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Disjoint supercyclic powers of weighted shifts on weighted sequence spaces

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Abstract: We characterize the disjoint supercyclicity of finitely many different powers of weighted shifts acting on the weighted sequence spaces $l^2(\mathbb{N}, w)$, $c_0(\mathbb{N}, w)$, and $l^2(\mathbb{Z}, w)$, $c_0(\mathbb{Z}, w)$, where $w = (w_i)_i$ is a positive weight sequence satisfying $w_i \geq 1$ for every $i \in \mathbb{N}$ (or $i \in \mathbb{Z}$).

Key words: Disjoint supercyclic, unilateral shifts, bilateral shifts, weighted sequence spaces

1. Introduction

Let $L(X)$ denote the space of linear continuous operators on a separable infinite dimensional Fréchet space X. For a positive integer n, the nth iterate of $T \in L(X)$, denoted by T^n, is the operator obtained by composing T with itself n times.

An operator $T \in L(X)$ is called *hypercyclic* (respectively, *supercyclic*) provided that there is some $f \in X$ such that the orbit $Orb(T, f) = \{T^n f : n = 0, 1, \cdots\}$ (respectively, the projective orbit $\{\lambda T^n f : \lambda \in \mathbb{C}, n = 0, 1, 2, \cdots\}$) is dense in X. Such a vector f is said to *hypercyclic* (respectively, *supercyclic*) for T. Supercyclicity was introduced in the 1960s by Hilden and Wallen [10]. They proved that every unilateral weighted shift is *supercyclic*. From 1991, this property was studied extensively; for example, see the work of Godefroy and Shapiro [8]. The first example of a *supercyclic* operator in infinite-dimensional Banach spaces (moreover, *hypercyclic*) was discovered by Rolewicz [13] in 1969. Apart from *supercyclicity*, other properties have also been studied in recent years. For example, Liang and Zhou [11] characterized the *hereditarily hypercyclicity* of the unilateral (or bilateral) weighted shifts. Zhang and Zhou [19] studied *disjoint mixing* weighted backward shifts on the space of all complex-valued square summable sequences. We refer the readers to these papers and the references therein. As we all know, the Hypercyclicity Criterion and the Supercyclicity Criterion play important roles in showing the dynamic behaviors of $T \in L(X)$. In this paper, we are mainly concerned with the disjoint supercyclicity of $T \in L(X)$. Thus, only the Supercyclicity Criterion and the Disjoint Supercyclicity Criterion are given. For other criteria, we refer the readers to the books [1, 9]. The following theorem is due to Salas [15].

Theorem 1.1 *[1, Definition 1.13](Supercyclicity Criterion) Let $T \in L(X)$. Suppose that there exist 2 dense subsets Y and Z in X, an increasing sequence (n_k) of positive integers, and a sequence of maps $S_{n_k} : Z \to X$*

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such that:

$$(1)\ \ T^{n_k}S_{n_k}z \to z \text{ for every } z \in Z.$$
$$(2)\ \ \|T^{n_k}y\|\|S_{n_k}z\| \to 0 \text{ for every } y \in Y \text{ and every } z \in Z.$$

Then T is supercyclic.

Remark 1.2 *The Supercyclicity Criterion is a sufficient condition for assuring the supercyclicity of $T \in L(X)$.*

It is well known that the space of all (real or complex) sequences is defined as follows:

$$\mathbb{K}^{\mathbb{N}} = \{(x_n)_n : \ x_n \in \mathbb{K},\ n \in \mathbb{N}\} \quad \text{or} \quad \mathbb{K}^{\mathbb{Z}} = \{(x_i)_i : \ x_i \in \mathbb{K},\ i \in \mathbb{Z}\},$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathbb{Z}$ (or $\mathbb{N}$) is the set of all (nonnegative) integers. The natural concept of convergence is that of *coordinatewise convergence*. As we all know, the spaces $l^p(\mathbb{N}), l^p(\mathbb{Z}), c_0(\mathbb{N})$, and $c_0(\mathbb{Z})$ are separable subspaces of $\mathbb{K}^{\mathbb{N}}$. We refer the readers to the book [9] by Grosse-Erdmann and Peris Manguillot . Similarly, we define the weighted sequence spaces $l^2(\mathbb{N}, w)$, $l^2(\mathbb{Z}, w)$, $c_0(\mathbb{N}, w)$, and $c_0(\mathbb{Z}, w)$, where $w = (w_i)_i$ is a positive weight sequence satisfying $w_i \geq 1$ for every $i \in \mathbb{N}$ (or $i \in \mathbb{Z}$).

The weighted space $l^2(\mathbb{N}, w)$, indexed over $\mathbb{N}$, is defined by

$$l^2(\mathbb{N}, w) := \Big\{x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} : \ \sum_{n\in\mathbb{N}} w_n^2 |x_n|^2 < \infty\Big\}. \tag{1.1}$$

The weighted space $l^2(\mathbb{Z}, w)$, indexed over $\mathbb{Z}$, is defined analogously:

$$l^2(\mathbb{Z}, w) := \Big\{x = (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \ \sum_{i\in\mathbb{Z}} w_i^2 |x_i|^2 < \infty\Big\}. \tag{1.2}$$

It is clear that $l^2(\mathbb{N}, w)$ and $l^2(\mathbb{Z}, w)$ are *separable* Banach spaces under the norms $\|x\| = \Big(\sum\limits_{n\in\mathbb{N}} w_n^2|x_n|^2\Big)^{1/2}$ and $\|x\| = \Big(\sum\limits_{i\in\mathbb{Z}} w_i^2|x_i|^2\Big)^{1/2}$, respectively.

The weighted space $c_0(\mathbb{N}, w)$, indexed over $\mathbb{N}$, is defined by

$$c_0(\mathbb{N}, w) := \Big\{x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} : \ \lim_{n\to\infty} w_n|x_n| = 0\Big\}. \tag{1.3}$$

The weighted space $c_0(\mathbb{Z}, w)$, indexed over $\mathbb{Z}$, is defined analogously:

$$c_0(\mathbb{Z}, w) := \Big\{x = (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \ \lim_{|i|\to\infty} w_i|x_i| = 0\Big\}. \tag{1.4}$$

It is obvious that $c_0(\mathbb{N}, w)$ and $c_0(\mathbb{Z}, w)$ are *separable* Banach spaces under the sup-norms $\|x\| = \sup\limits_{n\in\mathbb{N}} w_n|x_n|$ and $\|x\| = \sup\limits_{i\in\mathbb{Z}} w_i|x_i|$, respectively.

