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# Arithmetical rank of the edge ideals of some $n$-cyclic graphs with a common edge

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**Abstract:** In this paper, we present some lower bounds and upper bounds on the arithmetical rank of the edge ideals of some $n$-cyclic graphs with a common edge. For some special $n$-cyclic graphs with a common edge, we prove that the arithmetical rank equals the projective dimension of the corresponding quotient ring.

**Key words:** Arithmetical rank, edge ideal, projective dimension, set-theoretic complete intersection

## 1. Introduction

Let $R$ be a Noetherian commutative ring with identity and $I$ a proper ideal of $R$. The arithmetical rank (ara) of $I$ is defined as the minimal number $s$ of elements $a_1, \ldots, a_s$ of $R$ such that the ideal $(a_1, \ldots, a_s)$ has the same radical as $I$. In this case we will say that $a_1, \ldots, a_s$ generate $I$ up to radical. In general $\mathrm{ht}\,(I) \leq \mathrm{ara}\,(I)$. If equality holds, $I$ is called a set-theoretic complete intersection.

We consider the case where $R$ is a polynomial ring over any field $K$ and $I$ is the edge ideal of a graph whose vertices are the indeterminates. The set of its generators is formed by the products of the pairs of indeterminates that form the edges of the graph. Thus, $I$ is generated by square-free quadratic monomials and is therefore a radical ideal. The problem of the arithmetical rank of edge ideals or monomial ideals has been intensively studied by many authors over the past 3 decades (see [1, 2, 3, 5, 8, 10]).

According to a well-known result by Lyubeznik [9], if $I$ is a square-free monomial ideal, the projective dimension of the quotient ring $R/I$, denoted $\mathrm{pd}_R\,(R/I)$, provides a lower bound on the arithmetical rank of $I$. We define the big height of $I$, denoted $\mathrm{bight}\,(I)$, as the maximum height of the minimal prime ideals of $I$. In general, we have that

$$\mathrm{ht}\,(I) \leq \mathrm{bight}\,(I) \leq \mathrm{pd}_R\,(R/I) \leq \mathrm{ara}\,(I) \leq \mu(I),$$

where the second inequality on the left is due to Morey and Villarreal [11] and $\mu(I)$ is the minimum number of generators of the ideal $I$. If $I$ is not unmixed, then $I$ is not a set-theoretic complete intersection, but it could still be true that $\mathrm{bight}\,(I) = \mathrm{pd}_R\,(R/I) = \mathrm{ara}\,(I)$. This equality has been respectively established for the edge ideals of forests by Barile [1] and Kimura and Terai [8]. A weaker condition is the equality between the arithmetical rank and the projective dimension. This is the case for all cyclic and bicyclic graphs (see [3]) and for the graphs consisting of paths and cycles with a common vertex (see [7]). In all these cases, the arithmetical rank is independent of the field $K$.

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The computation of the arithmetical rank of the edge ideal of a graph is still an open problem. In this paper, we consider the class of some $n$-cyclic graphs with a common edge and give some lower bounds and upper bounds on the arithmetical rank of the edge ideals of these graphs. For some special $n$-cyclic graphs with a common edge, we also prove that the arithmetical rank equals the projective dimension of the corresponding quotient ring.

## 2. Preliminaries

We first recall some definitions and basic facts about graphs and their edge ideals in order to make this paper self-contained. However, for more details on the notions, we refer the reader to [4, 6, 13].

**Definition 2.1** *A finite graph $G$ is an ordered pair $G = (V(G), E(G))$ where $V(G) = \{x_1, \ldots, x_n\}$ is the set of vertices of $G$, and $E(G)$ is a collection of 2-element subsets of $V(G)$, usually called the edges of $G$.*

**Definition 2.2** *Let $G_i = (V(G_i), E(G_i))$ be some graphs with vertex set $V(G_i)$ and edge set $E(G_i)$, for $i = 1, \ldots, k$. The union of the graphs $G_1, G_2, \ldots, G_k$, written $G_1 \cup G_2 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$.*

**Definition 2.3** *Let $G = (V(G), E(G))$ be a graph. A walk of length $m$ in $G$ is an alternating sequence of vertices and edges $w = \{x_1, y_1, x_2, \ldots, x_m, y_m, x_{m+1}\}$, where $y_i = \{x_i, x_{i+1}\}$ is the edge joining $x_i$ and $x_{i+1}$. If $x_1 = x_{m+1}$, we call this walk closed. A walk may also be denoted $\{x_1, \ldots, x_{m+1}\}$, the edges being evident from the context.*

*A cycle of length $m$ ($m \geq 3$) is a closed walk in which the vertices $x_1, \ldots, x_m$ are distinct. We denote by $C_m$ the graph consisting of a cycle with $m$ vertices.*

**Definition 2.4** *A graph $G$ is called an $n$-cyclic graph with a common edge if $G$ is a graph consisting of $n$ cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$, $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$, $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ connected through a common edge, where $k_1 + k_2 + k_3 = n$.*

**Definition 2.5** *Let $G$ be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$, with $n \in \mathbb{N}$, $n \geq 1$, and whose edge set is $E(G)$. Suppose that $x_1, \ldots, x_n$ are indeterminates over the field $K$. The edge ideal of $G$ in the polynomial ring $R = K[x_1, \ldots, x_n]$ is the squarefree monomial ideal*

$$I(G) = (\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}).$$

*For the sake of simplicity, we will use the same notation $x_i x_j$ for the monomial and for the corresponding edge.*

Throughout the paper, we let $G$ be an $n$-cyclic graph with a common edge consisting of the union of $n$ cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$, $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$, $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ connected through a common edge $x_1x_2$, where $k_1 + k_2 + k_3 = n$. We consider the following labeling for the edges of $G$:

