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## On 4-dimensional almost para-complex pure-Walker manifolds

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**Abstract:** This paper is concerned with almost para-complex structures on Walker 4-manifolds. For these structures, we study some problems of Kähler manifolds. We also give an example of a flat almost para-complex manifold, which consists of a nonintegrable almost para-complex structure on Walker 4-manifolds.

**Key words:** Almost para-complex structure, pure metric, neutral metric, Walker metric, Kähler structure

### 1. Introduction

Let  $M_{2n}$  be a semi-Riemannian smooth manifold with the metric  $g$ , which is necessarily of neutral signature  $(n, n)$ , and let  $\mathfrak{S}_s^r(M_{2n})$  be the tensor field of  $M_{2n}$ , i.e. the field of all tensors of type  $(r, s)$  in  $M_{2n}$ .

An *almost para-complex structure* on  $M_{2n}$  is an affiner field  $\varphi$  on  $M_{2n}$ :  $\varphi^2 = I$ , and the 2 eigenbundles  $T^+M_{2n}$  and  $T^-M_{2n}$  corresponding to the two eigenvalues  $+1$  and  $-1$  have the same rank. The pair  $(M_{2n}, \varphi)$  is called an *almost para-complex manifold*.

Let  $(M_{2n}, \varphi)$  be an almost para-complex manifold with almost para-complex structure  $\varphi$ . If the Nijenhuis tensor of such a affiner field  $\varphi$  defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X, Y]$$

is equivalent to the vanish, for any vector fields  $X, Y$  on  $M_{2n}$ , then the almost para-complex structure  $\varphi$  is integrable and it is said to be a para-complex structure.

#### 1.1. Pure metrics

A *pure metric* with respect to the almost para-complex structure is a semi-Riemannian metric  $g$  such that

$$g(\varphi X, \varphi Y) = g(X, Y) \tag{1}$$

for any  $X, Y \in \mathfrak{S}_0^1(M_{2n})$ . For an almost para-complex manifold  $(M_{2n}, \varphi)$  with pure metric  $g$ , the triple  $(M_{2n}, \varphi, g)$  is called an almost para-complex manifold with pure metric  $g$  or almost para-complex pure metric manifold. If  $\varphi$  is integrable, then we say that  $(M_{2n}, \varphi, g)$  is a para-complex pure metric manifold. A similar geometry is generated if  $\varphi$  is an almost complex structure and acts as an antiisometry on the metric. Such metrics are known as B-metrics, Norden metrics, and anti-Hermitian metrics (see [1,2,4–10,16,17]).

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## 1.2. Holomorphic para-complex pure metric manifolds

Let the triple  $(M_{2n}, \varphi, g)$  be an almost para-complex manifolds with pure metric. A Tachibana operator applied to the pure metric  $g$  is given by [23]

$$(\Phi_{\varphi}g)(X, Y, Z) = g((\nabla_Y \varphi)Z, X) + g((\nabla_Z \varphi)X, Y) - g((\nabla_X \varphi)Y, Z), \quad (2)$$

for any  $X, Y \in \mathfrak{S}_0^1(M_{2n})$ . It is clear that a Tachibana operator  $\Phi_{\varphi}$  is an operator from  $\mathfrak{S}_2^0(M_{2n})$  to  $\mathfrak{S}_3^0(M_{2n})$ .

For an almost para-complex pure metric manifold  $(M_{2n}, \varphi, g)$ , if  $(\Phi_{\varphi}g)(X, Y, Z) = 0$  for any  $X, Y, Z \in \mathfrak{S}_0^1(M_{2n})$ , then a pure metric  $g$  is called a *holomorphic*. If the triple  $(M_{2n}, \varphi, g)$  is an almost para-complex pure metric manifold with holomorphic pure metric  $g$ , we say that  $(M_{2n}, \varphi, g)$  is a holomorphic para-complex pure metric manifold.

Now we give a theorem that we shall use later, which was proven in [17].