The basic model of all shifts is the *backward shift*, defined as follows:

$$B(x_0, x_1, x_2, \cdots) = (x_1, x_2, x_3, \cdots).$$

Rolewicz [13] showed that, for any λ with $|\lambda|>1$, the multiples of B,

$$\lambda B(x_n)_n = (\lambda x_{n+1})_n,$$

are *hypercyclic* on the sequence space $l^2(\mathbb{N})$. It is then a small step to let the weights vary from coordinate to coordinate, leading to the weighted shifts. Now given a *bounded* sequence $a=(a_l)_l$ of nonzero weights, let $B_a : X \to X$ be *unilateral weighted shift*

$$x=(x_0,x_1,\cdots)\overset{B_a}{\to}(a_1x_1,a_2x_2,\cdots).$$

Remark 1.3 *The unilateral weighted shift B_a is an operator on $l^2(\mathbb{N},w)$ (or $c_0(\mathbb{N},w)$) if and only if there is an $M>0$ such that, for all $x\in l^2(\mathbb{N},w)$ (or $c_0(\mathbb{N},w)$),*

$$\sum_{n=1}^{\infty}|x_{n+1}|^2|a_{n+1}|^2w_n^2 \le M\sum_{n=1}^{\infty}|x_n|^2w_n^2,$$
$$(or \quad \sup_n |x_{n+1}a_{n+1}|w_n < M\sup_n |x_n||w_n|),$$

which is equivalent to

$$\sup_{n\in\mathbb{N}}\frac{|a_{n+1}|w_n}{w_{n+1}}<\infty.$$

For other spaces we can obtain similar characterizations, so we omit them. In the following, we can always assume that B_a is an operator on the space X due to Proposition 3.1.

Salas [14, 15] characterized the hypercyclicity and supercyclicity of B_a in terms of the weighted sequence $a=(a_l)_l$. For the *bilateral weighted shift*, we define the following: let $(e_j)_{j\in\mathbb{Z}}$ be a basis in a separable Banach space X and $a=(a_l)_{l\in\mathbb{Z}}$ be a *bounded* bilateral sequence of nonzero scalars; then the associated backward shift on X given by

$$B_ae_j = a_je_{j-1}\ (j\in\mathbb{Z}).$$

Now if given $N\ge 2$ operators $T_1,\cdots,T_N\in L(X)$, it has been natural to study whether the *hypercyclic* (or *supercyclic*) properties of their direct sum $T_1\oplus\cdots\oplus T_N$ may inherit from those of $T_1,T_2,\cdots,T_N$. In 2007, Bès and Peris [7] and, independently, Bernal [2] investigated the property of the orbits

$$\{(z,z,\cdots,z),(T_1z,T_2z,\cdots,T_Nz),(T_1^2z,T_2^2z,\cdots,T_N^2z),\cdots\}\ (z\in X).$$

They studied the case when one of these orbits is dense in X^N endowed with the product topology. If there is some vector $z\in X$ satisfying the above condition, the operators $T_1,\cdots,T_N$ are called *disjoint hypercyclic* (*d-hypercyclic*). Similarly, if there is some vector $\hat{z}\in X$ such that the projective orbit

$$\mathbb{C}\{(\hat{z},\hat{z},\cdots,\hat{z}),(T_1\hat{z},T_2\hat{z},\cdots,T_N\hat{z}),(T_1^2\hat{z},T_2^2\hat{z},\cdots,T_N^2\hat{z}),\cdots\}$$

is dense in the product space X^N, then the operators $T_1,\cdots,T_N$ are called *disjoint supercyclic* (*d-supercyclic*). In recent years, there have been some papers that studied the *disjoint hypercyclic* of finitely many weighted shifts. We refer the readers to section 4 in [7] and chapter 4 in [12], respectively. The papers by Bès et al.

[3, 4, 5, 7, 6], Salas [16, 17], and Shkarin [18] further explored different aspects of disjoint hypercyclicity. In our paper, we mainly investigate the weaker property–*disjoint supercyclic* of *weighted shifts* acting on the *weighed sequence spaces* $l^2(\mathbb{N}, w)$, $l^2(\mathbb{Z}, w)$, $c_0(\mathbb{N}, w)$, and $c_0(\mathbb{Z}, w)$. From another standpoint, we extend the results about *disjoint supercyclic* of weighted shifts in the paper [12] to general weighted sequence spaces. The structure of this paper is as follows: Section 2 gives some definitions and propositions. The disjoint supercyclicity of powers of weighted shifts acting on the weighted sequence spaces $l^2(\mathbb{N}, w)$, $l^2(\mathbb{Z}, w)$, $c_0(\mathbb{N}, w)$, and $c_0(\mathbb{Z}, w)$ are given in Sections 3 and 4.

2. Some definitions

To be more precise, we need to quote some definitions and propositions for our further application. In particular, the d-Supercyclicity Criterion will be used to show our main results. In the remainder of our paper, we will write *d-supercyclic* as the shortened form of *disjoint supercyclic*.

Definition 2.1 *[12, Definition 1.3.1] We say that $N \geq 2$ sequences of operators $(T_{1,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$ in $L(X)$ are* d-supercyclic *provided that the sequence of direct sums $(T_{1,n} \oplus \cdots \oplus T_{N,n})_{n=1}^{\infty}$ has a* supercyclic *vector of the form $(z, \cdots, z) \in X^N$. Then z is called a* d-supercyclic *vector for the sequences $(T_{1,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$. The operators $T_1, \cdots, T_N$ in $L(X)$ are called* d-supercyclic *if the sequences of iterations $(T_1^n)_{n=1}^{\infty}, \cdots, (T_N^n)_{n=1}^{\infty}$ are* d-supercyclic.

The following *d-Supercyclicity Criterion* is a generalization of theorem 1.1 and is due to Martin.

Definition 2.2 *[12, Definition 4.1.1] Let X be a Banach space and $(n_k)_k$ be a strictly increasing sequence of positive integers and $N \geq 2$. We say that $T_1, \cdots, T_N$ in $L(X)$ satisfy the d-Supercyclicity Criterion with respect to $(n_k)_k$ provided there exist dense subsets $X_0, X_1, \ldots, X_N$ of X and mappings*

$$S_l : X_l \to X, \ (1 \leq l \leq N)$$

so that for $1 \leq i \leq N$

$$(i) \ \ (T_l^{n_k} S_i^{n_k} - \delta_{i,l} I_{X_i}) \underset{k\to\infty}{\to} 0 \ \ \text{pointwise on } X_i,$$

$$(ii) \ \ \lim_{k\to\infty} \|T_l^{n_k} x\| \cdot \|\sum_{j=1}^{N} S_j^{n_k} y_j\| = 0 \text{ for } x \in X_0 \text{ and } y_{-i} \in X_{-i}.$$

Remark 2.3 *The d-Supercyclicity Criterion is a sufficient condition for d-supercyclicity.*

In the following proposition, we give an important necessary condition for the d-Supercyclicity Criterion.

