < \infty$ , for all  $i$ , and for all  $0 \leq i \leq 2n$  the cup product

$$H^i(X, k) \times H^{2n-i}(X, k) \rightarrow H^{2n}(X, k) \cong k$$

is a nondegenerate bilinear form.) We say that  $X$  is cohomologically symplectic ( $c$ -symplectic for short) over  $k$  if there is a class  $w \in H^2(X, k)$ , which is called the  $c$ -symplectic class, such that  $w^n \neq 0$ .

**Definition 2.4** Let  $G$  be a compact connected Lie group and  $X$  a  $c$ -symplectic space. If  $G$  acts on  $X$ , then the action is said to be cohomologically Hamiltonian ( $c$ -Hamiltonian for short) if

$$w \in \text{im}\{i_G^* : H_G^*(X, k) \rightarrow H^*(X, k)\}.$$

**Remark 2.5** (1) A closed symplectic manifold is  $c$ -symplectic (over  $\mathbb{R}$ ) with  $w = [w]$ , the class of symplectic form.

(2) If  $X$  is a closed symplectic manifold,  $G$  is a compact connected Lie group,  $G$  is acting on  $X$ , and the action is symplectic, then the action is Hamiltonian if and only if it is  $c$ -Hamiltonian. Necessity follows from Frankel's theorem (see [9]). Sufficiency follows easily from the results and techniques of Atiyah-Bott (see [3], Section 4; Audin [4], Chapter 5, Proposition 3.1.1; and McDuff and Salamon [13], Section 5.2).

The next theorem is important for our main result.

**Theorem 2.6** (*Bredon et al.* [7, 5.1]; *Löwen* [12]). *If  $N$  is a totally disconnected compact group and  $X$  is a locally compact  $N$ -space, then the orbit map  $\pi : X \rightarrow X/N$  induces an isomorphism*

$$H_c^*(X/N, \mathbb{Q}) \simeq (H_c^*(X, \mathbb{Q}))^N$$

( $H_c^*$  denotes sheaf cohomology with compact supports. For the details, the reader is referred to Bredon's monograph [6]).

**Remark 2.7** *Let  $G$  be a finite-dimensional compact connected abelian group acting on a compact space  $X$ . Let  $N$  be a totally disconnected closed subgroup of  $G$  such that  $G/N$  is a torus. Since  $G$  is connected, its action (and hence that of  $N$ ) on  $H^*(X, \mathbb{Q})$  is trivial (see [6, II.10.6 cf. II.11.11]). Thus, Theorem 2.6 implies that*

$$H^*(X, \mathbb{Q}) \cong H^*(X/N, \mathbb{Q}).$$

### 3. Main result

Let  $\mathfrak{L}$  denote the category of locally compact abelian groups, whose morphisms are continuous homomorphisms. For an object  $G$  of the category  $\mathfrak{L}$ , the group  $Hom(G, T)$  of continuous homomorphisms from  $G$  to the circle group  $T$  endowed with the compact-open topology is an object of  $\mathfrak{L}$ . This group is called the character group of  $G$  and is denoted by  $\hat{G}$ .

The correspondence  $G \mapsto \hat{G}$  defines a contravariant functor  $\chi : \mathfrak{L} \rightarrow \mathfrak{L}$ . The Pontryagin duality theorem states that

$$G \simeq \chi(\chi(G)) = \hat{\hat{G}}.$$

This means that  $\chi$  is a contravariant category equivalence. (i.e. there are natural equivalences  $q_1 : 1 \rightarrow \chi^2$  and  $q_2 : \chi^2 \rightarrow 1$  where  $1 : \mathfrak{L} \rightarrow \mathfrak{L}$  is identity functor in  $\mathfrak{L}$ ). The next proposition is a well-known formal consequence of an equivalence of categories.