**Theorem 1** *An almost para-complex pure metric manifold is a holomorphic para-complex pure metric manifold if and only if the almost para-complex structure is parallel with respect to the Levi-Civita connection  $\nabla$  of  $g$ , i.e. the condition  $\Phi_{\varphi}g = 0$  is equivalent to  $\nabla\varphi = 0$ .*

Let  $(M_{2n}, \varphi, g)$  be an almost para-complex pure metric manifold. If  $\nabla\varphi = 0$ , where  $\nabla$  is the Levi-Civita connection of pure metric  $g$ , then the triple  $(M_{2n}, \varphi, g)$  is called a *Kähler para-complex pure metric manifold*. Therefore, from Theorem 1, the condition of being a holomorphic para-complex pure metric manifold of a manifold coincides with the condition of being a Kähler para-complex pure metric manifold of a manifold. From this, we can say that a holomorphic para-complex pure metric manifold is a Kähler para-complex pure metric manifold. In this paper, we shall use the term ‘Kähler para-complex pure metric manifold’ instead of ‘holomorphic para-complex pure metric manifold’.

## 2. A pure metric on a neutral 4-manifold

In the present paper, we shall focus our attention on 4-dimensional almost para-complex pure metric manifolds of neutral signature  $(+ + - -)$ . For the next step, it is appropriate to state a neutral metric  $g$  and the almost para-complex structure  $\varphi$  in terms of an orthonormal frame  $\{e_i\}$  ( $i = 1, \dots, 4$ ) of vectors, and its dual frame  $\{e^j\}$  ( $j = 1, \dots, 4$ ) of 1-forms. Actually, the metric  $g$  can be given by

$$g = (g(e_i, e_j)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3)$$

The almost para-complex structure  $\varphi$  can be written as

$$\varphi = (\varphi_i^j) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (4)$$

Furthermore, a simple form of pure metric  $g$ , which is pure with respect to  $\varphi$  in (4), can be written as

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

### 3. Pure-Walker metrics

In this section, using (4) and (5), we study some properties of para-complex pure metric manifolds on a Walker 4-manifold.

#### 3.1. Walker metrics

A triple  $(M_4, g, D)$  is said to be a 4-dimensional Walker manifold such that  $D$  is a 2-dimensional null plane and parallel distribution with respect to a neutral metric  $g$ . For such metrics a canonical form was obtained by Walker [22], showing the existence of suitable coordinates  $(x^1, x^2, x^3, x^4)$  where the metric is expressed as

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{6}$$

where  $a, b$ , and  $c$  are some functions of the coordinates  $(x^1, x^2, x^3, x^4)$ . The metric (6) is the most generic form of Walker metrics on the 4-dimensional Walker manifolds. Hereafter, we show by  $\partial_i = \partial/\partial x^i$  ( $i = 1, 2, 3, 4$ ) the coordinate tangent vectors and we use subscript for partial derivatives, i.e.  $h_i = \frac{\partial h}{\partial x^i}$ , for any function  $h$  depending on  $(x^1, x^2, x^3, x^4)$ . Such Walker metrics have been intensively investigated, e.g., [11,14,15,18,19].

In the present paper, we analyze a restricted form, as in [1,11,14], rather than the generic metric (6). The restricted Walker metric is the metric (6) with  $c = 0$ , i.e.

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \end{pmatrix}. \tag{7}$$

#### 3.2. Almost para-complex pure-Walker manifolds

Let  $(M_4, g)$  be a Walker 4-manifold with the Walker metric  $g$ , which is given in (7). If  $\{e_1, e_2, e_3, e_4\}$  and  $\{\partial_1, \partial_2, \partial_3, \partial_4\}$  are 2 orthonormal frames, the matrix  $A = (A_j^i)$  of change of coordinates satisfies

$$g = A^T g' A, \tag{8}$$

where the matrix  $A^T$  is the transpose matrix of the matrix  $A$ . In particular,  $\det(A) = \mp 1$ .

