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On p -schemes with the same degrees of thin radical and thin residue

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Abstract: Let p and $n > 1$ be a prime number and an integer, respectively. In this paper, first we show that any p -scheme whose thin radical and thin residue are equal is isomorphic to a fission of the wreath product of 2 thin schemes. In addition, we characterize association p -schemes whose thin radical and thin residue each have degree equal to p . We also characterize association p -schemes on p^n points whose thin radical and thin residue each have degree equal to p^{n-1} , and whose basis relations each have valency 1 or p^{n-1} . Moreover, we show that such schemes are Schurian.

Key words: Association scheme, p -scheme, thin radical, thin residue

1. Introduction

Association schemes are related to a variety of combinatorial objects (codes, designs, graphs, etc.). In [6], schemes are presented as a natural generalization of permutation groups. In this direction, p -schemes correspond to p -groups, where p is a prime number. The concept of p -schemes was given in [6] as follows: a scheme \mathcal{C} is called p -scheme if the cardinality of each basis relation of \mathcal{C} is a power of p . Recently some algebraic and combinatorial properties of p -schemes were studied in [2, 3, 6, 7, 8].

In this paper we deal with association schemes and refer to them as schemes. Given a scheme \mathcal{C} , one can define its *thin radical* $\mathcal{O}_\vartheta(\mathcal{C})$ and *thin residue* $\mathcal{O}^\vartheta(\mathcal{C})$. Suppose that \mathcal{C} is a p -scheme; this implies that the thin radical $\mathcal{O}_\vartheta(\mathcal{C})$ of \mathcal{C} is a nontrivial p -group [6, Theorem 2.2]. Thus, the degree of its thin radical is a power of p . All p -schemes of degree p are thin, and they are unique up to isomorphism; we denote this unique p -scheme by T_p . Moreover, the number of isomorphism classes of p -schemes of degree p^2 is 3, which are T_{p^2} , $T_p \otimes T_p$, and $T_p \wr T_p$. Thus, any p -scheme on V with $|V| \in \{1, p, p^2\}$ is Schurian [2, p. 2]. In [1], non-Schurian p -schemes of degree p^3 were constructed. In [2], p -schemes with $|V| = p^3$ and thin residue of degree p^2 were studied. In [7], it was shown that any p -scheme on V is isomorphic to a fission of the wreath product of n copies of T_p , where $|V| = p^n$.

Our main results show that any p -scheme whose thin radical and thin residue are equal is isomorphic to a fission of the wreath product of 2 thin schemes. Moreover, we provide a characterization of p -schemes whose degrees of thin radical and thin residue are equal, in terms of the wreath product of thin schemes. The following theorems are the main results of this paper.

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Theorem 1 *Let \mathcal{C} be a scheme whose thin radical and thin residue are equal. \mathcal{C} is then isomorphic to a fission of the wreath product of 2 thin schemes.*

Theorem 2 *Let \mathcal{C} be a p -scheme of degree p^n . The degrees of the thin radical and the thin residue of \mathcal{C} are then equal to p if and only if \mathcal{C} is isomorphic to the wreath product of T_p and a thin scheme of degree p^{n-1} .*

Theorem 3 *Let \mathcal{C} be a p -scheme of degree p^n . The degrees of the thin radical and the thin residue of \mathcal{C} are then equal to p^{n-1} and the valencies of its basis relations are 1 or p^{n-1} if and only if \mathcal{C} is isomorphic to the wreath product of a thin scheme of degree p^{n-1} and T_p .*

This paper is organized as follows. In Section 2, we present some notations and terminology on association schemes and p -schemes. In Section 3, we prove our main theorems.

2. Preliminaries

In this section, we prepare some notations and results for association schemes and p -schemes that will be used throughout the paper. We refer the reader to [5, 7, 10, 11] for more details about association schemes and p -schemes.

2.1. Association schemes

Let V be a nonempty finite set. Let $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ be a set of nonempty binary relations on V that partitions $V \times V$. The pair $\mathcal{C} = (V, \mathcal{R})$ is called an *association scheme* (or shortly a *scheme*) if it satisfies the following conditions:

- 1) $\Delta(V) = \{(v, v) \mid v \in V\} = R_0$.
- 2) For each $R_i \in \mathcal{R}$, $R_i^t := \{(u, v) \mid (v, u) \in R_i\} \in \mathcal{R}$. We denote R_i^t by $R_{i'}$.
- 3) For all $R_i, R_j, R_k \in \mathcal{R}$ there exists an *intersection number* p_{ij}^k such that $p_{ij}^k = |R_i(u) \cap R_{j'}(v)|$ for all $(u, v) \in R_k$, where $R(u) := \{v \in V \mid (u, v) \in R\}$.