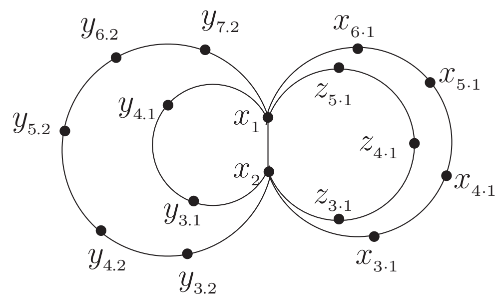
$E(C_{3r_i,i}) = \{x_1x_2, x_2x_{3,i}, \ldots, x_{3r_i,i}x_1\}$ for all $1 \leq i \leq k_1$ and $r_1 \geq r_2 \geq \cdots \geq r_{k_1}$,

$E(C_{3s_i+1,i}) = \{x_1x_2, x_2y_{3,i}, \ldots, y_{3s_i+1,i}x_1\}$ for all $1 \leq i \leq k_2$ and $s_1 \geq s_2 \geq \cdots \geq s_{k_2}$,

$E(C_{3t_i+2,i}) = \{x_1x_2, x_2z_{3,i}, \ldots, z_{3t_i+2,i}x_1\}$ for all $1 \leq i \leq k_3$ and $t_1 \geq t_2 \geq \cdots \geq t_{k_3}$.

Let $K$ be any field and $R = K[x_1, x_2, x_{3,1}, \ldots, x_{3r_1,1}, \ldots, x_{3,k_1}, \ldots, x_{3r_{k_1},k_1}, y_{3,1}, \ldots, y_{3s_1+1,1}, \ldots, y_{3,k_2}, \ldots, y_{3s_{k_2}+1,k_2}, z_{3,1}, \ldots, z_{3t_1+2,1}, \ldots, z_{3,k_3}, \ldots, z_{3t_{k_3}+2,k_3}]$ be the polynomial ring.

**Example 2.6** *The following graph $G$ is a 4-cyclic graph consisting of the union of 4 cycles $C_4$, $C_5$, $C_6$, and $C_7$ with a common edge $x_1x_2$.*




**Figure**

*The edge ideal of $G$ is* $I(G) = (x_1x_2, x_2x_{3,1}, x_{3,1}x_{4,1}, x_{4,1}x_{5,1}, x_{5,1}x_{6,1}, x_{6,1}x_1, x_2y_{3,1}, y_{3,1}y_{4,1}, y_{4,1}x_1, x_2y_{3,2}, y_{3,2}y_{4,2}, y_{4,2}y_{5,2}, y_{5,2}y_{6,2}, y_{6,2}y_{7,2}, y_{7,2}x_1, x_2z_{3,1}, z_{3,1}z_{4,1}, z_{4,1}z_{5,1}, z_{5,1}x_1)$.

**Definition 2.7** *Let $G = (V(G), E(G))$ be a graph. A vertex cover for $G$ is a subset $S$ of $V(G)$ that intersects every edge of $G$. If $S$ is a minimal element (under inclusion) of the set of vertex covers of $G$, it is called a minimal vertex cover.*

**Remark 2.8** *It is well known that the minimal primes of $I(G)$ in $R$ are the ideals generated by the minimal vertex covers of G. Hence, for any minimal vertex cover $S$ of $G$, we have*

$$|S| \leq \mathit{bight}(I(G)).$$

We can easily find the minimal vertex covers for the cycle graph $C_m$ on the vertex set $\{x_1, \ldots, x_m\}$. Note that a subset $S$ of $V(C_m)$ is a minimal vertex cover for $C_m$ if and only if at least 1 and at most 2 of $x_i$, $x_{i+1}$, and $x_{i+2}$ belong to $S$ for every $i \in \{1, \ldots, m-2\}$.
If $m = 3r$, then a minimal vertex cover of $C_m$ is

$$S = \{x_2, x_3, x_5, x_6, \ldots, x_{3r-4}, x_{3r-3}, x_{3r-1}, x_{3r}\}, \quad \text{where } |S| = 2r.$$

If $m = 3s+1$, then a minimal vertex cover of $C_m$ is

$$S = \{x_2, x_3, x_5, x_6, \ldots, x_{3s-4}, x_{3s-3}, x_{3s-1}, x_{3s+1}\}, \quad \text{where } |S| = 2s.$$

If $m = 3t+2$, then a minimal vertex cover of $C_m$ is

$$S = \{x_2, x_3, x_5, x_6, \ldots, x_{3t-1}, x_{3t}, x_{3t+2}\}, \quad \text{where } |S| = 2t+1.$$

By [3, Theorem 2], we obtain that the cardinalities of these minimal vertex covers are all maximum.

## 3. Main results

In this section, we present some lower bounds on bight $(I(G))$ and some upper bounds on ara $(I(G))$ of the edge ideals $I(G)$ of some $n$-cyclic graphs $G$ with a common edge.

The following useful technique that provides an upper bound for the arithmetical rank of some ideals is a result due to Schmitt and Vogel [12].

**Lemma 3.1** *[12, p. 249] Let $P$ be a finite subset of elements of $R$. Let $P_0, \ldots, P_r$ be subsets of $P$ such that*

*(i)* $\bigcup_{i=0}^{r} P_i = P$ *;*

*(ii)* $P_0$ *has exactly one element;*

*(iii) if $p$ and $p'$ are different elements of $P_i (0 < i \le r)$, then there is an integer $i'$ with $0 \le i' < i$ and an element in $P_{i'}$ that divides $pp'$.*

*We set $q_i = \sum_{p \in P_i} p^{e(p)}$, where $e(p) \ge 1$ are arbitrary integers. We will write $(P)$ for the ideal of $R$ generated by the elements of $P$. Then we get*

$$\sqrt{(P)} = \sqrt{(q_0, \ldots, q_r)}.$$

*In particular, ara $((P)) \le r + 1$.*

In the construction given in the above lemma, if we take all exponents $e(p) = 1$, then $q_0, \ldots, q_r$ are sums of generators.

The following lemma can be used for computing the arithmetical rank of some monomial ideals.

