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A Cohen type inequality for Laguerre–Sobolev expansions with a mass point outside their oscillatory regime

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Abstract: Let consider the Sobolev type inner product

$$\langle f,g\rangle_S=\int_0^\infty f(x)g(x)d\mu(x)+Mf(c)g(c)+Nf'(c)g'(c),$$

where $d\mu(x)=x^\alpha e^{-x}dx,\ \alpha>-1,$ is the Laguerre measure, $c<0,$ and $M,N\geq 0$. In this paper we get a Cohen-type inequality for Fourier expansions in terms of the orthonormal polynomials associated with the above Sobolev inner product. Then, as an immediate consequence, we deduce the divergence of Fourier expansions and Cesàro means of order δ in terms of this kind of Laguerre–Sobolev polynomials.

Key words: Sobolev-type orthogonal polynomials, Cohen-type inequality, Fourier–Sobolev expansions

1. Introduction

The aim of this paper is to establish a Cohen-type inequality when we deal with the following Sobolev inner product on the linear space $\mathbb{P}$ of polynomials with real coefficients

$$\langle f,g\rangle_S=\int_0^\infty f(x)g(x)d\mu(x)+Mf(c)g(c)+Nf'(c)g'(c),\tag{1.1}$$

where $d\mu(x)=x^\alpha e^{-x}dx,\ \alpha>-1,$ is the Laguerre measure, $M,N\geq 0,$ and the mass point c is located outside the support of μ. Such a kind of inner product is known in the literature as either a Sobolev-type inner product or a discrete Sobolev inner product.

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It is well known that the presentation of an approach leading to a simple proof of Cohen-type inequalities for Jacobi expansions and stating the corresponding Cohen-type inequalities for Laguerre and Hermite expansions is due to Markett (see [9] and the references therein). The crucial step behind the Markett approach is to give a family of suitable test functions Λ_n such that their Fourier coefficients $\hat{\Lambda}_n(k)$ satisfy the following properties:

(i) $\hat{\Lambda}_n(k) = 0$, for $0 \leq k < n$.

(ii) The exact rate of growth of $\hat{\Lambda}_n(n)$ can be determined.

In the setting of Sobolev orthogonality, the study of Cohen-type inequalities is most recent and has attracted considerable attention, mainly when it is possible to use the same (up to constant factor) test functions given in [9]. For instance, the authors of [5, 11] obtained Cohen-type inequalities for Laguerre orthonormal expansions with respect to Sobolev-type inner products with only one mass point at $c = 0$, i.e. the mass is located in the boundary of the support of the measure. Similar results for Laguerre–Sobolev orthonormal expansions with respect to a nondiscrete Sobolev inner product associated with the Laguerre weight function appear in [3].

The novelty of our approach comes from 2 directions: First, we consider a Sobolev-type inner product with only a mass point outside the support of the measure μ and, second, we incorporate new test functions different from those used in [9].

The outline of the paper is as follows. Section 2 provides a basic background dealing with structural and asymptotic properties of the classical and k-iterated Laguerre orthogonal polynomials, respectively, as well as some well known analytic properties of Laguerre–Sobolev type polynomials. In particular, the outer strong asymptotics (Perron's formula) for Laguerre polynomials and the outer relative ratio asymptotics for 2 consecutive iterated Laguerre polynomials are emphasized, taking into account that they will play a central role in the sequel. Section 3 contains some estimates for the norms of the k-iterated Laguerre polynomials and Laguerre–Sobolev-type polynomials (Propositions 3.1 and 3.3), respectively. We also state therein a new representation formula involving different families of k-iterated Laguerre polynomials in such a way that we are able to deduce the corresponding estimates for its coefficients. In Section 4 we prove our main result (Theorem 4.1). We obtain an estimate from below for the $S^p_{f(\alpha)}$-norm of the partial sums of some balanced Fourier expansions in terms of Laguerre–Sobolev-type orthonormal polynomials. As an immediate consequence (Corollaries 4.1 and 4.2), the divergence of such partial sums and Cesàro means of order δ when p is located outside the Pollard interval is deduced.

Throughout this manuscript, the notation $u_n \cong v_n$ means that the sequence $(u_n/v_n)_{n\geq 0}$ converges to 1 while the notation $u_n \sim v_n$ means that there exist positive real numbers C_1 and C_2 such that $C_1 u_n \leq v_n \leq C_2 u_n$ for n large enough. Any other standard notation will be properly introduced whenever needed.