Substituting (3) and (7) into (8), one of the matrices  $A$ , which we will apply in the present analysis, is

$$A = (A_j^i) = \begin{pmatrix} \frac{1}{2}(1-a) & 0 & -\frac{1}{2}(1+a) & 0 \\ 0 & \frac{1}{2}(1-b) & 0 & -\frac{1}{2}(1+b) \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{9}$$

Furthermore, for affinors, the matrix  $A = (A_j^i)$  of change of coordinates satisfies

$$\varphi = A^{-1}\varphi'A, \tag{10}$$

where the matrix  $A^{-1}$  is the inverse matrix of the matrix  $A$ .

The inverse of the matrix (9),  $A^{-1}$ , is given by

$$A^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2}(1+a) & 0 \\ 0 & 1 & 0 & \frac{1}{2}(1+b) \\ -1 & 0 & \frac{1}{2}(1-a) & 0 \\ 0 & -1 & 0 & \frac{1}{2}(1-b) \end{pmatrix}. \tag{11}$$

Substituting (5) and (9) into (8), the pure-Walker metric  $g'$  is written as

$$g' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & \frac{1}{2}(a-b) \\ 1 & 0 & \frac{1}{2}(a-b) & 0 \end{pmatrix} \tag{12}$$

with respect to the natural frame  $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ .

Similarly, substituting (4), (9), and (11) into (10), the almost para-complex structure  $\varphi'$  is obtained as

$$\varphi' = \begin{pmatrix} 0 & -a & 0 & \frac{1}{2}(1-ab) \\ -b & 0 & \frac{1}{2}(1-ab) & 0 \\ 0 & 2 & 0 & b \\ 2 & 0 & a & 0 \end{pmatrix} \tag{13}$$

with respect to the natural frame  $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ . Thus, the triple  $(M_4, \varphi', g')$  is an almost para-complex pure-Walker manifold.

### 3.3. Para-complex pure-Walker manifolds ( $\varphi'$ -integrability)

The almost para-complex structure  $\varphi'$  is integrable iff the torsion of  $\varphi'$  (Nijenhuis tensor) vanishes, i.e. the components

$$(N_{\varphi'})^i_{j k} = \varphi_j^m \partial_m \varphi_k^i - \varphi_k^m \partial_m \varphi_j^i - \varphi_m^i \partial_j \varphi_k^m + \varphi_m^i \partial_k \varphi_j^m \tag{14}$$

all vanish [12, p. 124].

From (13) and (14), we get the following integrability condition:

**Theorem 2** *An almost para-complex pure-Walker manifold  $(M_4, \varphi', g')$  is a para-complex pure-Walker manifold ( $\varphi'$ -integrability) if and only if*

$$a_1 = b_2 = 0, \quad ba_2 - 2a_4 = 0, \quad ab_1 - 2b_3 = 0. \tag{15}$$

From this theorem, we easily see that if  $a = b$  or  $a = -b$ , then  $\varphi'$  is integrable.

### 4. Kähler para-complex pure-Walker manifolds

In this part, we study some problems of Kähler manifolds on the almost para-complex pure-Walker manifold  $(M_4, \varphi', g')$ .

### 4.1. Kähler pure-Walker metrics

Let  $(M_4, \varphi', g')$  be an almost para-complex pure-Walker manifold. If

$$(\Phi_{\varphi'} g')_{kij} = \varphi'_k{}^m \partial_m g'_{ij} - \varphi'_i{}^m \partial_k g'_{mj} + g'_{mj} (\partial_i \varphi'_k{}^m - \partial_k \varphi'_i{}^m) + g'_{im} \partial_j \varphi'_k{}^m = 0, \tag{16}$$

then on account of Theorem 1,  $\varphi'$  is integrable and the triple  $(M_4, \varphi', g')$  is called a *Kähler para-complex pure-Walker manifold*.