Proposition 2.4 *[12, Theorem 4.1.2] Let $N \geq 2$ and $T_1, \cdots, T_N \in L(X)$ satisfy the d-Supercyclicity Criterion. Then $T_1, \cdots, T_N$ have a residual set (i.e. a dense G_δ set) of d-supercylic vectors.*

3. d-Supercyclicity on $c_0(\mathbb{N}, w)$ and $l^2(\mathbb{N}, w)$

The following proposition shows that a weighted shift defines an operator on a Fréchet sequence space X as soon as it maps X into itself. In addition, X should carry a topology that is compatible with the sequence space structure of X.

Proposition 3.1 *[9, Proposition 4.1] Let X be a Fréchet sequence space. Then every weighted shift $B_a : X \to X$ is continuous.*

From proposition 3.1, we know that the weighted shifts acting on the weighted sequence spaces in our main results are meaningful. Now, in this section, we establish a characterization for the *d-supercyclicity* of finitely many different powers of weighted shifts acting on the spaces $c_0(\mathbb{N}, w)$ and $l^2(\mathbb{N}, w)$. From (1.1) and (1.3), the norm in the following theorem is either $\|\|_2$ or $\|\|_\infty$. Besides, $w = (w_j)_{j\in\mathbb{N}}$ is a positive weight sequence satisfying $w_j \geq 1$ for every $j \in \mathbb{N}$.

Theorem 3.2 *Let $X = c_0(\mathbb{N}, w)$ or $l^2(\mathbb{N}, w)$, and let integers $1 \leq r_1 < r_2 < \cdots < r_N$ be given, where $N \geq 2$. For each $1 \leq l \leq N$, let $a_l = (a_{l,n})_{n=1}^\infty$ be a bounded sequence of nonzero scalars and let $B_{a_l} : X \to X$ be the associated unilateral backward shift on X:*

$$x = (x_0, x_1, \cdots) \overset{B_{a_l}}{\to} (a_{l,1}x_1, a_{l,2}x_2, \cdots).$$

The following statements are equivalent:

(a) *$B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ have a dense set of d-supercyclic vectors on X.*

(b) *For each $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that, for every $0 \leq j \leq q$,*

$$w_{j+(r_l-r_s)m}\left|\frac{\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{s,i}}{\prod_{i=j+1}^{j+r_l m} a_{l,i}}\right| < \varepsilon \ \ (1 \leq s < l \leq N).$$

(c) *$B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ satisfy the d-Supercyclicity Criterion.*

Proof (a)⇒(b). Let $\varepsilon > 0$ and $q \in \mathbb{N}$. Choose $0 < \delta < 1$ such that $\delta/(1-\delta) < \varepsilon$. Let $x = \sum\limits_{k\in\mathbb{N}} x_k e_k$ and $0 \neq \alpha \in \mathbb{C}$ and $m \in \mathbb{N}$ $(m > q)$ such that

$$\|\alpha B_{a_l}^{r_l m} x - \sum_{0\leq j\leq q} e_j\| < \delta \ (1 \leq l \leq N). \tag{3.1}$$

Considering the norms on the spaces $c_0(\mathbb{N}, w)$ and $l^2(\mathbb{N}, w)$ and employing (3.1), it follows that

$$1 - \frac{\delta}{w_j} < \left|\alpha\Big(\prod_{i=j+1}^{j+r_l m} a_{l,i}\Big)x_{j+r_l m}\right| < 1 + \frac{\delta}{w_j} \ \text{ if } \ 0 \leq j \leq q. \tag{3.2}$$

$$\left|\alpha\Big(\prod_{i=j+1}^{j+r_l m} a_{l,i}\Big)x_{j+r_l m}\right| w_j < \delta \ \text{ if } \ j > q. \tag{3.3}$$

Now, let $0 \leq j \leq q$ and $1 \leq s < l \leq N$ be fixed. By (3.3) (since $j+(r_l-r_s)m > q$), (3.2), and $w_j \geq 1$ $(j \in \mathbb{N})$, it follows that

$$\begin{aligned} & w_{j+(r_l-r_s)m} \frac{\left|\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{s,i}\right|}{\left|\prod_{i=j+1}^{j+r_l m} a_{l,i}\right|} \\ & = \frac{w_{j+(r_l-r_s)m}|\alpha| \left|\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{s,i}\right| |x_{j+r_l m}|}{|\alpha| \left|\prod_{i=j+1}^{j+r_l m} a_{l,i}\right| |x_{j+r_l m}|} \\ & < \frac{\delta}{1-\frac{\delta}{w_j}} \leq \frac{\delta}{1-\delta} < \varepsilon. \end{aligned} \tag{3.4}$$

From (3.4) we obtain (b).

(b)⇒ (c). From (b), for $0 \leq j \leq q$ and $1 \leq s < l \leq N$, there exist integers $1 \leq n_1 < n_2 < \cdots$ such that

$$w_{j+(r_l-r_s)n_q} \left| \frac{\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}}{\prod_{i=j+1}^{j+r_l n_q} a_{l,i}} \right| < \frac{1}{q}. \tag{3.5}$$

Let $X_0 = X_1 = \cdots = X_N = \operatorname{span}\{e_j : j \in \mathbb{N}\}$. Thus, each X_j is a dense subset of X. Define the operators $S_l : X_l \to X$ $(1 \leq l \leq N)$ as follows:

$$S_l e_j = \frac{1}{a_{l,j+1}} e_{j+1} \ (j \in \mathbb{N}).$$

It is obvious that

$$B_{a_l} S_l = Id_{X_l}. \tag{3.6}$$

For $1 \leq s < l \leq N$ and $n_q \in \mathbb{N}$ large enough, it follows that

$$\left\| B_{a_l}^{r_l n_q} S_s^{r_s n_q} e_j \right\| = 0. \tag{3.7}$$

By (3.5) we obtain that

$$\begin{aligned} \left\| B_{a_l}^{r_s n_q} S_l^{r_l n_q} e_j \right\| &= \left\| \frac{\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}}{\prod_{i=j+1}^{j+r_l n_q} a_{l,i}} e_{j+(r_l-r_s)n_q} \right\| \\ &= w_{j+(r_l-r_s)n_q} \left| \frac{\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}}{\prod_{i=j+1}^{j+r_l n_q} a_{l,i}} \right| < \frac{1}{q}. \end{aligned} \tag{3.8}$$

From (3.6) to (3.8) we obtain (i) of Definition 2.2.

Let $y_0, .., y_N \in \operatorname{span}\{e_j : j \in \mathbb{N}\}$. Choose k_0 large enough so that

$$y_i = \sum_{j=0}^{k_0} y_{i,j} e_j \ \ (0 \leq i \leq N).$$

Then for $n_q > k_0$, it is clear that

$$\lim_{q\to\infty} \|B_{a_l}^{r_l n_q} y_0\| \| \sum_{s=1}^{N} S_s^{r_s n_q} y_s\| = 0.$$

Hence, $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ satisfy the d-Supercyclicity Criterion with respect to $(n_q)_q$.