2. Background: structural and asymptotic properties

For $\alpha > -1$, let $\{L^\alpha_n(x)\}_{n\geq 0}$ and $\{L^{(\alpha)}_n(x)\}_{n\geq 0}$ be the sequences of orthonormal and normalized Laguerre polynomials with leading coefficient $\frac{(-1)^n}{n!}$, respectively. Thangavelu stated in [13], and see also [1, 10], that there are at least 4 types of Laguerre expansions on the positive half line $\mathbb{R}_+$ studied in the literature. The first ones are related to the standard Laguerre polynomials, for which it is well known that Fourier series in terms of them turn out to be nonconvergent for $p \neq 2$. The other types can be defined as follows:

$$\mathcal{L}_n^{\alpha}(x) := \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^{(\alpha)}(x)e^{-x/2}x^{\alpha/2}, \tag{2.2}$$

$$\psi_n^{\alpha}(x) := \mathcal{L}_n^{\alpha}\left(\frac{x^2}{2}\right)x^{-\alpha}, \tag{2.3}$$

$$\varphi_n^{\alpha}(x) := \mathcal{L}_n^{\alpha}\left(x^2\right)(2x)^{1/2}. \tag{2.4}$$

The functions $\mathcal{L}_n^{\alpha}$ and φ_n^{α} form orthonormal systems in the classical space $L^2(\mathbb{R}_+, dx)$, and the functions ψ_n^{α} form an orthonormal system in the weighted space $L^2(\mathbb{R}_+, x^{2\alpha+1}dx)$.

Using a k-iterated Christoffel transform of the measure μ, to the best of our knowledge, a fifth type of Laguerre expansions can be introduced. This family of functions is called k-iterated Laguerre polynomials, and it is constituted essentially by polynomials orthogonal with respect to the modified Laguerre measure $(x-c)^k d\mu(x)$, for $k \in \mathbb{N}$ fixed (see [2, 12].) Note that the modified Laguerre measure $(x-c)^k d\mu(x)$ is positive when either k is an even integer number or k is an odd integer number and c is located outside the support of μ. Furthermore, it is very well known that, when $k=1$ and c is outside $supp\,\mu$, these polynomials are actually the kernel polynomials corresponding to the moment functional associated with μ and the K-parameter c [2, Sec. I.7].

In the following, we will denote by $\{L_n^{\alpha,[k]}(x)\}_{n\geq 0}$ and $\{L_n^{(\alpha),[k]}(x)\}_{n\geq 0}$ the sequences of orthonormal and normalized k-iterated Laguerre polynomials with leading coefficient equal to $\frac{(-1)^n}{n!}$, respectively. It is clear that for $k=0$ these sequences coincide with the orthonormal and normalized Laguerre polynomials with leading coefficient $\frac{(-1)^n}{n!}$, respectively.

The next proposition summarizes some structural and asymptotic properties of the classical and k-iterated Laguerre polynomials (see [8, 7, 4, 6] and the references therein).

Proposition 2.1 *The following statements hold.*

(i) Hahn condition. For every $n \in \mathbb{N}$,

$$\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x). \tag{2.5}$$

(ii) [12, Theorem 8.22.3] Outer strong asymptotics or Perron asymptotics formula on $\mathbb{C}\setminus\mathbb{R}_+$. Let $\alpha \in \mathbb{R}$; then

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{1}{2}\pi^{-1/2}e^{x/2}(-x)^{-\alpha/2-1/4}n^{\alpha/2-1/4}\exp\left(2(-nx)^{1/2}\right) \\ &\quad \times\left\{\sum_{k=0}^{p-1}C_k(\alpha;x)n^{-k/2}+O(n^{-p/2})\right\}. \end{aligned} \tag{2.6}$$

Here $C_k(\alpha;x)$ is independent of n. This relation holds for x in the complex plane with a cut along the positive real semiaxis, and it also holds if x is in the cut plane mentioned. $(-x)^{-\alpha/2-1/4}$ and $(-x)^{1/2}$ must be taken as real and positive if $x<0$. The bound for the remainder holds uniformly in every compact subset of the complex plane with empty intersection with $\mathbb{R}_+$.

(iii) *For every* $n \in \mathbb{N}$,

$$h_n^{(\alpha)} := \int_0^\infty [L_n^{(\alpha)}(x)]^2 d\mu(x) \cong n^\alpha.$$

(iv) *[4, page 79] The limit*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha),[k]}(x)}{n^{1/2} L_n^{(\alpha),[k-1]}(x)} = \frac{1}{\sqrt{-x}+\sqrt{-c}} \tag{2.7}$$

holds uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$.

The following proposition will be useful in the sequel and it summarizes some recent structural and asymptotic properties of Laguerre–Sobolev-type polynomials.