**Proposition 3.1**  $\chi$  takes projective limits to direct limits.

**Proof** Let  $\{G, f_\alpha\}$  be the projective limit of the projective system  $\{G_\alpha, f_\alpha^\beta\}$ . Then  $\chi$  induces morphisms  $\hat{f}_\alpha : \hat{G}_\alpha \rightarrow \hat{G}$  where  $\{\hat{G}_\alpha, \hat{f}_\alpha^\beta\}$  is a direct system in  $\mathfrak{L}$  satisfying  $\hat{f}_\beta \hat{f}_\alpha^\beta = \hat{f}_\alpha$  whenever  $\alpha \leq \beta$ . Let  $H$  be a locally compact abelian group and suppose  $h_\alpha : \hat{G}_\alpha \rightarrow H$  are morphisms such that  $h_\beta \hat{f}_\alpha^\beta = h_\alpha$  whenever  $\alpha \leq \beta$ . We apply  $\chi$  and we see that  $\{\hat{G}, \hat{f}_\alpha\}$  is projective limit of the projective system  $\{\hat{G}_\alpha, \hat{f}_\alpha^\beta\}$  by Pontryagin's duality theorem. Therefore, there is a unique morphism  $f : \hat{H} \rightarrow \hat{G}$  such that  $\hat{f}_\alpha f = \hat{h}_\alpha$  for every  $\alpha$ . Since  $\chi$  is a contravariant category equivalence, there is a morphism  $f_0 : \hat{G} \rightarrow H$  such that  $\hat{f}_0 = f$  and  $f_0$  is the unique morphism in  $Hom(\hat{G}, H)$  such that  $f_0 \hat{f}_\alpha = h_\alpha$  for every  $\alpha$  (see [16, Prop. 10.1]). Thus,  $\{\hat{G}, \hat{f}_\alpha\}$  is the direct limit of the direct system  $\{\hat{G}_\alpha, \hat{f}_\alpha^\beta\}$ .  $\square$

(\*) For the rest of this section  $G$  will be assumed to be a solenoid.  $X$  will be a compact  $G$ -space and TNHZ in  $X_G \rightarrow B_G$  with respect to  $H^*(-, \mathbb{Q})$ .

**Theorem 3.2** *Let  $N \subseteq G$  be a totally disconnected closed subgroup such that  $G/N$  is the circle group. If  $X^G = \emptyset$ , then  $X/N$  is TNHZ in  $(X/N)_{G/N} \rightarrow B_{G/N}$  with respect to  $H^*(-, \mathbb{Q})$ .*

**Proof** Since  $X \rightarrow X/N$  induces a homeomorphism,

$$X^G \approx (X/N)^{G/N},$$

we have

$$(X/N)^{G/N} = \emptyset$$

where  $(X/N)^{G/N}$  is the fixed point set of induced action of the circle group  $G/N$  on the orbit space  $X/N$ . Thus, all isotropy subgroups,  $(G/N)_{Nx}$ , of  $G/N$  are finite. It is obvious that any finite subgroups of the circle group are the groups of the  $n$ th roots of unity for some  $n$ . For the isotropy subgroups,  $G_x$  of  $G$  is explicitly discussed in ([10, Prop.10.31ff]) and we have

$$(G/N)_{Nx} = NG_x/N \simeq G_x/(G_x \cap N).$$

In particular, it follows that

$$G_x = \varprojlim_{N \in \mathcal{N}} (G/N)_{Nx}$$

where  $\mathcal{N}$  is a filter basis of compact normal subgroups of  $G$  such that  $G/N$  is a circle for  $N \in \mathcal{N}$  and such that  $\bigcap \mathcal{N} = 1$ . (For  $M \subseteq N$  in  $\mathcal{N}$ , let  $f_N^M : G/M \rightarrow G/N$  denote the natural homomorphism given by  $f_N^M(gM) = gN$ . Then restriction of  $f_N^M$  to the  $(G/M)_{Mx}$  gives a homomorphism from  $(G/M)_{Mx}$  into the  $(G/N)_{Nx}$ . This restricted homomorphism constitutes a projective system.) This implies that all isotropy subgroups of  $G$  are projective limits of finite cyclic groups.