The elements of V and \mathcal{R} are called *points* and *basis relations* of \mathcal{C} , respectively.

The numbers $|V|$ and $|\mathcal{R}|$ are called the *degree* and the *rank* of \mathcal{C} and are denoted by $\text{deg}(\mathcal{C})$ and $\text{rk}(\mathcal{C})$, respectively. We define the *valency* of R_i as $d(R_i) = p_{i'i}^0$. We can describe each R_i by its $\{0, 1\}$ -*adjacency matrix* A_i defined by $(A_i)_{uv} = 1$ if $(u, v) \in R_i$, and 0 otherwise. We denote \mathcal{R}^\cup as the set of all unions of the elements of \mathcal{R} .

Let G be a transitive permutation group acting on a set V ; then G acts on $V \times V$ by the componentwise action. An orbit of this action is called an *orbital*. The set of orbitals of G is denoted by $\text{Orb}(G)$. It is well known that $\text{Orb}(G)$ forms an association scheme on V , denoted by $\text{Inv}(G)$. A given scheme \mathcal{C} is said to be *Schurian* if $\mathcal{C} = \text{Inv}(G)$ for some permutation group G .

For a given scheme $\mathcal{C} = (V, \mathcal{R})$, an *equivalence* of \mathcal{C} is an equivalence relation E on V such that E is a union of some basis relations of \mathcal{C} . Denote by $\mathcal{E}(\mathcal{C})$ the set of all equivalences of \mathcal{C} . For each $E \in \mathcal{E}(\mathcal{C})$ denote by $d(E)$ the degree of E , which is defined as the sum of the valencies of all basis relations of \mathcal{C} that lie in E .

Let E be an equivalence of the scheme \mathcal{C} . Denote by V/E the set of equivalence classes modulo E . For any $X, Y \in V/E$ and $R \in \mathcal{R}$ define

$$R_{X,Y} = R \cap (X \times Y), \quad R_X = R_{X,X}.$$

Moreover, we define

$$\begin{aligned} \mathcal{R}_X &= \{R_X \mid R_X \neq \emptyset\}, \\ R_{V/E} &:= \{(X, Y) \in (V/E) \times (V/E) \mid R_{X,Y} \neq \emptyset\}, \\ \mathcal{R}_{X/E} &:= \{R_{V/E} \mid R \in \mathcal{R}\}. \end{aligned}$$

It is well known that

$$\mathcal{C}_{V/E} = (V/E, \mathcal{R}_{V/E})$$

is a scheme and is called the *factor scheme* of \mathcal{C} modulo E . It is also clear that $\mathcal{C}_X = (X, \mathcal{R}_X)$ is a scheme and is called the *restriction of \mathcal{C}* with respect to X .

Let $\mathcal{C} = (V, \mathcal{R})$ be an association scheme, and $E \in \mathcal{E}(\mathcal{C})$. According to [9, Definition 2.2] we make an order on the elements of V as follows: consider an ordering on the elements of V/E and also suppose that for each $X \in V/E$ we have an ordering on X . First, we order the classes of the equivalence E on the elements of V using the ordering of V/E . Then, in each class $X \in V/E$ and for any $u, v \in V$ such that $u, v \in X$, we order u and v exactly in the same way as in X ; in this case, we say that the elements of V are ordered according to the equivalence E .

Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme. The set

$$\mathcal{O}_\emptyset(\mathcal{C}) = \{R \in \mathcal{R} : d(R) = 1\}$$

is called the *thin radical* of \mathcal{C} . We say that \mathcal{C} is a *thin scheme* if $\mathcal{O}_\emptyset(\mathcal{C}) = \mathcal{R}$. It is well known that any thin scheme is Schurian.