Proposition 2.2 *Let* $\{L_n^{(\alpha,M,N)}(x)\}_{n\geq 0}$ *be the sequence of normalized Laguerre–Sobolev-type polynomials with leading coefficient equal to* $\frac{(-1)^n}{n!}$, *associated with the Sobolev-type inner product (1.1). Then the following statements hold.*

(a) *[8, Theorem 4] Connection formula for* $L_n^{(\alpha,M,N)}(x)$.

$$L_n^{(\alpha,M,N)}(x) = B_{0,n} L_n^{(\alpha)}(x) + B_{1,n}(x-c) L_{n-1}^{(\alpha),[2]}(x) + B_{2,n}(x-c)^2 L_{n-2}^{(\alpha),[4]}(x), \tag{2.8}$$

where

(i) *If* $M>0$ *and* $N>0$, *then*

$$B_{0,n} \cong \frac{8cn^\alpha}{M\left(L_n^{(\alpha)}(c)\right)^2}, \qquad B_{1,n} \cong -\frac{32c\sqrt{|c|}\,n^{\alpha-1/2}}{M\left(L_n^{(\alpha)}(c)\right)^2}, \qquad B_{2,n} \cong \frac{1}{n^2}. \tag{2.9}$$

(ii) *If* $M=0$ *and* $N>0$, *then*

$$B_{0,n} \cong \frac{1}{4\sqrt{|c|n}}, \qquad B_{1,n} \cong -\frac{1}{n}, \qquad B_{2,n} \cong \frac{1}{4n^2\sqrt{|c|n}}.$$

(iii) *If* $M>0$ *and* $N=0$, *then*

$$B_{0,n} \cong \frac{\sqrt{|c|}}{Mn^{1/2-\alpha}\left(L_{n-1}^{(\alpha)}(c)\right)^2}, \qquad B_{1,n} \cong -\frac{1}{n}, \qquad B_{2,n} = 0.$$

(b) *[8, Theorem 5 (ii)] Mehler–Heine-type formula.*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M,N)}(\frac{x}{n})}{n^\alpha} = \begin{cases} x^{-\alpha/2} J_\alpha(2\sqrt{x}), & \text{if } M>0, N>0, \\ \\ -x^{-\alpha/2} J_\alpha(2\sqrt{x}), & \text{if } M=0, N>0 \text{ or } M>0, N=0, \end{cases} \tag{2.10}$$

uniformly on compact subsets of $\mathbb{C}$.

3. Estimates for the norms of k-iterated and Laguerre–Sobolev-type polynomials

In this section, we obtain some estimates for the norm of the k-iterated Laguerre orthogonal polynomials and Laguerre–Sobolev-type polynomials, respectively. In addition, we complete our study by deducing a connection formula involving different families of k-iterated Laguerre orthogonal polynomials. It is worth mentioning that this is a result of independent interest.

Proposition 3.1 *For $\alpha > -1$ we have*

$$h_n^{(\alpha),[k]} := \int_0^{\infty} [L_n^{(\alpha),[k]}(x)]^2 (x-c)^k d\mu(x) \cong n^{\alpha+k}, \quad k \geq 0. \tag{3.11}$$

Proof

First of all, we proceed by induction on k in order to prove

$$L_{n+1}^{(\alpha),[k]}(c) \cong L_n^{(\alpha),[k]}(c), \quad k \geq 0. \tag{3.12}$$

For $k=0$, from Perron's asymptotics formula (2.6), we obtain $L_{n+1}^{(\alpha)}(c) \cong L_n^{(\alpha)}(c)$. Assuming that (3.12) is true for $i \leq k-1$, we use this induction hypothesis for $i = k-1$ and (2.7), as follows

$$\frac{L_{n+1}^{(\alpha),[k]}(c)}{L_n^{(\alpha),[k]}(c)} = \frac{L_{n+1}^{(\alpha),[k]}(c)}{L_{n+1}^{(\alpha),[k-1]}(c)} \frac{L_n^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k]}(c)} \frac{L_{n+1}^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k-1]}(c)} \cong 1.$$

Finally, the estimate (3.12) together with [8, equation (9)] yields (3.11). $\square$

3.1. Representation formula for k-iterated Laguerre polynomials

We complete our study of k-iterated Laguerre orthogonal polynomials by giving a representation formula. Notice that even though the Askey inversion formula has been used repeatedly in order to obtain representation formulas involving different families of Laguerre orthogonal polynomials (up to multiplication for the corresponding weight functions), this method can not be applied in order to obtain a connection formula for k-iterated polynomials as that for classical Laguerre ones given in [9, equation (2.15)] because the function $\frac{(x-c)^{2j}}{x^j}$ does not belong to $L^1(x^\alpha e^{-x}dx)$. An alternative method is presented in the following proposition and, in addition, we obtain estimates for the coefficients appearing therein.