**Lemma 3.2** *Let $q_0, q_{11}, q_{12}, q_{21}, q_{22} \in R$ be such that $q_0 | q_{11}q_{22}, q_{21} | q_{11}q_{12}$, and $q_{12} | q_{21}q_{22}$. Set $q_1 = q_{11} + q_{12}$ and $q_2 = q_{21} + q_{22}$. Then*

$$\sqrt{(q_0, q_1, q_2)} = \sqrt{(q_0, q_{11}, q_{12}, q_{21}, q_{22})}.$$

**Proof** It suffices to show that $q_{11}, q_{12}, q_{21}, q_{22} \in \sqrt{(q_0, q_1, q_2)}$. Let $a, b \in R$ be such that $q_{11}q_{22} = aq_0$ and $q_{11}q_{12} = bq_{21}$, and then

$$\begin{aligned} q_{11}^3 &= q_{11}^2(q_{11} + q_{12}) - q_{11}^2 q_{12} = q_{11}^2(q_{11} + q_{12}) - q_{11} b q_{21} \\ &= q_{11}^2(q_{11} + q_{12}) - b q_{11}(q_{21} + q_{22}) + b q_{11} q_{22} \\ &= q_{11}^2(q_{11} + q_{12}) - b q_{11}(q_{21} + q_{22}) + b a q_0 \\ &= q_{11}^2 q_1 - b q_{11} q_2 + b a q_0 \in \sqrt{(q_0, q_1, q_2)}. \end{aligned}$$

This shows that $q_{11} \in \sqrt{(q_0, q_1, q_2)}$, from which $q_{12} = q_1 - q_{11} \in \sqrt{(q_0, q_1, q_2)}$. The claim for $q_{21}, q_{22} \in \sqrt{(q_0, q_1, q_2)}$ follows by symmetry. This completes the proof. □

In order to present some lower bounds on bight $(I(G))$ and some upper bounds on ara $(I(G))$ of the edge ideals $I(G)$ of some graphs $G$, which are the union of some cycles with a common edge, we consider 3 cases, depending on the residue modulo 3 of the lengths of the cycles. The cases $m \equiv 0, 1 \mod 3$ can be settled by a direct application of Lemma 3.1. The case $m \equiv 2 \mod 3$ is more interesting, since it needs some additional nontrivial computations on the generators. The idea of this paper is derived from Barile et al. [3]; we consider 3 cases that are treated separately in the following 3 theorems.

First, we consider the case where graph $G$ is the union of some cycles whose lengths are all multiples of 3.

**Theorem 3.3** *Let $G$ be a $k_1$-cyclic graph consisting of the union of $k_1$ cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$ with a common edge $x_1x_2$. Set $q_0 = x_1x_2$, and, for any $1 \leq j \leq r_i - 1$, $1 \leq i \leq k_1$, set*

$$\begin{aligned} q_{1,i} &= x_1x_{3r_i,i} + x_2x_{3,i}, \\ q_{2j,i} &= x_{3j+1,i}x_{3j+2,i}, \\ q_{2j+1,i} &= x_{3j,i}x_{3j+1,i} + x_{3j+2,i}x_{3j+3,i}. \end{aligned}$$

*Then*

$$I(G) = \sqrt{(q_0, q_{1,1}, \ldots, q_{2r_1-1,1}, \ldots, q_{1,k_1}, \ldots, q_{2r_{k_1}-1,k_1})}.$$

*In particular, $ara(I(G)) = ara(\sum_{i=1}^{k_1} I(C_{3r_i})) \leq 1 + \sum_{i=1}^{k_1}(2r_i - 1)$.*

**Proof** We show that the assumptions of Lemma 3.1 are fulfilled by the $1 + \sum_{i=1}^{k_1}(2r_i - 1)$ sets $P_0, P_{1,1}, \ldots, P_{2r_1-1,1}, \ldots, P_{1,k_1}, \ldots, P_{2r_{k_1}-1,k_1}$, where, for all $1 \leq j \leq 2r_i - 1$, $1 \leq i \leq k_1$, $P_{j,i}$ is the set of monomials appearing in $q_{j,i}$. It is straightforward to verify that conditions $(i)$ and $(ii)$ are satisfied. For any $1 \leq j \leq r_i - 1$, $1 \leq i \leq k_1$, we have that $x_1x_2|(x_1x_{3r_i,i})(x_2x_{3,i})$, $x_{3j+1,i}x_{3j+2,i}|(x_{3j,i}x_{3j+1,i})(x_{3j+2,i}x_{3j+3,i})$, where $x_1x_{3r_i,i}, x_2x_{3,i}$ are the monomials in $P_{1,i}$, $x_{3j+1,i}x_{3j+2,i}$ is the only monomial in $P_{2j,i}$, and $x_{3j,i}x_{3j+1,i}, x_{3j+2,i}x_{3j+3,i}$ are the monomials in $P_{2j+1,i}$. Note that $I(G) = \sum_{i=1}^{k_1} I(C_{3r_i})$; hence, the claim is true. □

**Theorem 3.4** *Let $G$ be a $k_1$-cyclic graph consisting of the union of $k_1$ cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$ with a common edge $x_1x_2$. Then $bight(I(G)) \geq 1 + \sum_{i=1}^{k_1}(2r_i - 1)$.*

**Proof** It is easy to prove that $S = \{x_2, x_{3,1}, x_{5,1}, x_{6,1}, \ldots, x_{3r_1-4,1}, x_{3r_1-3,1}, x_{3r_1-1,1}, x_{3r_1,1}, \ldots, x_{3,k_1}, x_{5,k_1}, x_{6,k_1}, \ldots, x_{3r_{k_1}-4,k_1}, x_{3r_{k_1}-3,k_1}, x_{3r_{k_1}-1,k_1}, x_{3r_{k_1},k_1}\}$ is a minimal vertex cover for $G$. By Remark 2.8, we obtain that

$$\text{bight}(I(G)) \geq |S| = \sum_{i=1}^{k_1} 2r_i - (k_1 - 1) = 1 + \sum_{i=1}^{k_1}(2r_i - 1).$$

□

As a consequence of the above 2 theorems, we have:

**Corollary 3.5** *Let $G$ be a graph as in Theorem 3.4. Then*

$$bight(I(G)) = pd_R(R/I(G)) = ara(I(G)).$$

Now we consider the case where graph $G$ is the union of some cycles whose lengths are all congruent to 1 modulo 3.