Since  $(\Phi_{\varphi'} g')_{kij} = (\Phi_{\varphi'} g')_{kji}$ , we need only consider  $(\Phi_{\varphi'} g')_{kij}$  ( $i < j$ ). Substituting (12) and (13) into (16), we find

$$\begin{aligned} (\Phi_{\varphi'} g')_{113} &= -a_1 + b_1, (\Phi_{\varphi'} g')_{123} = b_2, (\Phi_{\varphi'} g')_{124} = b_1, (\Phi_{\varphi'} g')_{133} = -aa_1 + 2b_3, \\ (\Phi_{\varphi'} g')_{134} &= -\frac{1}{2}b(a_2 - b_2) - a_4, (\Phi_{\varphi'} g')_{144} = bb_1, (\Phi_{\varphi'} g')_{213} = -a_2, (\Phi_{\varphi'} g')_{214} = -a_1, \\ (\Phi_{\varphi'} g')_{224} &= -a_2 + b_2, (\Phi_{\varphi'} g')_{233} = -aa_2, (\Phi_{\varphi'} g')_{234} = -\frac{1}{2}a(a_1 + b_1) - b_3, \\ (\Phi_{\varphi'} g')_{244} &= -2a_4 + bb_2, (\Phi_{\varphi'} g')_{311} = 2a_1, (\Phi_{\varphi'} g')_{312} = a_2, (\Phi_{\varphi'} g')_{313} = \frac{1}{2}a(a_1 + b_1), \\ (\Phi_{\varphi'} g')_{314} &= a_4, (\Phi_{\varphi'} g')_{323} = \frac{1}{2}a(a_2 + b_2), (\Phi_{\varphi'} g')_{324} = b_3, (\Phi_{\varphi'} g')_{333} = ab_3, \\ (\Phi_{\varphi'} g')_{334} &= \frac{1}{4}(1 - ab)(a_2 - b_2) + aa_4, (\Phi_{\varphi'} g')_{344} = bb_3, (\Phi_{\varphi'} g')_{412} = -b_1, \\ (\Phi_{\varphi'} g')_{413} &= -a_4, (\Phi_{\varphi'} g')_{414} = -\frac{1}{2}b(a_1 + b_1), (\Phi_{\varphi'} g')_{422} = -2b_2, (\Phi_{\varphi'} g')_{423} = -b_3, \\ (\Phi_{\varphi'} g')_{424} &= -\frac{1}{2}b(a_2 + b_2), (\Phi_{\varphi'} g')_{433} = -aa_4, (\Phi_{\varphi'} g')_{434} = \frac{1}{4}(1 - ab)(a_1 - b_1) - bb_3, \\ (\Phi_{\varphi'} g')_{444} &= -ba_4. \end{aligned} \tag{17}$$

From the above equations, we have:

**Theorem 3** *An almost para-complex pure-Walker manifold  $(M_4, \varphi', g')$  is a Kähler para-complex pure-Walker manifold, i.e.  $\Phi_{\varphi'} g' = 0$ , if and only if  $a$  is only dependent on  $x^3$  and  $b$  is only dependent on  $x^4$ . In fact, it holds that  $a$  and  $b$  satisfy the following PDEs:*

$$a_1 = a_2 = a_4 = b_1 = b_2 = b_3 = 0. \tag{18}$$

### 4.2. Quasi-Kähler para-complex pure-Walker metrics

The basic class of almost para-complex manifolds with pure metrics is the class of the quasi-Kähler manifolds with pure metrics, which is the basic class with nonintegrable almost para-complex structure [5,13]. A *quasi-Kähler manifold* is an almost para-complex pure metric manifold  $(M_{2n}, \varphi, g)$  such that

$$\sigma_{X,Y,Z} g((\nabla_X \varphi)Y, Z) = 0, \tag{19}$$

where  $\sigma$  is the cyclic sum over  $X, Y, Z$ .

From (2) and (19), we get

$$\sigma_{X,Y,Z} g((\nabla_X \varphi)Y, Z) = (\Phi_\varphi g)(X, Y, Z) + 2g((\nabla_X \varphi)Y, Z) = 0 \tag{20}$$

for all vector fields  $X, Y$ , and  $Z$  on  $M_{2n}$ .