Proposition 3.2 *The following connection formula holds.*

$$(x-c)^j L_n^{(\alpha),[k]}(x) = \sum_{m=0}^{j} a_{m,j,k}(\alpha,n) L_{n+m}^{(\alpha),[k-j]}(x), \quad \textit{for } 1 \leq j \leq k, \tag{3.13}$$

where $a_{m,j,k}(\alpha,n) \cong (-1)^m \binom{j}{m} n^j$. In particular, for $j=k$ we have (generalized Christoffel representation formula)

$$(x-c)^j L_n^{(\alpha),[j]}(x) = \sum_{m=0}^{j} a_{m,j}(\alpha,n) L_{n+m}^{(\alpha)}(x).$$

Proof We proceed by induction on j. For the case $j=1$, the Christoffel formula reads (see [2, Sec. I.7])

$$(x-c)L_n^{(\alpha),[k]}(x) = -(n+1)L_{n+1}^{(\alpha),[k-1]}(x) + (n+1)\frac{L_{n+1}^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k-1]}(c)}L_n^{(\alpha),[k-1]}(x), \tag{3.14}$$

and using (3.12), we obtain

$$a_{0,1,k}(\alpha,n) = (n+1)\frac{L_{n+1}^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k-1]}(c)} \cong n.$$

For $j=2$, it suffices to note that $(x-c)^2L_n^{(\alpha),[k]}(x) = (x-c)\left[(x-c)L_n^{(\alpha),[k]}(x)\right]$ and, according to (3.14), we have

$$(x-c)^2L_n^{(\alpha),[k]}(x) = a_{2,2,k}(\alpha,n)L_{n+2}^{(\alpha),[k-2]}(x) + a_{1,2,k}(\alpha,n)L_{n+1}^{(\alpha),[k-2]}(x) + a_{0,2,k}(\alpha,n)L_n^{(\alpha),[k-2]}(x),$$

where

$$\begin{aligned}
a_{1,2,k}(\alpha,n) &= (n+1)(n+2) \cong n^2 \\
a_{1,2,k}(\alpha,n) &= -(n+1)(n+2)\frac{L_{n+2}^{(\alpha),[k-2]}(c)}{L_{n+1}^{(\alpha),[k-2]}(c)} - (n+1)^2\frac{L_{n+1}^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k-1]}(c)} \cong -2n^2, \\
a_{0,2,k}(\alpha,n) &= (n+1)^2\frac{L_{n+1}^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k-1]}(c)}\frac{L_{n+1}^{(\alpha),[k-2]}(c)}{L_n^{(\alpha),[k-2]}(c)} \cong n^2.
\end{aligned}$$

Let us assume that

$$(x-c)^{j-1}L_n^{(\alpha),[k]}(x) = \sum_{m=0}^{j-1} a_{m,j-1,k}(\alpha,n)L_{n+m}^{(\alpha),[k-j+1]}(x),$$

where $a_{m,j-1,k}(\alpha,n) \cong (-1)^m\binom{j-1}{m}n^{j-1}$. Then,

$$\begin{aligned}
(x-c)^jL_n^{(\alpha),[k]}(x) &= \sum_{m=0}^{j-1} a_{m,j-1,k}(\alpha,n)(x-c)L_{n+m}^{(\alpha),[k-j+1]}(x) \\
&= \sum_{m=0}^{j-1} a_{m,j-1,k}(\alpha,n)\left(-(n+1)L_{n+m+1}^{(\alpha),[k-j]}(x) + (n+1)\frac{L_{n+m+1}^{(\alpha),[k-j]}(c)}{L_{n+m}^{(\alpha),[k-j]}(c)}L_{n+m}^{(\alpha),[k-j]}(x)\right) \\
&= \sum_{m=0}^{j} a_{m,j,k}(\alpha,n)L_{n+m}^{(\alpha),[k-j]}(x),
\end{aligned}$$

with

$$a_{0,j,k}(\alpha,n) = (n+1)a_{0,j-1,k}(\alpha,n)\frac{L_{n+1}^{(\alpha),[k-j]}(c)}{L_{n}^{(\alpha),[k-j]}(c)} \cong n^j,$$

$$a_{m,j,k}(\alpha,n) = -(n+1)a_{m-1,j-1,k}(\alpha,n) + (n+1)a_{m,j-1,k}(\alpha,n)\frac{L_{n+m+1}^{(\alpha),[k-j]}(c)}{L_{n+m}^{(\alpha),[k-j]}(c)}$$

$$\cong (-1)^m\binom{j-1}{m-1}n^j + (-1)^m\binom{j-1}{m}n^j = (-1)^m\binom{j}{m}n^j, \quad 1 \le m \le j-1,$$

$$a_{j,j,k}(\alpha,n) = -(n+1)a_{j-1,j-1,k}(\alpha,n) \cong (-1)^j n^j,$$

and this proves (3.13). □

Now we need to estimate the Laguerre–Sobolev-type norm

$$h_n^{(\alpha,M,N)} := \langle L_n^{(\alpha,M,N)}, L_n^{(\alpha,M,N)}\rangle_S.$$

The next proposition states that the estimate of this norm is the same as the estimate obtained for the norms of classical Laguerre polynomials.