The following 2 theorems can be shown by arguments similar to those used for Theorem 3.3, and so we omit their proofs.

**Theorem 3.6** *Let $G$ be a $k_2$-cyclic graph consisting of the union of $k_2$ cycles $C_{3s_1+1},\dots,C_{3s_{k_2}+1}$ with a common edge $x_1x_2$, where $s_i=1$ for any $i\in\{1,\dots,k_2-1\}$. Set $q_0=x_1x_2$, $q_1=x_1y_{4,1}+x_2y_{3,1}$; for $2\le i\le k_2$, set $q_i=x_1y_{4,i}+x_2y_{3,i}+y_{3,i-1}y_{4,i-1}$; and, for any $1\le j\le s_{k_2}-1$, set*

$$\begin{aligned} q'_{2j} &= y_{3j+1,k_2}y_{3j+2,k_2}, \\ q'_{2j+1} &= y_{3j,k_2}y_{3j+1,k_2}+y_{3j+2,k_2}y_{3j+3,k_2}, \end{aligned}$$

*and, finally, $q'_{2s_{k_2}}=y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}$. Then*

$$I(G)=\sqrt{(q_0,q_1,\dots,q_{k_2},q'_2,q'_3,\dots,q'_{2s_{k_2}})}.$$

*In particular, $ara(I(G))\le 1+k_2+2(s_{k_2}-1)+1=k_2+2s_{k_2}$.*

**Theorem 3.7** *Let $G$ be a $k_2$-cyclic graph consisting of the union of $k_2$ cycles $C_{3s_1+1},\dots,C_{3s_{k_2}+1}$ with a common edge $x_1x_2$. Set $q_0=x_1x_2$, and, for any $1\le j\le s_i-1$, $1\le i\le k_2$, set*

$$\begin{aligned} q'_{1,i} &= x_1y_{3s_i+1,i}+x_2y_{3,i}, \\ q'_{2j,i} &= y_{3j+1,i}y_{3j+2,i}, \\ q'_{2j+1,i} &= y_{3j,i}y_{3j+1,i}+y_{3j+2,i}y_{3j+3,i}, \\ q'_{2s_i,i} &= y_{3s_i,i}y_{3s_i+1,i}. \end{aligned}$$

*Then*

$$I(G)=\sqrt{(q_0,q'_{1,1},\dots,q'_{2s_1,1},\dots,q'_{1,k_2},\dots,q'_{2s_{k_2},k_2})}.$$

*In particular, $ara(I(G))=ara(\sum_{i=1}^{k_2}I(C_{3s_i+1}))\le 1+2\sum_{i=1}^{k_2}s_i$.*

**Theorem 3.8** *Let $G$ be a $k_2$-cyclic graph consisting of the union of $k_2$ cycles $C_{3s_1+1},\dots,C_{3s_{k_2}+1}$ with a common edge $x_1x_2$. Then $bight(I(G))=2-k_2+2\sum_{i=1}^{k_2}s_i$.*

**Proof** It is obvious that $S=\{x_2,y_{3,1},y_{5,1},y_{6,1},\dots,y_{3s_1-4,1},y_{3s_1-3,1},y_{3s_1-1,1},y_{3s_1,1};x_1,y_{4,2},y_{5,2},y_{7,2},y_{8,2},\dots,y_{3s_2-2,2},y_{3s_2-1,2},y_{3s_2+1,2};\dots,y_{4,k_2},y_{5,k_2},y_{7,k_2},y_{8,k_2},\dots,y_{3s_{k_2}-2,k_2},y_{3s_{k_2}-1,k_2},y_{3s_{k_2}+1,k_2}\}$ is a minimal vertex cover for $G$. By Remark 2.8, we can obtain that

$$\text{bight}\,(I(G))\ge|S|=2s_1+2s_2+\sum_{i=3}^{k_2}[2(s_i-1)+1]=2-k_2+2\sum_{i=1}^{k_2}s_i.$$

Now we prove that $\text{bight}\,(I(G))\le 2-k_2+2\sum_{i=1}^{k_2}s_i$. This is equivalent to showing that the cardinality of any minimal vertex cover $S$ of $G$ is at most $2-k_2+2\sum_{i=1}^{k_2}s_i$. For each $i\in\{1,\dots,k_2\}$, set $S_i=S\cap V(C_{3s_i+1})$.

As $x_1x_2$ is the edge of $C_{3s_i+1}$, by Definition 2.7, we obtain that at least one of $x_1$ and $x_2$ belong to $S_i$, and that $S_i$ is either a minimal vertex cover $\bar{S}_i$ of $C_{3s_i+1}$ or becomes a minimal cover $\bar{S}_i$ after removing $x_1$ or $x_2$. The vertex to be removed certainly belongs to $\bar{S}_j$ for some $j$, or otherwise $S$ would not be minimal. This shows that $S$ is the union of the sets $\bar{S}_i$. By the characterization of minimal vertex covers given below Remark 2.8, we obtain that $\bar{S}_i$ at most contain $2s_i$ elements, one of which is $x_1$ or $x_2$. We distinguish the following cases: (1) If $x_1 \in S_i$ and $x_2 \notin S_i$ for all $1 \le i \le k_2$, or $x_2 \in S_i$ and $x_1 \notin S_i$ for all $1 \le i \le k_2$. These two cases can be shown by similar arguments, so we only consider the case $x_1 \in S_i$ and $x_2 \notin S_i$ for all $1 \le i \le k_2$. In this case, we have that

$$|S| \le 2s_1 + (2s_2 - 1) + \cdots + (2s_{k_2} - 1) = \sum_{i=1}^{k_2} 2s_i - k_2 + 1.$$