The *thin residue* of \mathcal{C} is the smallest equivalence of \mathcal{C} containing basis relations R_k such that $p_{ii'}^k \neq 0$ for some $R_i \in \mathcal{R}$, and denoted by $\mathcal{O}^\emptyset(\mathcal{C})$. From [11, Lemma 4.2.7], we know that the thin residue of \mathcal{C} is the uniquely defined smallest equivalence of \mathcal{C} having a thin quotient scheme.

If there exists a bijection between the point sets of 2 schemes that induces a bijection between their sets of basis relations, then these 2 schemes are *isomorphic*.

For 2 schemes $\mathcal{C} = (V, \mathcal{R})$ and $\mathcal{C}' = (V, \mathcal{R}')$, we define $\mathcal{C} \leq \mathcal{C}'$ if and only if $\mathcal{R}^\cup \subseteq (\mathcal{R}')^\cup$. Then \mathcal{C} is a fusion of \mathcal{C}' and \mathcal{C}' is a fission of \mathcal{C} .

Given 2 schemes $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$, we put

$$\mathcal{R}_1 \wr \mathcal{R}_2 = \{\Delta(V_2) \otimes R : R \in \mathcal{R}_1\} \cup \{S \otimes V_1 \times V_1 : S \in \mathcal{R}_2 \setminus \{\Delta(V_2)\}\}.$$

Define the *wreath product* of \mathcal{C}_1 and \mathcal{C}_2 , denoted $\mathcal{C}_1 \wr \mathcal{C}_2$, as the scheme on $V_1 \times V_2$ with the set of basis relations $\mathcal{R}_1 \wr \mathcal{R}_2$. Moreover, by considering A_0, A_1, \dots, A_d and B_0, B_1, \dots, B_e as the adjacency matrices of basis relations of \mathcal{R}_1 and \mathcal{R}_2 , respectively, the elements of $V_1 \times V_2$ can be ordered such that the adjacency matrices of $\mathcal{C}_1 \wr \mathcal{C}_2$ are given by

$$\begin{aligned} C_0 &= B_0 \otimes A_0, C_1 = B_0 \otimes A_1, \dots, C_d = B_0 \otimes A_d, \\ C_{d+1} &= B_1 \otimes J_{|V_1|}, \dots, C_{d+e} = B_e \otimes J_{|V_1|}, \end{aligned}$$

where J_n is the $n \times n$ matrix whose entries are all equal to 1. It is well known that $\mathcal{C}_1 \wr \mathcal{C}_2$ is Schurian if and only if \mathcal{C}_1 and \mathcal{C}_2 are Schurian. Clearly, the degree of the wreath product of 2 schemes is equal to the product of their degrees.

A scheme \mathcal{C} is called a p -scheme if the cardinality of each basis relation of \mathcal{C} is a power of p , where p is a prime number. This implies that the degree of any p -scheme is a power of p , and so $\mathcal{O}_\vartheta(\mathcal{C})$ is a nontrivial p -group with respect to products of basis relations. Therefore, the order of the thin radical of \mathcal{C} is a power of p . In [8], it was shown that the class of p -schemes is closed with respect to taking quotients and restrictions.

3. Main theorems

In this section, we study the relationship between the degree of the thin residue of a scheme and the valency of its basis relations. Then we prove Theorems 1, 2, and 3.

Lemma 1 *Let \mathcal{C} be an association scheme. Then, for each basis relation R of the scheme \mathcal{C} , we have $d(R) \leq d(\mathcal{O}_\vartheta(\mathcal{C}))$.*

Proof Let $\mathcal{C} = (V, \mathcal{R})$ and $R \in \mathcal{R}$. From [11, Lemma 4.2.7], the scheme $\mathcal{C}_{V/\mathcal{O}_\vartheta(\mathcal{C})}$ is a thin scheme. Thus, $d(R_{V/\mathcal{O}_\vartheta(\mathcal{C})}) = 1$. It follows that for each $X \in V/\mathcal{O}_\vartheta(\mathcal{C})$ there exists exactly one block $Y \in V/\mathcal{O}_\vartheta(\mathcal{C})$, such that $R \cap (X \times Y) \neq \emptyset$. It follows that

$$d(R) \leq |Y| = d(\mathcal{O}_\vartheta(\mathcal{C})),$$

as desired. □

Lemma 2 *Let \mathcal{C} be a scheme that is isomorphic to the wreath product of 2 nontrivial thin schemes, \mathcal{C}_1 and \mathcal{C}_2 . We then have $\mathcal{O}_\vartheta(\mathcal{C}) = \mathcal{O}_\vartheta(\mathcal{C}_1)$. Moreover,*

$$d(\mathcal{O}_\vartheta(\mathcal{C})) = d(\mathcal{O}_\vartheta(\mathcal{C}_1)) = \deg(\mathcal{C}_1).$$