(c)$\Rightarrow$(a). This is immediate from proposition 2.4. This completes the proof. □

Remark 3.3 *Theorem 4.2.5 with* $p = 2$ *in [12] is a special case of theorem 3.2 with the positive weight sequence* $w = (w_j)_{j\in\mathbb{N}}$ *satisfying* $w_j = 1$ *for all* $j \in \mathbb{N}$.

When the shifts on theorem 3.2 are *invertible*, we have:

Corollary 3.4 *Let* $X = c_0(\mathbb{N}, w)$ *or* $l^2(\mathbb{N}, w)$, *and let integers* $1 \le r_1 < r_2 < \cdots < r_N$ *be given, where* $N \ge 2$. *For each* $1 \le l \le N$, *let* $B_{a_l} : X \to X$ *be an invertible unilateral backward shift on* X, *with weight sequence* $(a_{l,j})_{j\in\mathbb{N}}$,

$$x = (x_0, x_1, \cdots) \overset{B_{a_l}}{\to} (a_{l,1}x_1, a_{l,2}x_2, \cdots).$$

The following statements are equivalent:

(a) $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ *have a dense set of d-supercyclic vectors on* X.

(b) *There exist integers* $1 \le n_1 < n_2 < \cdots$ *such that, for* $1 \le s < l \le N$ *and* $j \in \mathbb{N}$,

$$\lim_{q\to\infty} w_{j+(r_l-r_s)n_q} \left| \frac{\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}}{\prod_{i=j+1}^{j+r_l n_q} a_{l,i}} \right| = 0.$$

(c) $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ *satisfy the d-Supercyclicity Criterion.*

In the following we obtain 2 special cases of theorem 3.2.

Corollary 3.5 *Let* $X = c_0(\mathbb{N}, w)$ *or* $l^2(\mathbb{N}, w)$, *and let integers* $1 \le r_1 < r_2 < \cdots < r_N$ *be given, where* $N \ge 2$. *Let* B_a *be a unilateral shift on* X, *with weight sequence* $a = (a_n)_{n\in\mathbb{N}}$. *The following statements are equivalent:*

(a) $B_a^{r_1}, B_a^{r_2}, \cdots, B_a^{r_N}$ *have a dense set of d-supercyclic vectors on* X.

(b) *For each* $\varepsilon > 0$ *and* $q \in \mathbb{N}$, *there exists* $m \in \mathbb{N}$ *such that, for every* $0 \le j \le q$,

$$\frac{w_{j+(r_l-r_s)m}}{|\prod_{i=j+1}^{j+(r_l-r_s)m} a_i|} < \varepsilon \ \ (1 \le s < l \le N).$$

(c) $B_a^{r_1}, B_a^{r_2}, \cdots, B_a^{r_N}$ *satisfy the d-Supercyclicity Criterion.*

Corollary 3.6 *Let* $X = c_0(\mathbb{N}, w)$ *or* $l^2(\mathbb{N}, w)$, *and let integers* $1 \le r_1 < r_2 < \cdots < r_N$ *be given, where* $N \ge 2$. *Let* $\lambda_l \in \mathbb{C}$ $(1 \le l \le N)$ *and* $B : X \to X$ *be the backward shift defined as follows:*

$$x = (x_0, x_1, \cdots) \overset{B}{\to} (x_1, x_2, \cdots).$$

Then the following statements are equivalent:

(a) $\lambda_1 B^{r_1}, \lambda_2 B^{r_2} \cdots, \lambda_N B^{r_N}$ *have a dense set of d-supercyclic vectors on* X.

(b) For each $\varepsilon > 0$ *and* $q \in \mathbb{N}$, *there exists* $m \in \mathbb{N}$ *such that, for every* $0 \leq j \leq q$,

$$w_{j+(r_l-r_s)m} \left| \frac{\lambda_s}{\lambda_l} \right|^m < \varepsilon \ \ (1 \leq s < l \leq N).$$

(c) $\lambda_1 B^{r_1}, \cdots, \lambda_N B^{r_N}$ *satisfy the d-Supercyclicity Criterion.*

Proof For each $1 \leq l \leq N$, let λ_l^{1/r_l} denote a fixed root of $z^{r_l} - \lambda_l = 0$, and let B_{a_l} denote the unilateral backward shift with constant weight sequence $a_l = (a_{l,n})_{n\in\mathbb{N}} = (\lambda_l^{1/r_l})_{n\in\mathbb{N}}$. Then $B_{a_l}^{r_l} = \lambda_l B^{r_l}$. It is clear that the result follows from theorem 3.2. □

Example 3.7 *Let* $w_0 = (3/2)_{j\in\mathbb{N}}$ *be a positive constant weight. Let* $a_n = 2$ *for all* $n \in \mathbb{N}$. *That is,* $a = ([2], 2, 2, \cdots)$, *where [.] denotes the zeroth coefficient. Then* B_a *is an* invertible *unilateral backward shift on the space* $X = c_0(\mathbb{N}, w_0)$ *or* $l^2(\mathbb{N}, w_0)$. *Then the operators* B_a *and* B_a^2 *have a dense set of d-supercyclic vectors.*

Proof Since

$$\begin{aligned} & \lim_{n\to\infty} w_{j+(r_l-r_s)n} \left| \frac{\prod_{i=j+(r_l-r_s)n+1}^{j+r_l n} a_i}{\prod_{i=j+1}^{j+r_l n} a_i} \right| \\ = & \lim_{n\to\infty} w_{j+(2-1)n} \left| \frac{\prod_{i=j+(2-1)n+1}^{j+2n} a_i}{\prod_{i=j+1}^{j+2n} a_i} \right| \\ = & \lim_{n\to\infty} \frac{3}{2} \frac{2^n}{2^{2n}} = 0, \end{aligned}$$

where $r_s = 1$ and $r_l = 2$, there exist integers $1 \leq n_1 < n_2 < \cdots$ such that, for $j \in \mathbb{N}$,

$$\lim_{q\to\infty} w_{j+(r_l-r_s)n_q} \left| \frac{\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}}{\prod_{i=j+1}^{j+r_l n_q} a_{l,i}} \right| = 0.$$

By corollary 3.4, it follows the operators B_a and B_a^2 have a dense set of d-supercyclic vectors. □

4. d-Supercyclicity on $c_0(\mathbb{Z}, w)$ and $l^2(\mathbb{Z}, w)$

In this section, we establish a characterization for the *d-supercyclicity* of finitely many different powers of weighted shifts acting on the spaces $c_0(\mathbb{Z}, w)$ and $l^2(\mathbb{Z}, w)$.