(2) If $x_1 \in S_i$ and $x_2 \in S_j$ for some $i, j \in \{1, \ldots, k_2\}$, then we get that

$$\begin{aligned} |S| &\le (2s_1 - 1) + \cdots + (2s_{i-1} - 1) + 2s_i + (2s_{i+1} - 1) + \cdots \\ &\quad +(2s_{j-1} - 1) + 2s_j + (2s_{j+1} - 1) + \cdots + (2s_{k_2} - 1) \\ &= \sum_{i=1}^{k_2} 2s_i - k_2 + 2. \end{aligned}$$

In conclusion, we have that $\text{bight}\,(I(G)) \le \sum_{i=1}^{k_2} 2s_i - k_2 + 2$. □

As a consequence of Theorems 3.6 and 3.8, we have:

**Corollary 3.9** *Let $G$ be a $k_2$-cyclic graph consisting of the union of $k_2$ cycles $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$ with a common edge $x_1x_2$, where $s_i = 1$ for any $i \in \{1, \ldots, k_2 - 1\}$. Then*

$$\textit{bight}\,(I(G)) = \textit{pd}_R\,(R/I(G)) = \textit{ara}\,(I(G)).$$

As a consequence of Theorems 3.7 and 3.8, we have:

**Corollary 3.10** *Let $G$ be a graph as in Theorem 3.8. Then*

$$\textit{ara}\,(I(G)) - \textit{bight}\,(I(G)) \le k_2 - 1.$$

Then the following open question occurs.

**Problem 3.11** *Is the upper bound in Corollary 3.10 sharp? In other words, can the upper bound in Corollary 3.10 be improved?*

Finally, we consider the case where graph $G$ is the union of some cycles whose lengths are all congruent to 2 modulo 3.

**Theorem 3.12** *Let $G$ be a $k_3$-cyclic graph consisting of the union of $k_3$ cycles $C_{3t_1+2},\ldots,C_{3t_{k_3}+2}$ with a common edge $x_1x_2$. Set $q_0=x_1x_2$, and, for any $1\le j\le t_i-1$, $1\le i\le k_3$, set*

$$\begin{aligned} q''_{1,i} &= x_2z_{3,i}+z_{4,i}z_{5,i},\\ q''_{2j,i} &= z_{3j,i}z_{3j+1,i}+z_{3j+2,i}z_{3j+3,i},\\ q''_{2j+1,i} &= z_{3j+2,i}z_{3j+3,i}+z_{3j+4,i}z_{3j+5,i},\\ q''_{2t_i,i} &= x_1z_{3t_i+2,i}+z_{3t_i,i}z_{3t_i+1,i}. \end{aligned}$$

*Then*

$$I(G)=\sqrt{(q_0,q''_{1,1},\ldots,q''_{2t_1,1},\ldots,q''_{1,k_3},\ldots,q''_{2t_{k_3},k_3})}.$$

*In particular,* $ara(I(G))=ara(\sum_{i=1}^{k_3}I(C_{3t_i+2}))\le 1+2\sum_{i=1}^{k_3}t_i$.

**Proof** It suffices to show that $I(C_{3t_i+2})$ can be generated, up to radical, by $2t_i+1$ polynomials, one of which is $q_0=x_1x_2$. We consider 2 cases:

(i) If there exists some $i\in\{1,\ldots,k_3\}$ such that $t_i=1$, set $q_0=x_1x_2$, $q''_{1,i}=x_2z_{3,i}+z_{4,i}z_{5,i}$, $q''_{2,i}=z_{3,i}z_{4,i}+x_1z_{5,i}$; then we get that $q_0|(x_2z_{3,i})(x_1z_{5,i})$, $z_{3,i}z_{4,i}|(x_2z_{3,i})(z_{4,i}z_{5,i})$ and $z_{4,i}z_{5,i}|(x_1z_{5,i})(z_{3,i}z_{4,i})$. Thus, by Lemma 3.2, we have that $I(C_5)=\sqrt{(q_0,q''_{1,i},q''_{2,i})}$.

(ii) If there exists some $1\le i\le k_3$ with $t_i\ge 2$, then set $J_{t_i,i}=(q_0,q''_{1,i},\ldots,q''_{2t_i,i})$. It suffices to show that $I(C_{3t_i+2})\subseteq\sqrt{J_{t_i,i}}$. For all $f,g\in R$, by abuse of notation, we will write $f\equiv^{t_i}g$ whenever $f-g$ or $f+g$ belongs to $J_{t_i,i}$, and $f\equiv_{q''_{j,i}}g$ whenever $f-g$ or $f+g$ is divisible by $q''_{j,i}$. In this way, $f\equiv^{t_i}g$ or $f\equiv_{q''_{j,i}}g$ assures that $f\in J_{t_i,i}$ if and only if $g\in J_{t_i,i}$.

Set

$$\begin{aligned} u_{t_i,i} &= x_1^{2^{t_i-1}}z_{3t_i+2,i}^{2^{t_i}},\\ v_{t_i,i} &= z_{3,i}z_{4,i}z_{5,i}\prod_{j=2}^{t_i}z_{3j,i}^{3\cdot 2^{j-2}},\\ w_{t_i,i} &= (z_{3t_i,i}z_{3t_i+1,i}z_{3t_i+2,i})^{2^{t_i-1}}. \end{aligned}$$

First we prove that

$$u_{t_i,i}\equiv_{q''_{2t_i,i}}w_{t_i,i}\qquad\text{and}\qquad v_{t_i,i}\equiv^{t_i}w_{t_i,i}. \tag{1}$$

For all $t_i\ge 2$, we have that $u_{t_i,i}=x_1^{2^{t_i-1}}z_{3t_i+2,i}^{2^{t_i}}=(x_1z_{3t_i+2,i})^{2^{t_i-1}}z_{3t_i+2,i}^{2^{t_i-1}}\equiv_{q''_{2t_i,i}}(z_{3t_i,i}z_{3t_i+1,i})^{2^{t_i-1}}z_{3t_i+2,i}^{2^{t_i-1}}=w_{t_i,i}$. This proves the first relation in (1).