For the cohomology of the universal classifying space  $B_{G_x}$  with integer coefficient, since  $G_x$  is the projective limit of a projective system of finite cyclic groups, we have

$$H^r(B_{G_x}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & r = 0, \\ \hat{G}_x & r = 2, \\ 0 & r \neq 0 \text{ or } 2. \end{cases}$$

(See [7, remarks for Theorem 1].) By Proposition 3.1, we have that  $H^2(B_{G_x}, \mathbb{Z}) = \hat{G}_x$  is the direct limit of the direct system  $\{(G/N)_{Nx}, \hat{f}_N^M\}$ . Since all  $(G/N)_{Nx}$  are finite cyclic groups,

$$(G/N)_{Nx} = (G/N)_{Nx} \text{ for all } N \in \mathcal{N}.$$

Thus,  $H^2(B_{G_x}, \mathbb{Q}) = \hat{G}_x \otimes_{\mathbb{Z}} \mathbb{Q}$  is the direct limit of the direct system

$$\{(G/N)_{Nx} \otimes_{\mathbb{Z}} \mathbb{Q}, \hat{f}_N^M \otimes 1_{\mathbb{Q}}\}.$$

It is a well-known fact that the tensor product of finite abelian groups and rationals over  $\mathbb{Z}$  is 0. Therefore, we have

$$(G/N)_{Nx} \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

It follows that

$$H^2(B_{G_x}, \mathbb{Q}) = 0.$$

Thus, for  $r \geq 1$ ,  $H^r(B_{G_x}, \mathbb{Q})$  are trivial for all  $x \in X$ . By the Vietoris–Begle mapping theorem (see [19]) the orbit projection  $\pi_2 : X_G \rightarrow X/G$  thus induces an isomorphism:

$$\pi_2^* : H^*(X/G, \mathbb{Q}) \cong H_G^*(X, \mathbb{Q}).$$

Similarly, we have

$$H^*(X/G, \mathbb{Q}) \cong H_{G/N}^*(X/N, \mathbb{Q})$$

by considering the orbit projection

$$(X/N)_{G/N} \rightarrow (X/N)/(G/N) \approx X/G.$$

On the other hand,

$$H^*(X, \mathbb{Q}) \cong H^*(X/N, \mathbb{Q})$$

by Remark 2.7. From the commutative diagram,

$$\begin{array}{ccc} H_G^*(X, \mathbb{Q}) & \xrightarrow{i_G^*} & H^*(X, \mathbb{Q}) \\ \uparrow \cong & & \uparrow \cong \\ H_{G/N}^*(X/N, \mathbb{Q}) & \xrightarrow{i_{G/N}^*} & H^*(X/N, \mathbb{Q}) \end{array}$$

we see that  $X/N$  is TNHZ in  $(X/N)_{G/N} \rightarrow B_{G/N}$  with respect to  $H^*(-, \mathbb{Q})$ . □

With the assumptions of (\*), we have 3 corollaries.

**Corollary 3.3** *If  $0 < \dim H^*(X, \mathbb{Q}) < \infty$ , then  $X^G \neq \emptyset$ .*

**Proof** Suppose  $X^G = \emptyset$ . Then  $X/N$  is TNHZ in  $(X/N)_{G/N} \rightarrow B_{G/N}$  by Theorem (3.2). Since  $X^G \approx (X/N)^{G/N}$ , we have

$$\dim H^*(X/N, \mathbb{Q}) = \dim H^*((X/N)^{G/N}, \mathbb{Q}) = 0$$

(see [2, Corollary 3.1.15]). Since  $H^*(X, \mathbb{Q}) \cong H^*(X/N, \mathbb{Q})$ , this contradicts the assumption. □

Next we need the notion of a rational cohomology  $n$ -manifold. A rational cohomology  $n$ -manifold is a locally compact space whose cohomological dimension over  $\mathbb{Q}$  is finite, and it has locally constant cohomologies over  $\mathbb{Q}$  such that it is equal to  $\mathbb{Q}$  for degree  $n$  and to zero in degrees other than  $n$ . A connected rational cohomology  $n$ -manifold over  $X$  is called orientable if  $H_c^n(X, \mathbb{Q}) \cong \mathbb{Q}$ . Details can be found in the work of Bredon (see [6], Section V.16).