Proposition 3.3 *For every $n \in \mathbb{N}$,*

$$h_n^{(\alpha,M,N)} \cong n^\alpha. \tag{3.15}$$

Proof

From the Sobolev-type orthogonality, we get

$$h_n^{(\alpha,M,N)} = \left\langle L_n^{(\alpha,M,N)}(x), \frac{(-1)^n}{n!}(x-c)^n \right\rangle_S, \quad n \ge 0. \tag{3.16}$$

Since the nonstandard component of the Sobolev-type inner product on the right side of (3.16) is equal to zero for $n \ge 2$, according to (2.8) we have

$$\left\langle L_n^{(\alpha,M,N)}(x), \frac{(-1)^n}{n!}(x-c)^n \right\rangle_S = \int_0^\infty L_n^{(\alpha,M,N)}(x)\frac{(-1)^n}{n!}(x-c)^n x^\alpha e^{-x}dx$$

$$= B_{0,n}h_n^{(\alpha)} - \frac{B_{1,n}}{n}h_{n-1}^{(\alpha),[2]} + \frac{B_{2,n}}{n(n-1)}h_{n-2}^{(\alpha),[4]}.$$

Finally, analyzing the asymptotic behavior given in (2.9) and using (3.11), the result follows. □

Notice that the above estimate for the norm of the Laguerre–Sobolev-type orthogonal polynomials together with (2.10) (resp. (2.8)) allows us to obtain the corresponding Mehler–Heine-type formula (resp. a connection formula for the orthonormal Sobolev-type polynomials $L_n^{\alpha,M,N}(x)$.)

We conclude this section with the analog of [5, Proposition 5] when $c < 0$.

Proposition 3.4 *Let $M, N \ge 0$ and $\{L_n^{\alpha,M,N}(x)\}_{n\ge 0}$ be the sequence of orthonormal Laguerre–Sobolev-type polynomials. For $\alpha > -1/2$ we have*

$$\left(\int_0^\infty \left|L_n^{\alpha,M,N}(x)e^{-x/2}\right|^p x^\alpha dx\right)^{1/p} \ge \begin{cases} Cn^{-1/4}(\log n)^{1/p}, & \text{if } p = \frac{4\alpha+4}{2\alpha+1}, \\ Cn^{\alpha/2-(\alpha+1)/p}, & \text{if } \frac{4\alpha+4}{2\alpha+1} < p < \infty, \end{cases} \tag{3.17}$$

and for $\alpha > -2/p,\ 1 < p < \infty,$ *we have*

$$\left(\int_0^\infty \left|L_n^{\alpha,M,N}(x)e^{-x/2}\right|^p x^\alpha dx\right)^{1/p} \geq \begin{cases} Cn^{-1/4}(\log n)^{1/p} & \text{if } p = 4, \\ Cn^{-1/p}, & \text{if } 4 < p < \infty. \end{cases} \tag{3.18}$$

Proof It suffices to follow the proof given in [5, Proposition 5] by making the corresponding modifications and using (3.15) as well as (2.10) for orthonormal polynomials. □

4. Cohen-type inequality for Fourier expansions with respect to Laguerre–Sobolev-type orthogonal polynomials associated with the inner product (1.1)

The goal of this section is to show a Cohen-type inequality for Fourier expansions with respect to Laguerre–Sobolev-type orthonormal polynomials associated with the Sobolev inner product (1.1). To this end, we will follow the Markett approach but, as was mentioned at the beginning, we will incorporate new test functions different from those used in [9].

Now we are going to introduce the notation concerning weighted L^p spaces, Sobolev-type spaces, test functions, and some usual elements from functional analysis, which will be needed in the following text.

We consider the following weighted L^p spaces:

$$L^p_{w(\alpha)} = \begin{cases} \left\{f : \{\int_0^\infty |f(x)e^{-x/2}|^p x^\alpha dx\}^{1/p} < \infty\right\}, & 1 \leq p < \infty, \\ \left\{f : \operatorname{ess\,sup}_{x>0}|f(x)e^{-x/2}| < \infty\right\}, & p = \infty, \end{cases}$$

for $\alpha > -1$. Furthermore,

$$L^p_{u(\alpha)} = \{f : ||f(x)u(x,\alpha)||_{L^p(0,\infty)} < \infty, u(x,\alpha) = e^{-x/2}x^{\alpha/2}\},$$

where $\alpha > -\frac{2}{p}$ if $1 \leq p < \infty$ and $\alpha \geq 0$ if $p = \infty$.