We prove the second relation by induction on $t_i\ge 2$. If $t_i=2$, then we have $q''_{2,i}=z_{3,i}z_{4,i}+z_{5,i}z_{6,i}$, $q''_{3,i}=z_{5,i}z_{6,i}+z_{7,i}z_{8,i}$, so that $v_{2,i}=z_{3,i}z_{4,i}z_{5,i}z_{6,i}^3\equiv_{q''_{2,i}}z_{5,i}^2z_{6,i}^2z_{6,i}^2\equiv_{q''_{3,i}}z_{6,i}^2z_{7,i}^2z_{8,i}^2=w_{2,i}$, which shows that

$v_{2,i} \equiv^2 w_{2,i}$, i.e. our claim is correct for $t_i = 2$. Now suppose that $t_i > 2$ and that the claim is true for $t_i - 1$. We have that

$$\begin{aligned} v_{t_i,i} &= v_{t_i-1,i} z_{3t_i,i}^{3\cdot 2^{t_i-2}} \equiv^{t_i-1} w_{t_i-1,i} z_{3t_i,i}^{3\cdot 2^{t_i-2}} \\ &= (z_{3t_i-3,i} z_{3t_i-2,i} z_{3t_i-1,i})^{2^{t_i-2}} z_{3t_i,i}^{3\cdot 2^{t_i-2}} = (z_{3t_i-3,i} z_{3t_i-2,i})^{2^{t_i-2}} z_{3t_i-1,i}^{2^{t_i-2}} z_{3t_i,i}^{3\cdot 2^{t_i-2}} \\ &\equiv_{q''_{2t_i-2,i}} (z_{3t_i-1,i} z_{3t_i,i})^{2^{t_i-2}} z_{3t_i-1,i}^{2^{t_i-2}} z_{3t_i,i}^{3\cdot 2^{t_i-2}} = (z_{3t_i-1,i} z_{3t_i,i})^{2^{t_i-1}} z_{3t_i,i}^{2\cdot 2^{t_i-2}} \\ &\equiv_{q''_{2t_i-1,i}} (z_{3t_i+1,i} z_{3t_i+2,i})^{2^{t_i-1}} z_{3t_i,i}^{2^{t_i-1}} = (z_{3t_i,i} z_{3t_i+1,i} z_{3t_i+2,i})^{2^{t_i-1}} = w_{t_i,i}. \end{aligned}$$

It follows that $v_{t_i,i} \equiv^{t_i} w_{t_i,i}$. Note that the symbols $q''_{2,i}, \ldots, q''_{2(t_i-1)-1,i}$ have the same meaning in $J_{t_i-1,i}$ and $J_{t_i,i}$. This completes the proof of (1).

Secondly, we show that

$$x_1^{2^{t_i}} z_{3t_i+2,i}^{2^{t_i+1}} \in J_{t_i,i}. \tag{2}$$

Noting that $x_1 z_{3t_i+2,i} z_{4,i} z_{5,i} \mid x_1 z_{3t_i+2,i} v_{t_i,i}$ and $x_1 z_{3t_i+2,i} z_{4,i} z_{5,i} \equiv_{q_0} x_1 z_{3t_i+2,i}(x_2 z_{3,i} + z_{4,i} z_{5,i}) \in J_{t_i,i}$, we deduce that $x_1 z_{3t_i+2,i} v_{t_i,i} \in J_{t_i,i}$. Thus, (1) will imply that $x_1^{2^{t_i}} z_{3t_i+2,i}^{2^{t_i+1}} = x_1^2 z_{3t_i+2,i}^2 u_{t_i,i} \in J_{t_i,i}$. Hence, we have shown that

$$x_1 z_{3t_i+2,i} \in \sqrt{J_{t_i,i}}. \tag{3}$$

Thus, we get that

$$z_{3t_i,i} z_{3t_i+1,i} = q''_{2t_i,i} - x_1 z_{3t_i+2,i} \in \sqrt{J_{t_i,i}}. \tag{4}$$

In general, whenever, for some $j \in \{2, \ldots, t_i\}$,

$$z_{3j,i} z_{3j+1,i} \in \sqrt{J_{t_i,i}}, \tag{5}$$

from the fact that $z_{3j,i} z_{3j+1,i} | (z_{3j-1,i} z_{3j,i})(z_{3j+1,i} z_{3j+2,i})$, by Lemma 3.1, one deduces that

$$z_{3j-1,i} z_{3j,i} \in \sqrt{J_{t_i,i}}. \tag{6}$$

Since $z_{3j-3,i} z_{3j-2,i} = q''_{2j-2,i} - z_{3j-1,i} z_{3j,i}$, this in turn implies that

$$z_{3j-3,i} z_{3j-2,i} \in \sqrt{J_{t_i,i}}. \tag{7}$$

Finally, since $z_{3j-3,i} z_{3j-2,i} | (z_{3j-4,i} z_{3j-3,i})(z_{3j-2,i} z_{3j-1,i})$, by Lemma 3.1, we again conclude that

$$z_{3j-2,i} z_{3j-1,i} \in \sqrt{J_{t_i,i}}. \tag{8}$$

Therefore, for all $j \in \{2, \ldots, t_i\}$, by (4) and descending induction on $h$, one can derive that $z_{h,i} z_{h+1,i} \in \sqrt{J_{t_i,i}}$ for all $h = 3, \ldots, 3t_i$.