Topological  $n$ -manifolds are examples of rational cohomology  $n$ -manifolds. A nonmanifold example is the open cone over the  $(n - 1)$ -manifold, which is not a sphere but has the rational cohomology of an  $(n - 1)$ -sphere (for example, a real projective space of odd dimensions).

The property of being a rational cohomology manifold passes to orbit spaces under some mild conditions.

**Theorem 3.4** (*See Raymond [18]*) *Let  $N$  be a second countable totally disconnected compact group acting on a connected orientable rational cohomology  $n$ -manifold  $X$ . Suppose the action of  $N$  on  $H_c^n(X, \mathbb{Q})$  is trivial. Then  $X/N$  is a rational cohomology  $n$ -manifold.*

The next corollary is an explicit application to compact ( $c$ -)symplectic (cohomology) manifolds without using any geometric concepts.

**Corollary 3.5** *If  $X$  is a compact connected  $c$ -symplectic orientable rational cohomology manifold, then the fixed point set is nonempty.*

**Proof** Let  $N$  be a compact, totally disconnected subgroup of  $G$  such that  $G/N$  is a circle.  $X/N$  is a compact connected  $c$ -symplectic rational cohomology manifold having the same dimensions as  $X$ . This follows by Raymond’s theorem (see [18]) and Remark 2.7. Since the induced action of the circle,  $G/N$ , on the orbit space  $X/N$  is almost effective, this action is cohomologically effective by Remark 2.2. Suppose  $X^G = \emptyset$ . Then  $X/N$  is TNHZ in  $(X/N)_{G/N} \rightarrow B_{G/N}$  by Theorem (3.2). Since TNHZ implies cohomologically Hamiltonian, by the result of Allday (see [1], Proposition 6.7 and Remark 6.8), the fixed point set  $(X/N)^{G/N}$  (which is homeomorphic to  $X^G$ ) is nonempty and it has at least 2 connected components, which contradicts the assumption.  $\square$

Next we need to recall some basic facts concerning commutative graded algebras. Let  $k$  be a field and  $A = \bigoplus_{i=0}^{\infty} A_i$  be an  $\mathbb{N}$ -graded  $k$ -algebra. We shall assume that  $A$  is connected: i.e.  $A_0 = k$ . We shall also assume that  $A$  is commutative in the graded sense: i.e. for any  $a \in A_i, b \in A_j, ba = (-1)^{ij}ab$ . In the category of connected commutative  $\mathbb{N}$ -graded  $k$ -algebras the free objects are those of the form  $k[x_i : i \in I] \otimes \bigwedge (y_j : j \in J)$  where  $k[x_i : i \in I]$  is the polynomial ring generated by  $\{x_i : i \in I\}$  where each  $x_i, i \in I$ , is homogeneous of positive even degree and  $\bigwedge (y_j : j \in J)$  is the exterior algebra generated by  $\{y_j : j \in J\}$  where each  $y_j, j \in J$ , is homogeneous of positive odd degree.

**Definition 3.6** *A connected commutative graded algebra  $A$  is said to be finitely generated if there is a homogeneous epimorphism of  $k$ -algebras of degree zero*

$$\pi : B = k[x_1, \dots, x_r] \otimes \bigwedge (y_1, \dots, y_s) \rightarrow A$$

where each  $x_i$  (resp  $y_j$ ) is homogeneous of positive even (resp. odd) degree. Then  $J = \text{Ker}\pi$  is called the ideal of relations. We shall refer to the exact sequence

$$0 \rightarrow J \rightarrow B \rightarrow A \rightarrow 0$$

as a presentation of  $A$ .