We also use the notation $L^p_{g(\alpha)}$, where the subscript $g(\alpha)$ means either $w(\alpha)$ or $u(\alpha)$. The Sobolev-type spaces are denoted by

$$S^p_{g(\alpha)} = \{f \in L^p_{g(\alpha)} \cap C^\infty : ||f||^p_{S^p_{g(\alpha)}} = ||f||^p_{L^p_{g(\alpha)}} + M|f(c)|^p + N|f'(c)|^p < \infty\}, \quad 1 \leq p < \infty, \tag{4.19}$$

$$S^\infty_{g(\alpha)} = \left\{f \in L^\infty_{g(\alpha)} \cap C^\infty : ||f||_{S^\infty_{g(\alpha)}} = \max\{||f||_{L^\infty_{g(\alpha)}}, |f(c)|, |f'(c)|\} < \infty\right\}, \quad p = \infty. \tag{4.20}$$

Let $[S^p_{g(\alpha)}]$ be the space of all bounded linear operators $T : S^p_{g(\alpha)} \to S^p_{g(\alpha)}$, with the standard operator norm

$$||T||_{[S^p_{g(\alpha)}]} = \sup_{0 \neq f \in S^p_{g(\alpha)}} \frac{||T(f)||_{S^p_{g(\alpha)}}}{||f||_{S^p_{g(\alpha)}}}.$$

For $f \in S^1_{g(\alpha)}$, the Fourier series in terms of the Laguerre–Sobolev-type orthonormal polynomials is given by

$$\sum_{k=0}^{\infty} \hat{f}(k) L_k^{\alpha,M,N}(x), \tag{4.21}$$

where $\hat{f}(k) = \langle f, L_k^{\alpha,M,N} \rangle_S, \quad k = 0, 1, \ldots$

The Cesáro means of order δ, a nonnegative integer number, of the series (4.21) is

$$\sigma_n^\delta f(x) := \sum_{k=0}^{n} \frac{A_{n-k}^\delta}{A_n^\delta} \hat{f}(k) L_k^{\alpha,M,N}(x),$$

where $A_k^\delta = \binom{k+\delta}{k}$.

For $f \in S_{g(\alpha)}^p$ and $\{c_{k,n}\}_{k=0}^n, n \in \mathbb{N} \cup \{0\}$, a family of complex numbers with $|c_{n,n}| > 0$, let us introduce the operators $T_n^{\alpha,M,N}$:

$$T_n^{\alpha,M,N}(f) := \sum_{k=0}^{n} c_{k,n} \hat{f}(k) L_k^{\alpha,M,N}.$$

The first technical step required for the proof of our main result is the choice of suitable test functions. For instance, in the setting of Laguerre–Sobolev-type expansions, see [5, 9, 11]; the authors consider (up to a constant factor) the following test functions:

$$g_n^{\alpha,j}(x) := n^{-\alpha/2} \left[x^j L_n^{(\alpha+j)}(x) - \left(\frac{(n+1)(n+2)}{(n+\alpha+j+1)(n+\alpha+j+2)} \right)^{1/2} x^j L_{n+2}^{(\alpha+j)}(x) \right]. \tag{4.22}$$

These functions and their derivatives vanish at 0 and this fact is a key property in the development of the ideas of [9, 5, 11]. Unfortunately, they do not vanish at the mass point $c < 0$. For this reason, it seems to be natural to consider the following slight modification of the functions (4.22):

$$G_n^{\alpha,j}(x) := n^{-\alpha/2} \left[(x-c)^2 x^j L_{n+2}^{(\alpha+j)}(x) - A_{n,\alpha}(x-c)^2 x^j L_{n+4}^{(\alpha+j)}(x) \right] \tag{4.23}$$

with $A_{n,\alpha} = \left(\frac{(n+3)(n+4)}{(n+\alpha+j+3)(n+\alpha+j+4)} \right)^{1/2}$.

As a consequence, it is well known that the test polynomials $G_n^{\alpha,j}(x)$ can be expressed as (see [9, equation (2.15)])

$$G_n^{\alpha,j}(x) = n^{-\alpha/2} (x-c)^2 \sum_{m=0}^{j+2} a_{m,j}(\alpha, n) L_{n+2+m}^{(\alpha)}(x), \tag{4.24}$$

with

$$a_{0,j}(\alpha, n) \cong n^j.$$

Finally, the last technical step is to estimate the norm of the test functions (4.23).