Proof Let $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$ be 2 nontrivial thin schemes. Let \mathcal{C} be the wreath product of \mathcal{C}_1 and \mathcal{C}_2 . The basis relations of \mathcal{C} are then

$$\mathcal{R} := \{\Delta(V_2) \otimes R : R \in \mathcal{R}_1\} \cup \{S \otimes V_1 \times V_1 : S \in \mathcal{R}_2 \setminus \{\Delta(V_2)\}\}.$$

Let $R \in \mathcal{R}_1$. Then

$$d(\Delta(V_2) \otimes R) = d(R). \tag{1}$$

On the other hand, for each $S \in \mathcal{R}_2 \setminus \{\Delta(V_2)\}$, we have

$$d(S \otimes (V_1 \times V_1)) = \deg(\mathcal{C}_1)d(S). \tag{2}$$

Now, since \mathcal{C}_1 is a nontrivial thin scheme, all basis relations of \mathcal{C}_1 are of degree 1, and we also have $\deg(\mathcal{C}_1) > 1$. Thus, from (1) and (2) we conclude that

$$\mathcal{O}_\vartheta(\mathcal{C}) = \{\Delta(V_2) \otimes R : R \in \mathcal{R}_1\}.$$

It follows that $d(\mathcal{O}_\vartheta(\mathcal{C})) = \deg(\mathcal{C}_1)$.

To complete the proof of the lemma, it is enough to show that

$$\mathcal{O}^\vartheta(\mathcal{C}) = \{\Delta(V_2) \otimes R : R \in \mathcal{R}_1\}. \tag{3}$$

Let $R \in \mathcal{R}_1$. Then, for each $S \in \mathcal{R}_2$, we have

$$\Delta(V_2) \otimes R \in \widehat{S}^t \widehat{S},$$

where $\widehat{S} = S \otimes (V_1 \times V_1)$. From the definition of $\mathcal{O}^\vartheta(\mathcal{C})$ we get

$$\Delta(V_2) \otimes R \in \mathcal{O}^\vartheta(\mathcal{C}). \tag{4}$$

On the other hand, since \mathcal{C}_2 is a nontrivial scheme, we have $\deg(\mathcal{C}_2) > 1$. Let $S \in \mathcal{R}_2 \setminus \{\Delta(V_2)\}$. Then we have

$$S \otimes (V_1 \times V_1) \notin \mathcal{O}^\vartheta(\mathcal{C}). \tag{5}$$

Otherwise, $S \in \mathcal{O}^\vartheta(\mathcal{C}_2)$. Since \mathcal{C}_2 is a thin scheme, it follows that $\mathcal{O}^\vartheta(\mathcal{C}_2) = \{\Delta(V_2)\}$. Thus, $S = \{\Delta(V_2)\}$, which is a contradiction. Therefore, from (4) and (5) we get (3), as desired. \square

For a given scheme \mathcal{C} , it is not necessary to have the equality $\mathcal{O}^\vartheta(\mathcal{C}) = \mathcal{O}_\vartheta(\mathcal{C})$ even if $d(\mathcal{O}^\vartheta(\mathcal{C})) = d(\mathcal{O}_\vartheta(\mathcal{C}))$. Considering this condition, we show that \mathcal{C} is isomorphic to a fission of the wreath product of 2 thin schemes.

To prove the main theorems, we need the following remarks:

Remark 1 *Since each thin scheme is Schurian, using the corresponding permutation group, one can choose an ordering in the set of points of this scheme.*

Remark 2 *Let \mathcal{C} be a scheme on V and $E \in \mathcal{E}(\mathcal{C})$. If the scheme \mathcal{C}_X is a thin scheme for $X \in V/E$, then from separability of thin schemes (see [4, Theorem 2.1] and [9, p. 1908]), for any $X, Y \in V/E$ the schemes \mathcal{C}_X and \mathcal{C}_Y are isomorphic.*

Remark 3 *Given a bijection φ from a set V to a set W , and a binary relation R on the set V , we will let R^φ denote the induced binary relation on the set W .*

Proof of Theorem 1

Let \mathcal{C} be a scheme on V with the set of basis relations \mathcal{R} . By assumption, suppose that $\mathcal{O}^\vartheta(\mathcal{C}) = \mathcal{O}_\vartheta(\mathcal{C})$. If the thin radical and the thin residue are both trivial, then the scheme itself must be trivial, and the proof is complete. Thus, we assume that $|\mathcal{O}_\vartheta(\mathcal{C})| > 1$.