Theorem 4.1 *Let* $X = c_0(\mathbb{Z}, w)$ *or* $l^2(\mathbb{Z}, w)$, *and let integers* $1 \leq r_1 < r_2 < \cdots < r_N$ *be given, where* $N \geq 2$. *For each* $1 \leq l \leq N$, *let* $a_l = (a_{l,j})_{j\in\mathbb{Z}}$ *be a bounded bilateral sequence of nonzero scalars, and let* B_{a_l} *be the associated backward shift on* X *defined by*

$$B_{a_l} e_j = a_{l,j} e_{j-1} \ (j \in \mathbb{Z}).$$

Then the following statements are equivalent:

(a) $B_{a_1}^{r_1}, B_{a_2}^{r_2}, \cdots, B_{a_N}^{r_N}$ *have a dense set of d-supercyclic vectors on* X.

(b) For each $\varepsilon > 0$ *and* $q \in \mathbb{N}$, *there exists* $m \in \mathbb{N}$ $(m > 2q)$, *such that, for* $|j|, |k| \leq q$ *and* $1 \leq l, s \leq N$, *we have that*

$$\frac{w_{j-r_l m}\left|\prod_{i=j-r_l m+1}^{j} a_{l,i}\right| w_{k+r_s m}}{\left|\prod_{i=k+1}^{k+r_s m} a_{s,i}\right|} < \varepsilon \;\; (1 \leq l, s \leq N), \tag{4.1}$$

and for $1 \leq s < l \leq N$,

$$w_{j+(r_l-r_s)m}\left|\frac{\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{s,i}}{\prod_{i=j+1}^{j+r_l m} a_{l,i}}\right| < \varepsilon, \tag{4.2}$$

$$w_{j+(r_s-r_l)m}\left|\frac{\prod_{i=j+(r_s-r_l)m+1}^{j+r_s m} a_{l,i}}{\prod_{i=j+1}^{j+r_s m} a_{s,i}}\right| < \varepsilon. \tag{4.3}$$

(c) $B_{a_1}^{r_1}, B_{a_2}^{r_2}, \cdots, B_{a_N}^{r_N}$ *satisfy the d-Supercyclicity Criterion.*

Proof (a)⇒(b). Let $\varepsilon > 0$ and $q \in \mathbb{N}$. Choose $0 < \delta < 1/2$ such that $\delta/(1-\delta) < \varepsilon$. Let $x = \sum_{k\in\mathbb{Z}} x_k e_k$ and $0 \neq \alpha \in \mathbb{C}$ and $m \in \mathbb{N}$ $(m > 2q)$ such that

$$\|x - \sum_{|j|\leq q} e_j\| < \delta,$$
$$\|\alpha B_{a_l}^{r_l m} x - \sum_{|j|\leq q} e_j\| < \delta \; (1 \leq l \leq N).$$

Considering the norms on $c_0(\mathbb{Z}, w)$ and $l^2(\mathbb{Z}, w)$, it follows that

$$|x_j - 1| w_j < \delta \text{ if } |j| \leq q, \tag{4.4}$$

$$|x_j| w_j < \delta \text{ if } |j| > q. \tag{4.5}$$

$$1 - \frac{\delta}{w_j} < \left|\alpha\Big(\prod_{i=j+1}^{j+r_l m} a_{l,i}\Big) x_{j+r_l m}\right| < 1 + \frac{\delta}{w_j} \text{ if } |j| \leq q, \tag{4.6}$$

$$\left|\alpha\Big(\prod_{i=j+1}^{j+r_l m} a_{l,i}\Big) x_{j+r_l m}\right| w_j < \delta \text{ if } |j| > q. \tag{4.7}$$

Now, fix $|j| \leq q$ and $1 \leq s < l \leq N$. By (4.7) (since $j + (r_l - r_s)m > q$), (4.6), and $w_j \geq 1$ ($j \in \mathbb{Z}$),

$$
\begin{aligned}
& w_{j+(r_l-r_s)m} \frac{\left|\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{s,i}\right|}{\left|\prod_{i=j+1}^{j+r_l m} a_{l,i}\right|} \\
&= \frac{|\alpha| \left|\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{s,i}\right| |x_{j+r_l m}| w_{j+(r_l-r_s)m}}{|\alpha| \left|\prod_{i=j+1}^{j+r_l m} a_{l,i}\right| |x_{j+r_l m}|} \\
&< \frac{\delta}{1-\frac{\delta}{w_j}} \leq \frac{\delta}{1-\delta} < \varepsilon.
\end{aligned}
$$

Similarly, by (4.7) (since $j + (r_s - r_l)m < -q$), (4.6), and $w_j \geq 1$ ($j \in \mathbb{Z}$),

$$
\begin{aligned}
& w_{j+(r_s-r_l)m} \frac{\left|\prod_{i=j+(r_s-r_l)m+1}^{j+r_s m} a_{l,i}\right|}{\left|\prod_{i=j+1}^{j+r_s m} a_{s,i}\right|} \\
&= \frac{|\alpha| \left|\prod_{i=j+(r_s-r_l)m+1}^{j+r_s m} a_{l,i}\right| |x_{j+r_s m}| w_{j+(r_s-r_l)m}}{|\alpha| \left|\prod_{i=j+1}^{j+r_s m} a_{s,i}\right| |x_{j+r_s m}|} \\
&< \frac{\delta}{1-\frac{\delta}{w_j}} \leq \frac{\delta}{1-\delta} < \varepsilon.
\end{aligned}
$$

From the above 2 inequalities we get (4.2) and (4.3). Next we show (4.1). Fix $|j|, |k| \leq q$ and $1 \leq s, l \leq N$. By (4.4), $w_j \geq 1$ ($j \in \mathbb{Z}$), and $0 < \delta < 1/2$, it follows that for all $j \in \mathbb{Z}$

$$
|x_j| > 1 - \frac{\delta}{w_j} \geq 1 - \delta > \frac{1}{2}. \tag{4.8}
$$

Thus, by (4.7) (since $j - r_l m < -q$), (4.8), (4.5) (since $k + r_s m > q$), and (4.6),

$$
\begin{aligned}
& \frac{w_{j-r_l m} \left|\prod_{i=j-r_l m+1}^{j} a_{l,i}\right| w_{k+r_s m}}{\left|\prod_{i=k+1}^{k+r_s m} a_{s,i}\right|} = \frac{\left(|\alpha| \left|\prod_{i=j-r_l m+1}^{j} a_{l,i}\right| |x_j| w_{j-r_l m}\right) w_{k+r_s m}}{|x_j||\alpha| \left|\prod_{i=k+1}^{k+r_s m} a_{s,i}\right|} \\
&< 2\delta \frac{w_{k+r_s m}}{|\alpha| \left|\prod_{i=k+1}^{k+r_s m} a_{s,i}\right|} = 2\delta \frac{w_{k+r_s m} |x_{k+r_s m}|}{|\alpha| \left|\prod_{i=k+1}^{k+r_s m} a_{s,i}\right| |x_{k+r_s m}|} \\
&< \frac{2\delta^2}{1-\delta/w_k} \leq \frac{2\delta^2}{1-\delta} < 2\varepsilon \quad (1 \leq l, s \leq N).
\end{aligned}
$$

From this we get (4.1).