In particular, we have that $z_{4,i} z_{5,i} \in \sqrt{J_{t_i,i}}$, which, together with $q''_{1,i} \in \sqrt{J_{t_i,i}}$ and $x_2 z_{3,i} = q''_{1,i} - z_{4,i} z_{5,i}$, yields $x_2 z_{3,i} \in \sqrt{J_{t_i,i}}$. Moreover, $z_{3t_i-1,i} z_{3t_i,i} \in \sqrt{J_{t_i,i}}$, $q''_{2t_i-1,i} \in \sqrt{J_{t_i,i}}$ and $z_{3t_i+1,i} z_{3t_i+2,i} = q''_{2t_i-1,i} -$

$z_{3t_i-1,i}z_{3t_i,i}$ imply that $z_{3t_i+1,i}z_{3t_i+2,i} \in \sqrt{J_{t_i,i}}$ . This, together with (3) and $x_1x_2 = q_0 \in \sqrt{J_{t_i,i}}$ , shows that $I(C_{3t_i+2}) \subseteq \sqrt{J_{t_i,i}}$, as claimed. □

**Theorem 3.13** *Let $G$ be a $k_3$-cyclic graph consisting of the union of $k_3$ cycles $C_{3t_1+2},\dots,C_{3t_{k_3}+2}$ with a common edge $x_1x_2$. Then $bight(I(G)) \geq 1+2\sum_{i=1}^{k_3} t_i$.*

**Proof** It is obvious that $S = \{x_2, z_{3,1}, z_{5,1}, z_{6,1},\dots, z_{3t_1-1,1}, z_{3t_1,1}, z_{3t_1+2,1},\dots, z_{3,k_3}, z_{5,k_3}, z_{6,k_3},\dots, z_{3t_{k_3}-1,k_3}, z_{3t_{k_3},k_3}, z_{3t_{k_3}+2,\,k_3}\}$ is a minimal vertex cover for $G$. By Remark 2.8, we obtain that

$$\text{bight}\,(I(G)) \geq |S| = \sum_{i=1}^{k_3}(2t_i+1) - (k_3-1) = 1+2\sum_{i=1}^{k_3} t_i.$$

□

As a consequence of the above 2 theorems, we have:

**Corollary 3.14** *Let $G$ be a graph as in Theorem 3.12. Then*

$$bight(I(G)) = pd_R\,(R/I(G)) = ara\,(I(G)).$$

From Theorems 3.3, 3.7, and 3.12, we can derive an upper bound on $\text{ara}\,(I(G))$ when $G$ is an $n$-cyclic graphs with a common edge. We adopt the following convention: whenever, in a sum, the index runs from 1 to 0, the sum has to be taken equal to zero.

**Theorem 3.15** *Let $G$ be an $n$-cyclic graph consisting of the union of $n$ cycles $C_{3r_1},\dots,C_{3r_{k_1}}$, $C_{3s_1+1},\dots, C_{3s_{k_2}+1}$, $C_{3t_1+2},\dots,C_{3t_{k_3}+2}$ with a common edge $x_1x_2$, where $k_1+k_2+k_3=n$.*

*Then $ara\,(I(G)) \leq \sum_{i=1}^{k_1}(2r_i-1) + 2\sum_{i=1}^{k_2} s_i + 2\sum_{i=1}^{k_3} t_i + 1$.*

Now we present a lower bound on $\text{bight}\,(I(G))$ of $n$-cyclic graphs with a common edge.

**Theorem 3.16** *Let $G$ be an $n$-cyclic graph consisting of the union of $n$ cycles $C_{3r_1},\dots,C_{3r_{k_1}}$, $C_{3s_1+1},\dots, C_{3s_{k_2}+1}$, $C_{3t_1+2},\dots,C_{3t_{k_3}+2}$ with a common edge $x_1x_2$, where $k_1+k_2+k_3=n$. Then*

$$bight(I(G)) \geq \begin{cases} 1+\sum_{i=1}^{k_1}(2r_i-1)+\sum_{i=1}^{k_2}(2s_i-1)+2\sum_{i=1}^{k_3} t_i & \text{if } k_2 \in \{0,1\}; \\ 2+\sum_{i=1}^{k_1}(2r_i-1)+\sum_{i=1}^{k_2}(2s_i-1)+2\sum_{i=1}^{k_3} t_i & \text{if } k_2 \geq 2. \end{cases}$$

**Proof** We distinguish the following cases:
(1) If $k_2 = 0$, then it is easy to check that

$$\begin{aligned} S \;=\; & \{x_2\} \cup \{x_{3,1}, x_{5,1}, x_{6,1},\dots, x_{3r_1-4,1}, x_{3r_1-3,1}, x_{3r_1-1,1}, x_{3r_1,1}\} \cup \cdots \\ & \cup\{x_{3,\,k_1}, x_{5,\,k_1}, x_{6,\,k_1},\dots, x_{3r_{k_1}-4,\,k_1}, x_{3r_{k_1}-3,\,k_1}, x_{3r_{k_1}-1,\,k_1}, x_{3r_{k_1},\,k_1}\} \\ & \cup\{z_{3,1}, z_{5,1}, z_{6,1}, z_{8,1},\dots, z_{3t_1-1,1}, z_{3t_1,1}, z_{3t_1+2,1}\} \cup \cdots \\ & \cup\{z_{3,k_3}, z_{5,k_3}, z_{6,k_3}, z_{8,k_3},\dots, z_{3t_{k_3}-1,\,k_3}, z_{3t_{k_3},\,k_3}, z_{3t_{k_3}+2,\,k_3}\} \end{aligned}$$

is a minimal vertex cover for $G$. Thus, by Remark 2.8, we get that

$$\text{bight}\,(I(G)) \quad \geq \quad |S| = 1 + \sum_{i=1}^{k_1}(2r_i - 1) + 2\sum_{i=1}^{k_3} t_i.$$