Clearly, $\mathcal{O}^\vartheta(\mathcal{C})$ is an equivalence of \mathcal{C} ; thus, for each $X \in V/\mathcal{O}^\vartheta(\mathcal{C})$, the scheme \mathcal{C}_X is a thin scheme of degree $|X|$. Moreover, the scheme $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$ is a thin scheme of degree $|V|/|X|$. Let $V/\mathcal{O}^\vartheta(\mathcal{C}) = \{X_1, \dots, X_{|V|/|X|}\}$ be the set of points of the scheme $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$. Thus, by Remark 1, we have an ordering on $V/\mathcal{O}^\vartheta(\mathcal{C})$; also, for each $X_i \in V/\mathcal{O}^\vartheta(\mathcal{C})$, $1 \leq i \leq |V|/|X|$, we have an ordering on X_i . On the other hand, by Remark 2, for $X_i, X_j \in V/\mathcal{O}^\vartheta(\mathcal{C})$, $1 \leq i, j \leq |V|/|X|$, the schemes \mathcal{C}_{X_i} and \mathcal{C}_{X_j} are isomorphic, and so there is a bijection between the elements of X_i and X_j . Therefore, we can assume the corresponding order of X_1 for each X_i , $1 \leq i \leq |V|/|X|$.

Now, suppose that the elements of the set V are ordered according to the equivalence $\mathcal{O}^\vartheta(\mathcal{C})$. Thus, we have a bijection, φ , from V to the Cartesian product of the sets X_1 and $V/\mathcal{O}^\vartheta(\mathcal{C})$.

We claim that \mathcal{C} is isomorphic to a fission of the wreath product of \mathcal{C}_{X_1} and $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$. Suppose that $\mathcal{C}_{X_1} \wr \mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})} = (X_1 \times V/\mathcal{O}^\vartheta(\mathcal{C}), \mathcal{S})$. It is sufficient to show that φ induces a bijection such that each element of \mathcal{S} corresponds to a union of some elements of \mathcal{R} .

Let $S \in \mathcal{S}$ and $d(S) = 1$. Then, since $|X_1| > 1$, it follows from the definition of the wreath product that $S \cap (X_1 \times X_1) \neq \emptyset$. Thus, there is a relation $R \in \mathcal{R}$ such that

$$S = R^\varphi. \tag{6}$$

Now let $S \in \mathcal{S}$ and $d(S) \neq 1$. We show that S corresponds to a union of some basis relations of \mathcal{R} . Let \sim be the equivalence relation on \mathcal{R} induced by the canonical map to $\mathcal{R}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$. For each $R \in \mathcal{R}$, define $\hat{R} := R_{V/\mathcal{O}^\vartheta(\mathcal{C})} \in \mathcal{R}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$. Then \hat{R} corresponds to a permutation $g_{\hat{R}}$ on the set $V/\mathcal{O}^\vartheta(\mathcal{C})$ as follows:

$$X_i^{g_{\hat{R}}} = X_j \iff R_{X_i, X_j} \neq \emptyset.$$

Thus, for each $R, T \in \mathcal{R}$, we have $R \sim T$ if and only if $\hat{R} = \hat{T}$. Let $[R]$ be the equivalence class of R . Then $T \in [R]$ if and only if $g_{\hat{T}} = g_{\hat{R}}$. Suppose that $[R_0]$ contains the diagonal relation. Let $X_i, X_j \in V/\mathcal{O}^\vartheta(\mathcal{C})$, considering $[R_k]$ such that $R_k \cap (X_i \times X_j) \neq \emptyset$. For each $(x, y) \in X_i \times X_j$, there exists a basis relation $R \in \mathcal{R}$ such that $(x, y) \in R$, so $X_i^{g_{\hat{R}}} = X_j$. Therefore, $R \in [R_k]$. This shows that $X_i \times X_j \subseteq \bigcup_{R \in [R_k]} R^\varphi$. It then follows that the union of relations R such that $R \in [R_k]$ corresponds to the union of $X_i \times X_j$ such that $X_i^{g_{\hat{R}}} = X_j$. This implies that

$$\bigcup_{R \in [R_k]} R^\varphi = P_{g_{\hat{R}_k}} \otimes J_{|X_1|}, \tag{7}$$

where $P_{g_{\hat{R}_k}}$ is a permutation matrix corresponding to $g_{\hat{R}_k}$.