(b)$\Rightarrow$ (c). From (b), there exist integers $1 \leq n_1 < n_2 < \cdots$ such that for every $q \in \mathbb{N}$,

$$
\frac{w_{j-r_l n_q} \left|\prod_{i=j-r_l n_q+1}^{j} a_{l,i}\right| w_{k+r_s n_q}}{\left|\prod_{i=k+1}^{k+r_s n_q} a_{s,i}\right|} < \frac{1}{q} \quad (1 \leq l, s \leq N), \tag{4.9}
$$

and for $1 \leq s < l \leq N$,

$$w_{j+(r_l-r_s)n_q} \left| \frac{\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}}{\prod_{i=j+1}^{j+r_l n_q} a_{l,i}} \right| < \frac{1}{q}, \tag{4.10}$$

$$w_{j+(r_s-r_l)n_q} \left| \frac{\prod_{i=j+(r_s-r_l)n_q+1}^{j+r_s n_q} a_{l,i}}{\prod_{i=j+1}^{j+r_s n_q} a_{s,i}} \right| < \frac{1}{q}. \tag{4.11}$$

Let $X_0 = X_1 = \cdots = X_N = \text{span}\{e_j : \ j \in \mathbb{Z}\}$. Thus, each X_j is a dense subset of X. Define $S_l : X_l \to X$ as follows:

$$S_l e_j = \frac{e_{j+1}}{a_{l,j+1}} \ (j \in \mathbb{Z}).$$

It is obvious that

$$B_{a_l} S_l = Id_{X_l} \ (1 \leq l \leq N).$$

For $1 \leq s < l \leq N$, by (4.10) and (4.11), it follows that

$$\|B_{a_l}^{r_l n_q} S_s^{r_s n_q} e_j\| = w_{j+(r_s-r_l)n_q} \frac{\left|\prod_{i=j+(r_s-r_l)n_q+1}^{j+r_s n_q} a_{l,i}\right|}{\left|\prod_{i=j+1}^{j+r_s n_q} a_{s,i}\right|} < \frac{1}{q}.$$

$$\|B_{a_s}^{r_s n_q} S_l^{r_l n_q} e_j\| = w_{j+(r_l-r_s)n_q} \frac{\left|\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}\right|}{\left|\prod_{i=j+1}^{j+r_l n_q} a_{l,i}\right|} < \frac{1}{q}.$$

Thus, from the above 3 inequalities, we obtain (i) of Definition 2.2.

Now, let $y_0, \cdots, y_N \in \text{span}\{e_j : \ j \in \mathbb{Z}\}$. Take $k_0 \in \mathbb{N}$ large enough such that

$$y_i = \sum_{|j| \leq k_0} y_{i,j} e_j \ \ (0 \leq i \leq N).$$

Denote $C := \max\{|y_{i,j}| : 0 \le i \le N, |j| \le k_0\}$ and $q > k_0$; considering the norms on the spaces $c_0(\mathbb{Z}, w)$ and $l^2(\mathbb{Z}, w)$ and employing (4.9), it follows that

$$
\begin{aligned}
&\|B_{a_l}^{r_l n_q} y_0\| \|\sum_{s=1}^{N} S_s^{r_s n_q} y_s\| \\
&= \left\| \sum_{|j|\le k_0} \Big(\prod_{i=j-r_l n_q+1}^{j} a_{l,i} \Big) y_{0,j} e_{j-r_l n_q} \right\| \left\| \sum_{s=1}^{N} \sum_{|k|\le k_0} \frac{y_{s,k} e_{k+r_s n_q}}{\prod_{i=k+1}^{k+r_s n_q} a_{s,i}} \right\| \\
&\le C^2 \begin{cases} \left(\sum_{|j|\le k_0} \left| \prod_{i=j-r_l n_q+1}^{j} a_{l,i} \right|^2 w_{j-r_l n_q}^2 \right)^{1/2} \\ \quad \cdot \left(\sum_{s=1}^{N} \sum_{|k|\le k_0} \frac{2 w_{k+r_s n_q}^2}{\left| \prod_{i=k+1}^{k+r_s n_q} a_{s,i} \right|^2} \right)^{1/2}, \text{ on } l^2(\mathbb{Z}, w), \\ \left(\sum_{|j|\le k_0} \left| \prod_{i=j-r_l n_q+1}^{j} a_{l,i} \right| w_{j-r_l n_q} \right) \\ \quad \cdot \left(\sum_{s=1}^{N} \sum_{|k|\le k_0} \frac{w_{k+r_s n_q}}{\left| \prod_{i=k+1}^{k+r_s n_q} a_{s,i} \right|} \right), \text{ on } c_0(\mathbb{Z}, w), \end{cases} \\
&\le \sqrt{2} C^2 \left(\sum_{|j|\le k_0} \left| \prod_{i=j-r_l n_q+1}^{j} a_{l,i} \right| w_{j-r_l n_q} \right) \cdot \left(\sum_{s=1}^{N} \sum_{|k|\le k_0} \frac{w_{k+r_s n_q}}{\left| \prod_{i=k+1}^{k+r_s n_q} a_{s,i} \right|} \right) \underset{q\to\infty}{\to} 0.
\end{aligned}
$$

From this (ii) of definition 2.2 follows. Hence, $B_{a_1}^{r_1}, B_{a_2}^{r_2}, \cdots, B_{a_N}^{r_N}$ satisfy the d-Supercyclicity Criterion with respect to $(n_q)_q$.

(c)$\Rightarrow$(a). This follows from proposition 2.4. The proof is finished. □

Remark 4.2 *Theorem 4.2.1 with $p = 2$ in [12] is a special case of theorem 4.1 with the positive weight sequence $w = (w_i)_{i\in\mathbb{Z}}$ satisfying $w_i = 1$ for all $i \in \mathbb{Z}$.*

When the shifts on theorem 4.1 are *invertible*, we have:

Corollary 4.3 *Let $X = c_0(\mathbb{Z}, w)$ or $l^2(\mathbb{Z}, w)$, and let integers $1 \le r_1 < r_2 < \cdots < r_N$ be given, where $N \ge 2$. For each $1 \le l \le N$, let $B_{a_l} e_j = a_{l,j} e_{j-1}$ $(j \in \mathbb{Z})$ be an invertible bilateral backward shift on X, with weight sequence $(a_{l,j})_{j\in\mathbb{Z}}$. Then the following statements are equivalent:*

(a) *$B_{a_1}^{r_1}, B_{a_2}^{r_2}, \cdots, B_{a_N}^{r_N}$ have a dense set of d-supercyclic vectors on X.*