(2) If $k_2 = 1$, then by similar argumentation we obtain that

$$\begin{aligned} S \quad = \quad & \{x_2\} \cup \{x_{3,1}, x_{5,1}, x_{6,1}, \ldots, x_{3r_1-4,1}, x_{3r_1-3,1}, x_{3r_1-1,1}, x_{3r_1,1}\} \cup \cdots \\ & \cup\{x_{3,\,k_1}, x_{5,\,k_1}, x_{6,\,k_1}, \ldots, x_{3r_{k_1}-4,\,k_1}, x_{3r_{k_1}-3,\,k_1}, x_{3r_{k_1}-1,\,k_1}, x_{3r_{k_1},\,k_1}\} \\ & \cup\{y_{3,1}, y_{5,1}, y_{6,1}, \ldots, y_{3s_1-4,1}, y_{3s_1-3,1}, y_{3s_1-1,1}, y_{3s_1+1,1}\} \\ & \cup\{z_{3,1}, z_{5,1}, z_{6,1}, z_{8,1}, \ldots, z_{3t_1-1,1}, z_{3t_1,1}, z_{3t_1+2,1}\} \cup \cdots \\ & \cup\{z_{3,k_3}, z_{5,k_3}, z_{6,k_3}, z_{8,k_3}, \ldots, z_{3t_{k_3}-1,\,k_3}, z_{3t_{k_3},\,k_3}, z_{3t_{k_3}+2,\,k_3}\} \end{aligned}$$

is a minimal vertex cover for $G$. Thus, by Remark 2.8, we get that

$$\begin{aligned} \text{bight}\,(I(G)) \quad & \geq \quad |S| = 1 + \sum_{i=1}^{k_1}(2r_i - 1) + 2s_1 - 1 + \sum_{i=1}^{k_3} 2t_i \\ & = \quad 1 + \sum_{i=1}^{k_1}(2r_i - 1) + \sum_{i=1}^{1}(2s_i - 1) + 2\sum_{i=1}^{k_3} t_i. \end{aligned}$$

(3) Now suppose that $k_2 \geq 2$. It is then obvious that

$$\begin{aligned} S \quad = \quad & \{x_2\} \cup \{x_{3,1}, x_{5,1}, x_{6,1}, \ldots, x_{3r_1-4,1}, x_{3r_1-3,1}, x_{3r_1-1,1}, x_{3r_1,1}\} \cup \cdots \\ & \cup\{x_{3,\,k_1}, x_{5,\,k_1}, x_{6,\,k_1}, \ldots, x_{3r_{k_1}-4,\,k_1}, x_{3r_{k_1}-3,\,k_1}, x_{3r_{k_1}-1,\,k_1}, x_{3r_{k_1},\,k_1}\} \\ & \cup\{y_{3,1}, y_{5,1}, y_{6,1}, \ldots, y_{3s_1-4,1}, y_{3s_1-3,1}, y_{3s_1-1,1}, y_{3s_1,1}\} \\ & \cup\{x_1, y_{4,2}, y_{5,2}, y_{7,2}, y_{8,2}, \ldots, y_{3s_2-2,2}, y_{3s_2-1,2}, y_{3s_2+1,2}\} \cup \cdots \\ & \cup\{y_{4,\,k_2}, y_{5,\,k_2}, y_{7,\,k_2}, y_{8,\,k_2}, \ldots, y_{3s_{k_2}-2,\,k_2}, y_{3s_{k_2}-1,\,k_2}, y_{3s_{k_2}+1,\,k_2}\} \\ & \cup\{z_{3,1}, z_{5,1}, z_{6,1}, z_{8,1}, \ldots, z_{3t_1-1,1}, z_{3t_1,1}, z_{3t_1+2,1}\} \cup \cdots \\ & \cup\{z_{3,\,k_3}, z_{5,\,k_3}, z_{6,\,k_3}, z_{8,\,k_3}, \ldots, z_{3t_{k_3}-1,\,k_3}, z_{3t_{k_3},\,k_3}, z_{3t_{k_3}+2,\,k_3}\} \end{aligned}$$

is a minimal vertex cover for $G$. Thus, by Remark 2.8, we get that

$$\begin{aligned} \text{bight}\,(I(G)) \quad & \geq \quad |S| = 1 + \sum_{i=1}^{k_1}(2r_i - 1) + \sum_{i=1}^{k_2} 2s_i - k_2 + 1 + \sum_{i=1}^{k_3} 2t_i \\ & = \quad 2 + \sum_{i=1}^{k_1}(2r_i - 1) + \sum_{i=1}^{k_2}(2s_i - 1) + 2\sum_{i=1}^{k_3} t_i. \end{aligned}$$

□

As a consequence of the above 2 theorems, we have:

**Corollary 3.17** *Let $G$ be an $n$-cyclic graph consisting of the union of $n$ cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$ and $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ with a common edge $x_1x_2$, where $k_1 + k_3 = n$. Then*

$$bight(I(G)) = pd_R(R/I(G)) = ara(I(G)).$$

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## References

[1] Barile M. On the arithmetical rank of the edge ideals of forests. Comm Algebra 2008; 36: 4678–4703.

[2] Barile M. On the arithmetical rank of an intersection of ideals. Alg Coll 2012; 19: 797–806.

[3] Barile M, Kiani D, Mohammadi F, Yassemi S. Arithmetical rank of the cyclic and bicyclic graphs. J Algebra Appl 2012; 11: 1–14.

[4] Harary F. Graph Theory. Reading, MA, USA: Addison-Wesley, 1994.

[5] He J, Van Tuyl A. Algebraic properties of the path ideal of a tree. Comm Algebra 2010; 38: 1725–1742.

[6] Herzog J, Hibi T. Monomial Ideals. New York, NY, USA: Springer-Verlag, 2011.

[7] Kiani D, Mohammadi F. On the arithmetical rank of the edge ideals of some graphs. Alg Coll 2012; 19: 797–806.

[8] Kimura K, Terai N. Binomial arithmetical rank of edge ideals of forests. Proc Amer Math Soc 2013; 141: 1925–1932.

[9] Lyubeznik G. On the local cohomology modules $H^i_{\mathfrak{a}}(R)$ for ideals $\mathfrak{a}$ generated by monomials in an R-sequence. Lect Notes Math 1984; 1092: 214–220.

[10] Lyubeznik G. On the arithmetical rank of monomial ideals. J Algebra 1988; 112: 86–89.

[11] Morey S, Villarreal RH. Edge ideals: algebraic and combinatorial properties. Progress in Commutative Algebra, Combinatorics and Homology 2012; 1: 85–126.

[12] Schmitt T, Vogel W. Note on set-theoretic intersections of subvarieties of projective space. Math Ann 1979; 245: 247–253.

[13] Villarreal RH. Monomial Algebras. New York, NY, USA: Marcel Dekker, 2001.