Let  $G$  be a circle group and  $R_G = H^*(B_G, \mathbb{Q})$ . It is well known that  $R_G = \mathbb{Q}[w]$ ,  $\text{deg} w = 2$ . Furthermore,  $\mathbb{Q}$  is an  $R_G$ -module via the standard augmentation homomorphism  $R_G \rightarrow \mathbb{Q}$  defined by  $w \rightarrow 1$ . The next theorem, proven by Chang (see [8, pp. 245-246]), is one of the steps of the corollary of Theorem 2 in [8].

**Theorem 3.7** (See Chang [8]) *Let  $X$  be a space, and suppose that  $H^*(X, \mathbb{Q})$  has a  $\mathbb{Q}$ -algebra presentation*

$$0 \rightarrow J \rightarrow \mathbb{Q}[x_1, \dots, x_g] \otimes \bigwedge (y_1, \dots, y_h) \rightarrow H^*(X, \mathbb{Q}) \rightarrow 0$$

where  $x_1, \dots, x_g$  are generators of positive even degree,  $y_1, \dots, y_h$  are generators of odd degree, and the ideal of relations  $J = (f_1, \dots, f_m, e_1, \dots, e_n)$ , where  $f_1, \dots, f_m$  are relations of even degree and  $e_1, \dots, e_n$  are relations of odd degree.



Let  $G$  be a circle group, and suppose that  $G$  is acting on  $X$  so that  $X$  is TNHZ in  $X_G \rightarrow B_G$  with respect to  $H^*(-, \mathbb{Q})$ .

Then  $H_G^*(X, \mathbb{Q})$  has an  $R_G$ -algebra presentation as follows, and there is a commutative diagram

$$\begin{CD} 0 @>>> J_G @>>> R_G[X_1, \dots, X_g] \otimes \bigwedge(Y_1, \dots, Y_h) @>>> H_G^*(X, \mathbb{Q}) @>>> 0 \\ @. @VVV @VV\varphi V @VVi_G^* V @. \\ 0 @>>> J @>>> \mathbb{Q}[x_1, \dots, x_g] \otimes \bigwedge(y_1, \dots, y_h) @>>> H^*(X, \mathbb{Q}) @>>> 0 \end{CD}$$

where  $J_G = (F_1, \dots, F_m, E_1, \dots, E_n), \varphi(X_i) = x_i$  for  $1 \leq i \leq g, \varphi(Y_i) = y_i$  for  $1 \leq i \leq h, \varphi(F_i) = f_i$  for  $1 \leq i \leq m$ , and  $\varphi(E_i) = e_i$  for  $1 \leq i \leq n$ .

We will prove that a similar result holds for solenoid actions on a compact space without a fixed point.

Recall that for a solenoid  $G$ , there is a totally disconnected closed subgroup  $N$  such that  $G/N$  is a circle group. It is well known that  $H^*(B_{G/N}, \mathbb{Q}) = \mathbb{Q}[w]$ ,  $\deg w = 2$ .

Let  $B_\pi : B_G \rightarrow B_{G/N}$  be the mapping induced by canonical epimorphism  $\pi : G \rightarrow G/N$ . In the following, let  $R_{G/N} = \mathbb{Q}[w]$  and  $R_G = \mathbb{Q}[v]$ ,  $v = B_\pi^*(w)$  where  $B_\pi^* : H^*(B_{G/N}, \mathbb{Q}) \rightarrow H^*(B_G, \mathbb{Q})$ .