Let  $(M_{2n}, \varphi, g)$  be an almost para-complex pure metric manifold. The associated pure metric of the almost para-complex pure metric manifold is defined by  $G(X, Y) = (g \circ \varphi)(X, Y)$  for any  $X, Y, Z \in \mathfrak{S}_0^1(M_{2n})$ . We can easily see that  $G$  is a new pure metric, which is also called the twin metric of  $g$  (see [3]). Then, on account of (20), an almost para-complex pure-Walker manifold  $(M_4, \varphi', g')$  is a *quasi-Kähler para-complex pure-Walker manifold* if

$$\Phi_k g'_{ij} + 2\nabla_k G_{ij} = 0, \tag{21}$$

where  $G$  is defined by  $G_{ij} = \varphi_i^h g'_{hj}$ .

From (12) and (13), for the twin pure metric  $G$ , we obtain

$$G = (G_{ij}) = \begin{pmatrix} 2 & 0 & a & 0 \\ 0 & -2 & 0 & -b \\ a & 0 & \frac{1}{2}(a^2 - 1) & 0 \\ 0 & -b & 0 & -\frac{1}{2}(b^2 - 1) \end{pmatrix}. \tag{22}$$

Therefore, we can say that the twin pure metric  $G$  is not Walker.

Now, using a straightforward calculation, the inverse of the pure-Walker metric  $g'$  is given by

$$(g')^{-1} = \begin{pmatrix} 0 & \frac{1}{2}(a-b) & 0 & 1 \\ \frac{1}{2}(a-b) & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Furthermore, the Levi-Civita connection of a pure-Walker metric (12) is given by

$$\begin{aligned} \nabla_{\partial_1} \partial_3 &= \frac{1}{4}(a_1 - b_1)\partial_1, \nabla_{\partial_1} \partial_4 = -\frac{1}{4}(a_1 - b_1)\partial_2, \nabla_{\partial_2} \partial_3 = \frac{1}{4}(a_2 - b_2)\partial_1, \\ \nabla_{\partial_2} \partial_4 &= -\frac{1}{4}(a_2 - b_2)\partial_2, \nabla_{\partial_3} \partial_1 = \frac{1}{4}(a_1 - b_1)\partial_1, \nabla_{\partial_3} \partial_2 = \frac{1}{4}(a_2 - b_2)\partial_1, \\ \nabla_{\partial_3} \partial_3 &= \frac{1}{2}(a_3 - b_3)\partial_1, \nabla_{\partial_4} \partial_1 = -\frac{1}{4}(a_1 - b_1)\partial_2, \nabla_{\partial_4} \partial_2 = -\frac{1}{4}(a_2 - b_2)\partial_2, \\ \nabla_{\partial_3} \partial_4 &= -\frac{1}{8}(a-b)(a_2 - b_2)\partial_1 - \frac{1}{8}(a-b)(a_1 - b_1)\partial_2 + \frac{1}{4}(a_2 - b_2)\partial_3 \\ &\quad - \frac{1}{4}(a_1 - b_1)\partial_4, \\ \nabla_{\partial_4} \partial_3 &= -\frac{1}{8}(a-b)(a_2 - b_2)\partial_1 - \frac{1}{8}(a-b)(a_1 - b_1)\partial_2 + \frac{1}{4}(a_2 - b_2)\partial_3 \\ &\quad - \frac{1}{4}(a_1 - b_1)\partial_4, \\ \nabla_{\partial_4} \partial_4 &= -\frac{1}{2}(a_4 - b_4)\partial_2. \end{aligned} \tag{24}$$

For the covariant derivative  $\nabla G$  of the twin pure metric  $G$ , put  $(\nabla G)_{ijk} = \nabla_i G_{jk}$ . Then, from (22) and (24), after some calculations, we obtain