From Eq. (7) we conclude that for each $S \in \mathcal{S}$ there exists an equivalence class $[R_k]$ such that

$$S = \bigcup_{R \in [R_k]} R^\varphi. \tag{8}$$

This completes the proof of the theorem. □

Proof of Theorem 2.

We first prove the necessity condition of the theorem. Let $\mathcal{C} = (V, \mathcal{R})$ be a p -scheme and $|V| = p^n$, such that

$$d(\mathcal{O}^\vartheta(\mathcal{C})) = d(\mathcal{O}_\vartheta(\mathcal{C})) = p. \tag{9}$$

By Lemma 1, for each basis relation $R \in \mathcal{R}$ we have $d(R) \leq d(\mathcal{O}^\vartheta(\mathcal{C})) = p$. Moreover, since \mathcal{C} is a p -scheme, the valency of each basis relation of \mathcal{C} is a power of p . It follows that, for each basis relation R in \mathcal{R} , we have

$$d(R) \in \{1, p\}. \tag{10}$$

Now we claim that

$$\mathcal{O}^\vartheta(\mathcal{C}) = \mathcal{O}_\vartheta(\mathcal{C}). \tag{11}$$

Indeed, since $\Delta(V) \in \mathcal{O}^\theta(\mathcal{C})$ and $d(\mathcal{O}^\theta(\mathcal{C})) = \sum_{R \in \mathcal{O}^\theta(\mathcal{C})} d(R)$, we get

$$d(\mathcal{O}^\theta(\mathcal{C})) = 1 + \sum_{R \in \mathcal{O}^\theta(\mathcal{C}) \setminus \Delta(V)} d(R).$$

From (9), we have

$$1 + \sum_{R \in \mathcal{O}^\theta(\mathcal{C}) \setminus \Delta(V)} d(R) = p.$$

Now from (10) it is clear that the valency of each basis relation of \mathcal{C} is equal to 1 or p . It follows that the basis relation R belongs to $\mathcal{O}^\theta(\mathcal{C})$ if and only if $d(R) = 1$. Thus, Eq. (11) holds as claimed.

On the other hand, $|V| = p^n$ and $d(\mathcal{O}^\theta(\mathcal{C})) = p$ imply that $|V/\mathcal{O}^\theta(\mathcal{C})| = p^{n-1}$. Let $V/\mathcal{O}^\theta(\mathcal{C}) = \{X_0, X_1, \dots, X_{p^{n-1}-1}\}$. Thus, by Remark 1, we have an ordering on $V/\mathcal{O}^\theta(\mathcal{C})$; also, for each $X_i \in V/\mathcal{O}^\theta(\mathcal{C})$, $0 \leq i \leq p^{n-1} - 1$, we have an ordering on X_i . On the other hand, by Remark 2, for $X_i, X_j \in V/\mathcal{O}^\theta(\mathcal{C})$, $0 \leq i, j \leq p^{n-1} - 1$, the schemes \mathcal{C}_{X_i} and \mathcal{C}_{X_j} are isomorphic, and so there is a bijection between the elements of X_i and X_j . Therefore, we can assume the corresponding order of X_0 for each X_i , $0 \leq i \leq p^{n-1} - 1$. Thus, we may assume that the elements of V are ordered according to the equivalence $\mathcal{O}^\theta(\mathcal{C})$.

Thus, we have a bijection, φ , from V to the Cartesian product of the sets X_0 and $V/\mathcal{O}^\theta(\mathcal{C})$. Now we claim that \mathcal{C} is isomorphic to the wreath product of \mathcal{C}_{X_0} and $\mathcal{C}_{V/\mathcal{O}^\theta(\mathcal{C})}$.