Lemma 4.1 *For some $j > \alpha - 1/2 - 2(\alpha+1)/p$, we have*

$$||G_n^{\alpha,j}||_{S^p_{g(\alpha)}} \le C \begin{cases} n^{j+2-\alpha/2-1/2+(\alpha+1)/p}, & \text{if } g(\alpha) = w(\alpha), \\ n^{j+2-1/2+1/p}, & \text{if } g(\alpha) = u(\alpha). \end{cases} \tag{4.25}$$

Proof Taking into account that the Sobolev norm of $G_n^{\alpha,j}(x)$ coincides with its $L^p_{g(\alpha)}$-norm (for $g(\alpha) = w(\alpha)$ or $g(\alpha) = u(\alpha)$) and also considering the following expression for $G_n^{\alpha,j}(x)$,

$$G_n^{\alpha,j}(x) = g_{n+2}^{\alpha-2,j+2}(x) - 2cg_{n+2}^{\alpha-1,j+1}(x) + c^2 g_{n+2}^{\alpha,j}(x),$$

where $g_n^{\alpha,j}(x)$ is the test polynomial given in (4.22), so we only need to use [9, Lemma 1] in order to obtain the estimates (4.25). □

According to the notation in [5], let us denote $q_0 = \frac{4\alpha+4}{2\alpha+1}$, when $\beta = \alpha$, and $q_0 = 4$, when $\beta = p\alpha/2$, and let p_0 be the conjugate of q_0. We are ready to state our main result.

Theorem 4.1 *Let $M, N \ge 0$ and $1 \le p \le \infty$. For $\alpha > -1/2$,*

$$||T_n^{\alpha,M,N}||_{[S^p_{w(\alpha)}]} \ge C|c_{n,n}| \begin{cases} n^{\frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}} & \text{if } a \le p < p_0, \\ (\log n)^{\frac{2\alpha+1}{4\alpha+4}} & \text{if } p = p_0, p = q_0, \\ n^{\frac{2\alpha+1}{2} - \frac{2\alpha+2}{p}} & \text{if } q_0 < p \le b. \end{cases}$$

For $\alpha > -2/p$ if $1 \le p < \infty$ and $\alpha \ge 0$ if $p = \infty$,

$$||T_n^{\alpha,M,N}||_{[S^p_{u(\alpha)}]} \ge C|c_{n,n}| \begin{cases} n^{\frac{2}{p} - \frac{3}{2}} & \text{if } a \le p < p_0, \\ (\log n)^{\frac{1}{4}} & \text{if } p = p_0, p = q_0, \\ n^{\frac{1}{2} - \frac{2}{p}} & \text{if } q_0 < p \le b, \end{cases}$$

where

(i) if $M = 0$, $N \ge 0$, then $a = 1$ and $b = \infty$,

(ii) if $M > 0$, $N \ge 0$, then $a > 1$, $b < \infty$, and $1/a + 1/b = 1$.

Proof Applying the operator $T_n^{\alpha,M,N}$ to the test functions $G_n^{\alpha,j}(x)$ we get

$$T_n^{\alpha,N,M}(G_n^{\alpha,j}) = \sum_{k=0}^{n} c_{k,n} (G_n^{\alpha,j})^\wedge(k) L_k^{\alpha,M,N}, \tag{4.26}$$

where

$$(G_n^{\alpha,j})^\wedge(k) = \langle G_n^{\alpha,j}, L_k^{\alpha,M,N} \rangle_S, \qquad k = 0, \dots, n.$$

From (4.24) and the Sobolev orthogonality it follows in a straightforward way that

$$(G_n^{\alpha,j})^\wedge(k) = 0 \quad \text{if } k < n.$$

When $k = n$, we get

$$\begin{aligned}(G_n^{\alpha,j})^\wedge(n) =& n^{-\alpha/2} a_{0,j}(\alpha,n) \int_0^\infty L_{n+2}^{(\alpha)}(x) L_n^{\alpha,M,N}(x)(x-c)^2 x^\alpha e^{-x} dx \\ =& n^{-\alpha/2} a_{0,j}(\alpha,n) \left(h_n^{(\alpha,M,N)}\right)^{-1/2} \int_0^\infty L_{n+2}^{(\alpha)}(x) L_n^{(\alpha,M,N)}(x)(x-c)^2 x^\alpha e^{-x} dx.\end{aligned}$$

We can expand the polynomial $(x-c)^2 L_n^{(\alpha,M,N)}(x)$ in terms of the classical Laguerre polynomials,

$$(x-c)^2 L_n^{(\alpha,M,N)}(x) = \sum_{k=0}^{n+2} \alpha_{n+2,k} L_k^{(\alpha)}(x).$$