$$\begin{aligned}
 \nabla_1 G_{13} &= \frac{1}{2}(a_1 + b_1), \nabla_1 G_{24} = -\frac{1}{2}(a_1 + b_1), \nabla_1 G_{33} = \frac{1}{2}a(a_1 + b_1), \\
 \nabla_1 G_{44} &= -\frac{1}{2}b(a_1 + b_1), \nabla_2 G_{24} = -\frac{1}{2}(a_2 + b_2), \nabla_2 G_{33} = \frac{1}{2}a(a_2 + b_2), \\
 \nabla_2 G_{44} &= -\frac{1}{2}b(a_2 + b_2), \nabla_3 G_{11} = -a_1 + b_1, \nabla_3 G_{12} = \frac{1}{2}(-a_2 + b_2), \\
 \nabla_3 G_{13} &= \frac{1}{4}a(-a_1 + b_1) + b_3, \nabla_3 G_{14} = -\frac{1}{4}b(a_2 - b_2), \nabla_3 G_{23} = \frac{1}{4}a(-a_2 + b_2), \\
 \nabla_3 G_{24} &= \frac{1}{4}a(-a_1 + b_1) - b_3, \nabla_3 G_{33} = ab_3, \nabla_3 G_{34} = \frac{1}{8}(1 - ab)(a_2 - b_2), \\
 \nabla_3 G_{44} &= \frac{1}{4}(1 - ab)(a_1 - b_1) - bb_3, \nabla_4 G_{12} = \frac{1}{2}(-a_1 + b_1), \\
 \nabla_4 G_{13} &= \frac{1}{4}b(-a_2 + b_2) + a_4, \nabla_4 G_{14} = \frac{1}{4}b(-a_1 + b_1), \nabla_4 G_{22} = -a_2 + b_2, \\
 \nabla_4 G_{23} &= \frac{1}{4}a(-a_1 + b_1), \nabla_4 G_{24} = \frac{1}{4}b(-a_2 + b_2) - a_4, \nabla_4 G_{33} = \frac{1}{4}(1 - ab)(a_2 - b_2) + aa_4, \\
 \nabla_4 G_{34} &= \frac{1}{4}b(-a_2 + b_2), \nabla_4 G_{44} = -ba_4.
 \end{aligned} \tag{25}$$

From (17), (21), and (25), we get:

**Theorem 4** *An almost para-complex pure-Walker manifold  $(M_4, \varphi', g')$  is a quasi-Kähler para-complex pure-Walker manifold, i.e.  $\Phi_k g'_{ij} + 2\nabla_k G_{ij} = 0$ , if and only if*

$$a_1 = a_2 = a_4 = b_1 = b_2 = b_3 = 0. \tag{26}$$

### 4.3. Isotropic-Kähler structures on almost para-complex pure-Walker 4-manifolds

Let  $(M_4, \varphi', g')$  be an almost para-complex pure-Walker manifold. If  $\nabla\varphi' = 0$ , then  $(M_4, \varphi', g')$  becomes a Kähler para-complex pure-Walker manifold. If  $(M_4, \varphi', g')$  is a Kähler para-complex pure-Walker manifold, then  $\|\nabla\varphi'\|^2$  vanishes. However, the inverse position is not always being true. That is, in general, the vanishing of the square norm  $\|\nabla\varphi'\|^2$  does not always imply the Kähler condition  $\nabla\varphi' = 0$ .

An almost para-complex pure metric manifold satisfying the condition  $\|\nabla\varphi'\|^2 = 0$  ( $\nabla\varphi' \neq 0$ ) is called an *isotropic-Kähler manifold*. Here, the square norm  $\|\nabla\varphi'\|^2$  of  $\nabla\varphi'$  is defined by

$$\|\nabla\varphi'\|^2 = g'^{ij} g'^{kl} g'_{ms} \nabla_i \varphi'_k{}^m \nabla_j \varphi'_l{}^s. \tag{27}$$

Now, with long but straightforward calculations, we find the following condition:

**Theorem 5** *An almost para-complex pure-Walker manifold  $(M_4, \varphi', g')$  is a isotropic-Kähler para-complex pure-Walker manifold, i.e.  $\|\nabla\varphi'\|^2 = 0$ , if and only if*

$$a_1 + b_1 = 0, \quad a_2 + b_2 = 0. \tag{28}$$



#### 4.4. Nearly-Kähler structures on almost para-complex pure-Walker 4-manifolds

Let  $(M_4, \varphi', g')$  be an almost para-complex pure-Walker manifold. If

$$(\nabla_X \varphi') Y + (\nabla_Y \varphi') X = 0, \quad (\nabla_i \varphi'_j{}^k + \nabla_j \varphi'_i{}^k = 0) \tag{29}$$

for all vector fields  $X, Y$ , and  $Z$  on  $M_{2n}$ , then the triple  $(M_4, \varphi', g')$  is called a *nearly-Kähler manifold* with pure metric [20,24].