Suppose that $R \in \mathcal{O}^\theta(\mathcal{C})$. From (11), the scheme \mathcal{C}_{X_i} is a thin scheme of degree p for each $0 \leq i \leq p^{n-1} - 1$, and so it is isomorphic to T_p . Let A_R and $A_{R_{X_0}}$ be the adjacency matrices of R and R_{X_0} , respectively. Then we have $A_R = I_{p^{n-1}} \otimes A_{R_{X_0}}$. It follows that

$$R^\varphi = \Delta(V/\mathcal{O}^\theta(\mathcal{C})) \otimes R_{X_0}. \tag{12}$$

Now suppose that $R \notin \mathcal{O}^\theta(\mathcal{C})$. From (10) and (11) we conclude that $d(R) = p$. On the other hand, the scheme $\mathcal{C}_{V/\mathcal{O}^\theta(\mathcal{C})}$ is a thin scheme. Thus, we have

$$d(R_{V/\mathcal{O}^\theta(\mathcal{C})}) = 1. \tag{13}$$

Moreover, since $R \notin \mathcal{O}^\theta(\mathcal{C})$, we have

$$R_{V/\mathcal{O}^\theta(\mathcal{C})} \neq \Delta(V/\mathcal{O}^\theta(\mathcal{C})). \tag{14}$$

Let $X_i \in V/\mathcal{O}^\theta(\mathcal{C})$. From (13), there exists exactly one element $X_j \in V/\mathcal{O}^\theta(\mathcal{C})$, such that

$$R \cap (X_i \times X_j) \neq \emptyset. \tag{15}$$

It is a well-known fact that all classes of an equivalence relation of an association scheme have the same size; thus, we get $|X_i| = |X_j| = d(\mathcal{O}^\theta(\mathcal{C}))$. From (9) we have

$$|X_i| = |X_j| = p. \tag{16}$$

On the other hand, since $d(R) = p$, from (15) and (16) we conclude that

$$X_i \times X_j \subseteq R. \tag{17}$$

Moreover, from (14) we have $X_i \neq X_j$. Let A_R and $\overline{A_R}$ be the adjacency matrices of R and $R_{V/\mathcal{O}^\vartheta(\mathcal{C})}$, respectively. Then, from (17) and by the above ordering on V , we obtain $A_R = \overline{A_R} \otimes J_p$. Thus, we have

$$R^\varphi = R_{V/\mathcal{O}^\vartheta(\mathcal{C})} \otimes (X_0 \times X_0). \quad (18)$$

Therefore, from (12) and (18) we conclude that \mathcal{C} is isomorphic to the wreath product of \mathcal{C}_{X_0} and $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$. On the other hand, \mathcal{C}_{X_0} is isomorphic to T_p and the scheme $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$ is a thin scheme on p^{n-1} points. Thus,

$$\mathcal{C} \cong T_p \wr \mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}.$$

Conversely, let $T_p = (V_1, \mathcal{R}_1)$ and $\mathcal{C}' = (V_2, \mathcal{R}_2)$ be a thin scheme of degree p^{n-1} . Let \mathcal{C} be the wreath product of T_p and \mathcal{C}' . From Lemma 2, we have

$$d(\mathcal{O}^\vartheta(\mathcal{C})) = d(\mathcal{O}_\vartheta(\mathcal{C})) = \deg(T_p).$$

Since T_p is a thin p -scheme on p points, we have $\deg(T_p) = p$, and the proof is complete. \square

Since each thin scheme is Schurian and the wreath product of 2 Schurian schemes is Schurian, the following corollary is a direct consequence of Theorem 2:

Corollary 1 *Any p -scheme whose degrees of thin radical and thin residue are equal to p is Schurian.*

Proof of Theorem 3.

We first assume that $\mathcal{C} = (V, \mathcal{R})$ is a p -scheme and $|V| = p^n$, and that

$$d(\mathcal{O}^\vartheta(\mathcal{C})) = d(\mathcal{O}_\vartheta(\mathcal{C})) = p^{n-1}, \quad (19)$$

and for each $R \in \mathcal{R}$ we have $d(R) \in \{1, p^{n-1}\}$. Since $\Delta(V) \in \mathcal{O}^\vartheta(\mathcal{C})$ and $d(\mathcal{O}^\vartheta(\mathcal{C})) = \sum_{R \in \mathcal{O}^\vartheta(\mathcal{C})} d(R)$, it is easy to check that $R \in \mathcal{O}^\vartheta(\mathcal{C})$ if and only if $d(R) = 1$. Hence,

$$\mathcal{O}^\vartheta(\mathcal{C}) = \mathcal{O}_\vartheta(\mathcal{C}). \quad (20)$$

By the same argument as the proof of Theorem 2, we have a bijection, φ , from V to the Cartesian product of the sets X_0 and $V/\mathcal{O}^\vartheta(\mathcal{C})$, where $V/\mathcal{O}^\vartheta(\mathcal{C}) = \{X_0, X_1, \dots, X_{p-1}\}$.