The comparison of the leading coefficient of both hand sides yields

$$\alpha_{n+2,n+2} = (n+2)(n+1).$$

On the other hand,

$$\begin{aligned}(G_n^{\alpha,j})^\wedge(n) =& n^{-\alpha/2} a_{0,j}(\alpha,n) \left(h_n^{(\alpha,M,N)}\right)^{-1/2} \int_0^\infty L_{n+2}^{(\alpha)}(x) L_n^{(\alpha,M,N)}(x)(x-c)^2 x^\alpha e^{-x} dx \\ =& n^{-\alpha/2} a_{0,j}(\alpha,n) \left(h_n^{(\alpha,M,N)}\right)^{-1/2} \alpha_{n+2,n+2} \int_0^\infty \left(L_{n+2}^{(\alpha)}(x)\right)^2 x^\alpha e^{-x} dx \\ =& n^{-\alpha/2} a_{0,j}(\alpha,n) \left(h_n^{(\alpha,M,N)}\right)^{-1/2} (n+2)(n+1) h_{n+2}^{(\alpha)} \\ \cong& n^{j+2}.\end{aligned}$$

As a conclusion,

$$\begin{cases} (G_n^{\alpha,j})^\wedge(k) = 0, & 0 \le k \le n-1, \\ (G_n^{\alpha,j})^\wedge(n) \cong n^{j+2}. \end{cases}$$

Now we follow the proof given in [5, Theorem 1], taking into account that

$$|L_n^{\alpha,0,0}(c)| \sim n^{\alpha/2-1/4} e^{2\sqrt{-nc}} \left\{ \sum_{k=0}^{p-1} C_k(\alpha;c) n^{-k/2} + O(n^{-p/2}) \right\},$$

$$|L_n^{\alpha,0,N}(c)| \sim \frac{1}{\sqrt{-cn}} n^{\alpha/2-1/4} e^{2\sqrt{-nc}} \left\{ \sum_{k=0}^{p-1} C_k(\alpha;c) n^{-k/2} + O(n^{-p/2}) \right\}.$$

□

Corollary 4.1 *Let β, p_0, q_0, and p be the same as in Theorem 4.1. For $c_{k,n} = 1$, $k = 0, \ldots, n$, and for p outside the Pollard interval (p_0, q_0) we get*

$$||S_n||_{[S^p_{g(\beta)}]} \to \infty, \quad n \to \infty,$$

where S_n denotes the nth partial sum of the expansion (4.21).

It is worth pointing out that Corollary 4.1 says that as for the results of [9, 5, 11], the divergence of Fourier expansions in terms of this kind of Laguerre–Sobolev-type orthonormal polynomials remains true.

For $c_{k,n} = \frac{A^{\delta}_{n-k}}{A^{\delta}_{n}}$, $k = 0, \ldots, n$, from Theorem 4.1 we also get the divergence of Cesàro means of order δ when p is located outside the Pollard interval.

Corollary 4.2 *Let $M, N \geq 0$ and $1 \leq p \leq \infty$. For $\alpha > -1/2$,*

$$\begin{cases} 0 \leq \delta < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}, & \text{if } a \leq p < p_0, \\ 0 \leq \delta < \frac{2\alpha+1}{2} - \frac{2\alpha+2}{p}, & \text{if } q_0 < p \leq b, \end{cases}$$

and $p \notin [p_0, q_0]$, then

$$||\sigma^{\delta}_{n}||_{[S^{p}_{w(\alpha)}]} \to \infty, \quad n \to \infty.$$

For $\alpha > -2/p$ if $1 \leq p < \infty$, and $\alpha \geq 0$, if $p = \infty$,

$$\begin{cases} 0 \leq \delta < \frac{2}{p} - \frac{3}{2}, & \text{if } a \leq p < p_0, \\ 0 \leq \delta < \frac{1}{2} - \frac{2}{p}, & \text{if } q_0 < p \leq b, \end{cases}$$

and $p \notin [p_0, q_0]$, then we get

$$||\sigma^{\delta}_{n}||_{[S^{p}_{u(\alpha)}]} \to \infty, \quad n \to \infty.$$

Remark 4.1 *It still remains an open question the study of Cohen-type inequalities for the Laguerre–Sobolev-type orthonormal polynomials with respect to the inner product*

$$\langle f, g \rangle_S = \int_0^{\infty} f(x)g(x)d\mu(x) + \sum_{j=0}^{N} M_j f^{(j)}(c)g^{(j)}(c), \tag{4.27}$$

where $d\mu(x) = x^{\alpha}e^{-x}dx$ is the Laguerre measure, $c < 0$, and $M_j \geq 0$ for $j = 0, \ldots, N$ assuming that $M_N > 0$. The main difficulties in this case would be how to choose suitable test functions as well as the possibility of having gaps in the Sobolev-type inner products, i.e. $M_j = 0$ for some $j = 0, \ldots, N-1$. This means that the matrix $diag(M_0, ..., M_N)$ does not have full rank (see, for instance, [11] and the references therein.)

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