(b) *There exists integers $1 \le n_1 < n_2 < \cdots$ such that, for $1 \le s < l \le N$, and $j \in \mathbb{N}$, we have that*

$$\lim_{q\to\infty} w_{j+(r_l-r_s)n_q} \left| \frac{\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}}{\prod_{i=j+1}^{j+r_l n_q} a_{l,i}} \right| = 0,$$

$$\lim_{q\to\infty} w_{j+(r_s-r_l)n_q} \left| \frac{\prod_{i=j+(r_s-r_l)n_q+1}^{j+r_s n_q} a_{l,i}}{\prod_{i=j+1}^{j+r_s n_q} a_{s,i}} \right| = 0.$$

$$\lim_{q\to\infty} \max \left\{ \frac{w_{-r_l n_q} \left|\prod_{i=-r_l n_q}^{1} a_{l,i}\right| w_{r_s n_q}}{\left|\prod_{i=1}^{r_s n_q} a_{s,i}\right|} : \ 1 \le l, s \le N \right\} = 0.$$

(c) $B_{a_1}^{r_1}, B_{a_2}^{r_2}, \cdots, B_{a_N}^{r_N}$ *satisfy the d-Supercyclicity Criterion.*

Similarly, we obtain 2 special cases of theorem 4.1.

Corollary 4.4 *Let* $X = c_0(\mathbb{Z}, w)$ *or* $l^2(\mathbb{Z}, w)$, *and let integers* $1 \le r_1 < r_2 < \cdots < r_N$ *be given, where* $N \ge 2$. *Let* B_a *be a bilateral shift on* X, *with a bounded weight sequence* $a = (a_n)_{n\in\mathbb{Z}}$. *Then the following statements are equivalent:*

(a) $B_a^{r_1}, B_a^{r_2}, \cdots, B_a^{r_N}$ *have a dense set of d-supercyclic vectors on* X.

(b) For each $\varepsilon > 0$ *and* $q \in \mathbb{N}$, *there exists* $m \in \mathbb{N}$ $(m > 2q)$ *so that for* $|j|, |k| \le q$ *and* $1 \le l, s \le N$ *we have that*

$$\frac{w_{j-r_l m} \left|\prod_{i=j-r_l m+1}^{j} a_i\right| w_{k+r_s m}}{\left|\prod_{i=k+1}^{k+r_s m} a_i\right|} < \varepsilon \ \ (1 \le l, s \le N),$$

and for $1 \le s < l \le N$,

$$\frac{w_{j+(r_l-r_s)m}}{|\prod_{i=j+1}^{j+(r_l-r_s)m} a_i|} < \varepsilon \quad and \quad w_{j+(r_s-r_l)m}|\prod_{i=j+(r_s-r_l)m+1}^{j} a_i| < \varepsilon.$$

(c) $B_a^{r_1}, B_a^{r_2}, \cdots, B_a^{r_N}$ *satisfy the d-Supercyclicity Criterion.*

Corollary 4.5 *Let* $X = c_0(\mathbb{Z}, w)$ *or* $l^2(\mathbb{Z}, w)$, *and let integers* $1 \le r_1 < r_2 < \cdots < r_N$ *be given, where* $N \ge 2$. *Let* $\lambda_l \in \mathbb{C}$ $(1 \le \lambda \le N)$. *Then the following statements are equivalent:*

(a) $\lambda_1 B^{r_1}, \lambda_2 B^{r_2}, \cdots, \lambda_N B^{r_N}$ *have a dense set of d-supercyclic vectors on* X.

(b) For each $\varepsilon > 0$ *and* $q \in \mathbb{N}$, *there exists* $m \in \mathbb{N}$ $(m > 2q)$ *so that for* $|j|, |k| \le q$ *and* $1 \le l, s \le N$ *we have that*

$$w_{j-r_l m} w_{k+r_s m} \left|\frac{\lambda_l}{\lambda_s}\right|^m < \varepsilon \ \ (1 \le l, s \le N),$$

and for $1 \le s < l \le N$,

$$w_{j+(r_l-r_s)m} \left|\frac{\lambda_s}{\lambda_l}\right|^m < \varepsilon \ \ and \ \ w_{j+(r_s-r_l)m} \left|\frac{\lambda_l}{\lambda_s}\right|^m < \varepsilon.$$

(c) $\lambda_1 B^{r_1}, \lambda_2 B^{r_2}, \cdots, \lambda_N B^{r_N}$ *satisfy the d-Supercyclicity Criterion.*

Example 4.6 *Let* $w_0 = (3/2)_{j\in\mathbb{Z}}$ *be a positive constant weight. Define the* invertible *bilateral weighted shift* B_a *with the weight sequence* $a = (a_k)_{k\in\mathbb{Z}}$ *such that*

$$a_k = \begin{cases} 1/2, & \text{if } k \in \{-2^n+1, \cdots, -2^n+n\} \text{ for some odd } n \in \mathbb{N} \\ 2, & \text{if } k \in \{-2^n-n+1, \cdots, -2^n\} \text{ or } k = 2^n \text{ for some odd } n \in \mathbb{N} \\ 1, & \text{otherwise.} \end{cases}$$

where [.] denotes the zeroth coefficient and $a = (a_k)_{k\in\mathbb{Z}}$ *looks like*

$$a = (\cdots, 1, 2, 2, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \cdots, 1, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \\ 1, 1, 2, \frac{1}{2}, [1], 1, 2, 1, \cdots, 1, 2, 1, \cdots).$$

Proof Notice that the shifts are *invertible*, since $1/2 \leq a_k \leq 2$ $(k \in \mathbb{Z})$. Choose $n_q = 2^q$, for some *odd* $q \in \mathbb{N}$, and then we have that

$$\begin{aligned}
&\lim_{q\to\infty} w_{j+(r_l-r_s)n_q} \left| \frac{\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}}{\prod_{i=j+1}^{j+r_l n_q} a_{l,i}} \right| \\
= \quad &\lim_{q\to\infty} w_{j+n_q} \left| \frac{\prod_{i=j+n_q+1}^{j+2n_q} a_i}{\prod_{i=j+1}^{j+2n_q} a_i} \right| \\
= \quad &\lim_{q\to\infty} \frac{3}{2} \frac{1}{\prod_{i=j+1}^{j+2^q} a_i} \\
= \quad &\lim_{q\to\infty} \frac{3}{2} \frac{1}{2^{k(q)}} = 0 \; (1 \leq s < l \leq 2, \;\; j \in \mathbb{N}), \\
&\text{where } k(q) \text{ depends only on } q \text{ and } k(q) \to \infty \text{ as } q \to \infty \\
&(\text{since } \; a_i = 2 \; \text{ for } \; i = 2^n, \;\; n \text{ is odd}).
\end{aligned}$$