Therefore, from (29), we have:

**Theorem 6** *A triple  $(M_4, \varphi', g')$  is a nearly-Kähler para-complex pure-Walker manifold if and only if*

$$a_1 = a_2 = a_4 = b_1 = b_2 = b_3 = 0. \tag{30}$$

#### 5. On flat nonintegrable almost para-complex pure-Walker manifold

In this section, we give an example of a 4-dimensional flat almost para-complex pure-Walker manifold whose almost para-complex structure is not integrable. Examples of 4-dimensional almost Hermitian manifolds whose almost complex structures are not integrable are given in [21]. Additionally, examples of flat almost Norden manifolds with nonintegrable almost complex structures are given in [1].

Let  $(M_4, \varphi', g')$  be an almost para-complex pure-Walker manifold with pure-Walker metric  $g'$  in (12) and almost para-complex structure  $\varphi'$  in (13). We see that  $g'$  depends on the difference  $a - b$ . If we put

$$a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = \alpha(x^1, x^2, x^3, x^4), \tag{31}$$

then the nonzero components  $R'_{ijkl}$  of the Riemannian curvature tensor  $R'$  with respect to pure-Walker metric  $g'$  are given by

$$\begin{aligned} R'_{1314} &= -\frac{1}{2}\alpha_{11}, R'_{1324} = R'_{1423} = -\frac{1}{2}\alpha_{12}, R'_{1334} = -\frac{1}{2}\alpha_{13} + \frac{1}{4}\alpha_1\alpha_2, \\ R'_{1434} &= \frac{1}{2}\alpha_{14} - \frac{1}{4}(\alpha_1)^2, R'_{2324} = -\frac{1}{2}\alpha_{22}, R'_{2334} = -\frac{1}{2}\alpha_{23} + \frac{1}{4}(\alpha_2)^2, \\ R'_{2434} &= \frac{1}{2}\alpha_{24} - \frac{1}{4}\alpha_1\alpha_2, R'_{3434} = \alpha_{34} - \frac{1}{2}\alpha\alpha_1\alpha_2. \end{aligned} \tag{32}$$

For seeking a flat metric, we suppose that

$$a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = 2\alpha(x^3, x^4). \tag{33}$$

In this case, the nonzero component of the curvature tensor  $R'$  is

$$R'_{3434} = \alpha_{34}. \tag{34}$$

We see that the pure-Walker metric  $g'$  is still not flat. If  $\alpha(x^3, x^4) = \beta(x^3) + \gamma(x^4)$ , then the pure-Walker metric  $g'$  is flat.

**Corollary 1** *The pure-Walker metric*

$$g' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & \frac{1}{2}(\alpha(x^3) + \beta(x^4)) \\ 1 & 0 & \frac{1}{2}(\alpha(x^3) + \beta(x^4)) & 0 \end{pmatrix} \quad (35)$$

is flat.

From (15), for almost para-complex structure  $\varphi'$ , we have:

**Corollary 2** If  $a$  and  $b$  satisfy

$$a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = 2(\beta(x^3) + \gamma(x^4)), \quad (36)$$

then  $\varphi'$  is not integrable in general. Moreover,  $\varphi'$  is integrable if and only if  $a_1 = a_2 = a_4 = 0$ ,  $a_3 - 2\beta_3 = 0$ .

Therefore, we have:

**Theorem 7** Let  $(M_4, \varphi', g')$  be an almost para-complex pure-Walker manifold endowed with the metric  $g'$  as in (35) and an almost para-complex structure  $\varphi'$  as in (13). If  $a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = 2(\beta(x^3) + \gamma(x^4))$  as in (36), then the almost para-complex pure-Walker manifold  $(M_4, \varphi', g')$  admits a flat pure-Walker metric  $g'$  and nonintegrable almost para-complex structure  $\varphi'$ .