**Corollary 3.8** Suppose we have assumptions of (\*) and suppose that  $H^*(X, \mathbb{Q})$  has a  $\mathbb{Q}$ -algebra presentation

$$0 \rightarrow J \rightarrow \mathbb{Q}[x_1, \dots, x_g] \otimes \bigwedge(y_1, \dots, y_h) \rightarrow H^*(X, \mathbb{Q}) \rightarrow 0$$

where  $x_1, \dots, x_g$  are generators of positive even degree,  $y_1, \dots, y_h$  are generators of odd degree, and the ideal of relations  $J = (f_1, \dots, f_m, e_1, \dots, e_n)$ , where  $f_1, \dots, f_m$  are relations of even degree and  $e_1, \dots, e_n$  are relations of odd degree.

If  $X^G = \emptyset$ , then  $H_G^*(X, \mathbb{Q})$  has an  $R_G$ -algebra presentation as follows and there is a commutative diagram

$$\begin{CD} 0 @>>> J_G @>>> R_G[\bar{X}_1, \dots, \bar{X}_g] \otimes \bigwedge(\bar{Y}_1, \dots, \bar{Y}_h) @>>> H_G^*(X, \mathbb{Q}) @>>> 0 \\ @. @VVV @VV\varphi V @VVi_G^* V @. \\ 0 @>>> J @>>> \mathbb{Q}[x_1, \dots, x_g] \otimes \bigwedge(y_1, \dots, y_h) @>>> H^*(X, \mathbb{Q}) @>>> 0 \end{CD}$$

where  $J_G = (F_1, \dots, F_m, E_1, \dots, E_n), \varphi(\bar{X}_i) = x_i$  for  $1 \leq i \leq g, \varphi(\bar{Y}_i) = y_i$  for  $1 \leq i \leq h, \varphi(F_i) = f_i$  for  $1 \leq i \leq m$ , and  $\varphi(E_i) = e_i$  for  $1 \leq i \leq n$ .

**Proof** Since the orbit projection  $\pi : X \rightarrow X/N$  induces an isomorphism  $\pi^* : H^*(X/N, \mathbb{Q}) \simeq H^*(X, \mathbb{Q})$ , we consider

$$0 \rightarrow J \rightarrow \mathbb{Q}[x_1, \dots, x_g] \otimes \bigwedge(y_1, \dots, y_h) \rightarrow H^*(X/N, \mathbb{Q}) \rightarrow 0$$

as a  $\mathbb{Q}$ -algebra presentation of  $H^*(X/N, \mathbb{Q})$ . On the other hand,  $X/N$  is TNHZ in  $(X/N)_{G/N} \rightarrow B_{G/N}$  with respect to  $H^*(-, \mathbb{Q})$  by Theorem 3.2. It follows by Chang's result that  $H_{G/N}^*(X/N, \mathbb{Q})$  has an  $R_{G/N}$ -presentation

$$f : R_{G/N}[X_1, \dots, X_g] \otimes \bigwedge(Y_1, \dots, Y_h) \rightarrow H_{G/N}^*(X/N, \mathbb{Q})$$

as in Theorem 3.7. Since  $X/N$  is TNHZ, it follows that

$$(\alpha, \pi)^* : H_{G/N}^*(X/N, \mathbb{Q}) \rightarrow H_G^*(X, \mathbb{Q})$$

is surjective where  $\alpha : G \rightarrow G/N, \pi : X \rightarrow X/N$ .

Let  $(\alpha, \pi)^*(X_i) = \bar{X}_i, i = 1, \dots, g$  and  $(\alpha, \pi)^*(Y_j) = \bar{Y}_j, j = 1, \dots, h$ . We define a homogeneous epimorphism of  $\mathbb{Q}$ -algebras of degree zero:

$$R_G[\bar{X}_1, \dots, \bar{X}_g] \otimes \bigwedge (\bar{Y}_1, \dots, \bar{Y}_h) \rightarrow H_G^*(X, \mathbb{Q}),$$

$$\bar{X}_i \mapsto (\alpha, \pi)^*(f(X_i)),$$

$$\bar{Y}_j \mapsto (\alpha, \pi)^*(f(Y_j)).$$

It is easy to check that this epimorphism satisfies all the conditions we need. □

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