$$\begin{aligned}
&\lim_{q\to\infty} w_{j+(r_s-r_l)n_q} \left| \frac{\prod_{i=j+(r_s-r_l)n_q+1}^{j+r_s n_q} a_{l,i}}{\prod_{i=j+1}^{j+r_s n_q} a_{s,i}} \right| \\
= \quad &\lim_{q\to\infty} w_{j-n_q} \left| \frac{\prod_{i=j-n_q+1}^{j+n_q} a_i}{\prod_{i=j+1}^{j+n_q} a_i} \right| \\
= \quad &\lim_{q\to\infty} \frac{3}{2} \prod_{i=j-n_q+1}^{j} a_i \\
= \quad &\lim_{q\to\infty} \frac{3}{2} \prod_{i=-2^q+j+1}^{j} a_i \\
< \quad &\frac{3}{2} \lim_{q\to\infty} \frac{2^j}{2^{q-j-1}} = 0.
\end{aligned}$$

$$\begin{aligned}
&\lim_{q\to\infty} \max\left\{ \frac{w_{-r_l n_q}\left|\prod_{i=-r_l n_q}^{1} a_{l,i}\right| w_{r_s n_q}}{\left|\prod_{i=1}^{r_s n_q} a_{s,i}\right|} : \ 1 \le l, s \le 2 \right\} \\
= &\lim_{q\to\infty} \max\left\{ \frac{w_{-2n_q}\left|\prod_{i=-2n_q}^{1} a_i\right| w_{n_q}}{\left|\prod_{i=1}^{n_q} a_i\right|}, \ \frac{w_{-n_q}\left|\prod_{i=-n_q}^{1} a_i\right| w_{2n_q}}{\left|\prod_{i=1}^{2n_q} a_i\right|} \right\} \\
= &\frac{9}{4}\lim_{q\to\infty} \max\left\{ \frac{\left|\prod_{i=-2n_q}^{1} a_i\right|}{\left|\prod_{i=1}^{n_q} a_i\right|}, \ \frac{\left|\prod_{i=-n_q}^{1} a_i\right|}{\left|\prod_{i=1}^{2n_q} a_i\right|} \right\} \\
= &\frac{9}{4}\lim_{q\to\infty} \max\left\{ \frac{\left|\prod_{i=-2^{q+1}}^{1} a_i\right|}{\left|\prod_{i=1}^{2^q} a_i\right|}, \ \frac{\left|\prod_{i=-2^q}^{1} a_i\right|}{\left|\prod_{i=1}^{2^{q+1}} a_i\right|} \right\} \\
= &\frac{9}{4}\lim_{q\to\infty} \max\left\{ \frac{1}{2^{\frac{q-1}{2}}}, \ \frac{\frac{1}{2^{q-1}}}{2^{\frac{q-1}{2}}} \right\} = 0.
\end{aligned}$$

By corollary 4.3 we obtain that the operators B_a and B_a^2 satisfy the condition (b); thus, they are d-supercyclic. □

For $p \ge 1$, the weighted spaces $l^p(\mathbb{N}, w)$ and $l^p(\mathbb{Z}, w)$ are defined by

$$l^p(\mathbb{N}, w) := \Big\{ x = (x_n) \in \mathbb{K}^{\mathbb{N}} : \ \sum_{n\in\mathbb{N}} w_n^p |x_n|^p < \infty \Big\},$$
$$l^p(\mathbb{Z}, w) := \Big\{ x = (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \ \sum_{i\in\mathbb{Z}} w_i^p |x_i|^p < \infty \Big\},$$

where $w = (w_i)_i$ is still a positive weight sequence satisfying $w_i \ge 1$ for every $i \in \mathbb{N}$ (or $i \in \mathbb{Z}$). It is clear that $l^p(\mathbb{N}, w)$ and $l^p(\mathbb{Z}, w)$ are *separable* Banach spaces under the norms

$$\|x\| = \Big(\sum_{n\in\mathbb{N}} w_n^p |x_n|^p \Big)^{1/p} \quad \text{and} \quad \|x\| = \Big(\sum_{i\in\mathbb{Z}} w_i^p |x_i|^p \Big)^{1/p},$$

respectively. Then we have the following remark.

Remark 4.7 *Theorem 3.2 and corollaries 3.4–3.6 hold for the space* $l^p(\mathbb{N}, w)$ $(p \ge 1)$*, and theorem 4.1 and corollaries 4.3–4.5 are also true for the space* $l^p(\mathbb{Z}, w)$ $(p \ge 1)$.

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References

[1] Bayart F, Matheron E. Dynamics of Linear Operators. Cambridge, UK: Cambridge University Press, 2009.

[2] Bernal-González L. Disjoint hypercyclic operators. Studia Math 2007; 2: 113–131.

[3] Bès J, Martin Ö. Compositional disjoint hypercyclicity equals disjoint supercyclicity. Houston J Math 2012; 38: 1149–1163.

[4] Bès J, Martin Ö, Peris A. Disjoint hypercyclic linear fractional composition operators. J Math Anal Appl 2010; 381: 713–715.

[5] Bès J, Martin Ö, Peris A, Shkarin S. Disjoint mixing operators. J Funct Anal 2012; 263: 1283–1322.

[6] Bès J, Martin Ö, Sanders R. Weighted shifts and disjoint hypercyclicty. J Operator Theory (in press).

[7] Bès J, Peris A. Disjointness in hypercyclicity. J Math Anal Appl 2007; 336: 297–315.

[8] Godefroy G, Shapiro JH. Operators with dense, invariant, cyclic vector manifolds. J Funct Anal 1991; 98: 229–269.

[9] Grosse-Erdmann KG, Peris Manguillot A. Linear Chaos. New York, NY, USA: Springer, 2011.

[10] Hilden HM, Wallen LJ. Some cyclic and non-cyclic vectors for certain operators. Indiana Univ Math J 1974; 23: 557–565.

[11] Liang YX, Zhou ZH. Hereditarily hypercyclicity and supercyclicity of different weighted shifts. J Korean Math Soc 2014; 2: 363–382.

[12] Martin Ö. Disjoint hypercyclic and supercyclic composition operators. PhD, Bowling Green State University, Bowling Green, OH, USA, 2011.

[13] Rolewicz S. On orbits of elements. Studia Math 1969; 32: 17–22.

[14] Salas HN. Hypercyclic weighted shifts. Trans Amer Math Soc 1995; 347: 993–1004.

[15] Salas HN. Supercyclicity and weighted shifts. Studia Math 1999; 135: 55–74.

[16] Salas HN. Dual disjoint hypercyclic operators. J Math Anal Appl 2011; 374: 106–117.

[17] Salas HN. The strong disjoint below-up/collapse property. J Funct Spaces Appl 2013; 2013: 14–517.

[18] Shkarin S. A short proof of existence of disjoint hypercyclic operators. J Math Anal Appl 2010; 367: 713–715.

[19] Zhang L, Zhou ZH. Disjoint mixing weighted backward shifts on the space of all complex valued square summable sequences. J Comput Anal Appl 2014; 16: 618–625.