From (20), it follows that $R \notin \mathcal{O}^\vartheta(\mathcal{C})$ if and only if $d(R) = p^{n-1}$. For such a basis relation R , the valency of $R_{V/\mathcal{O}^\vartheta(\mathcal{C})}$ is equal to 1, because $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$ is a thin scheme. Moreover, from (20) the relation $R_{V/\mathcal{O}^\vartheta(\mathcal{C})}$ is a nondiagonal basis relation. Thus, for each $X_i \in V/\mathcal{O}^\vartheta(\mathcal{C})$ there exists exactly one element $X_j \in V/\mathcal{O}^\vartheta(\mathcal{C})$, $i \neq j$, such that $R \cap (X_i \times X_j) \neq \emptyset$. Since $|X_i| = |X_j| = p^{n-1}$ and $d(R) = p^{n-1}$, we conclude that $X_i \times X_j \subseteq R$. Thus, we have

$$R^\varphi = R_{V/\mathcal{O}^\vartheta(\mathcal{C})} \otimes (X_0 \times X_0). \quad (21)$$

Now suppose $R \in \mathcal{O}^\vartheta(\mathcal{C})$. From (20), the scheme \mathcal{C}_{X_i} is a thin scheme for each $0 \leq i \leq p-1$. Moreover, for each i and j the scheme \mathcal{C}_{X_i} is isomorphic to \mathcal{C}_{X_j} . It follows that

$$R^\varphi = \Delta(V/\mathcal{O}^\vartheta(\mathcal{C})) \otimes R_{X_0}. \quad (22)$$

Therefore, from (21) and (22) we conclude that

$$\mathcal{C} \cong \mathcal{C}_{X_0} \wr \mathcal{C}_{V/\mathcal{O}^\theta(\mathcal{C})},$$

where \mathcal{C}_{X_0} is a thin scheme on p^{n-1} points and $\mathcal{C}_{V/\mathcal{O}^\theta(\mathcal{C})}$ is a thin scheme on p points. Since any thin scheme of degree p is uniquely isomorphic to T_p :

$$\mathcal{C} \cong \mathcal{C}_{X_0} \wr T_p.$$

Conversely, let $\mathcal{C}' = (V_1, \mathcal{R}_1)$ be a thin scheme of degree p^{n-1} and $T_p = (V_2, \mathcal{R}_2)$. Define $\mathcal{C} = \mathcal{C}' \wr T_p$. From Lemma 2, we have

$$d(\mathcal{O}^\theta(\mathcal{C})) = d(\mathcal{O}_{\mathcal{O}^\theta(\mathcal{C})}) = \deg(\mathcal{C}') = p^{n-1}.$$

Now let R be a basis relation of \mathcal{C} . Then, from (1) and (2) in the proof of Lemma 2, we have $d(R) = 1$ or $d(R) = \deg(\mathcal{C}') = p^{n-1}$. This completes the proof of the theorem. \square

The following corollary is a direct consequence of Theorem 3:

Corollary 2 *Any p -scheme of degree p^n whose degrees of thin radical and thin residue are equal to p^{n-1} and the valency of each basis relation is either 1 or p^{n-1} is Schurian.*

Note that if \mathcal{C} is a p -scheme of degree p^n whose degrees of thin radical and thin residue are equal to p^i , $1 < i < n - 1$, then \mathcal{C} is not necessarily Schurian. For example, consider the scheme of degree 16, No. 173 in Hanaki's classification of association schemes (<http://math.shinshu-u.ac.jp/hanaki/as/>). The degree of the thin radical and the thin residue of this 2-scheme is 4, but it is not Schurian. One can study some conditions on basis relations of such p -schemes to ensure that these be Schurian.

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