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## References

- [1] Bonome A, Castro R, Hervella LM, Matsushita Y. Construction of Norden structures on neutral 4-manifolds. JP J Geom Topol 2005; 5: 121–140.
- [2] Bonome A, Castro R, Hervella LM, Matsushita Y. Flat almost Norden metrics with nonintegrable almost complex structures in dimension four. JP J Geom Topol 2005; 5: 141–153.
- [3] Borowiec A, Ferraris M, Francaviglia M, Volovich I. Almost complex and almost-product Einstein manifolds from a variational principle. J Math Phys 1999; 40: 3446–3464.
- [4] Gadea PM, Grifone J, Munoz Masque J. Manifolds modelled over free modules over the double numbers. Acta Math Hungar 2003; 100: 187–203.
- [5] Ganchev GT, Borisov AV. Note on the almost complex manifolds with a Norden metric. CR Acad Bulgarie Sci 1986; 39: 31–34.
- [6] Gribachev KI, Mekerov DG, Djelepov GD. Generalized B-manifolds. Compt Rend Acad Bulg Sci 1985; 38: 299–302.
- [7] Gribachev KI, Djelepov GD, Mekerov DG. On some subclasses of generalized B-manifolds. Compt Rend Acad Bulg Sci 1985; 38: 437–440.
- [8] Gribachev KI, Mekerov DG, Djelepov GD. On the geometry of almost B-manifolds. Compt Rend Acad Bulg Sci 1985; 38: 563–566.
- [9] Iscan M, Salimov AA. On Kähler-Norden manifolds. Proc Indian Acad Sci Math Sci 2009; 119: 71–80.
- [10] Iscan M, Magden A. On B-manifolds defined by algebra of plural numbers. The Arabian Journal for Science and Engineering 2010; 35: 57–63.
- [11] Iscan M. On Norden structures on neutral 4-manifolds with almost paracomplex structures. International Journal of Geometric Methods in Modern Physics 2010; 10: 1350052.

- [12] Kobayashi S, Nomizu K. *Foundations of Differential Geometry, Vol. 2*. New York, NY, USA: John Wiley, 1969.
- [13] Manev M, Mekerov D. On Lie groups as quasi-Kähler manifolds with Killing Norden metric. *Adv Geom* 2008; 8: 343–352.
- [14] Matsushita Y. Four-dimensional Walker metrics and symplectic structure. *J Geom Phys* 2004; 52: 89–99.
- [15] Matsushita Y. Walker 4-manifolds with proper almost complex structure. *J Geom Phys* 2005; 55: 385–398.
- [16] Mekerov DG. On some classes of almost B-manifolds. *Compt Rend Acad Bulg Sci* 1985; 38: 559–561.
- [17] Salimov AA, Iscan M, Etayo F. Paraholomorphic B-manifold and its properties. *Topol Appl* 2007; 154: 925–933.
- [18] Salimov AA, Iscan M. Some properties of Norden Walker metrics. *Kodai Math J* 2010; 33: 283–293.
- [19] Salimov AA, Iscan M, Akbulut K. Notes on para-Norden-Walker 4-manifolds. *International Journal of Geometric Methods in Modern Physics* 2010; 7: 1331–1347.
- [20] Schäfer L. Conical Ricci-flat nearly para-Kähler manifolds. *Ann Glob Anal Geom* 2014; 45: 11–24.
- [21] Tricerri F, Vanhecke L. Flat almost Hermitian manifolds which are not Kähler manifolds. *Tensor (NS)* 1977; 31: 249–254.
- [22] Walker AG. Canonical form for a Riemannian space with a parallel field of null planes. *Quart J Math Oxford* 1950; 1: 69–79.
- [23] Yano K, Ako M. On certain operators associated with tensor fields. *Kodai Math Sem Rep* 1968; 20: 414–436.
- [24] Yano K, Kon M. *Structure on Manifolds*. Singapore: World Scientific, 1984.