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## The Ext-strongly Gorenstein projective modules

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**Abstract:** In this paper, we introduce and study Ext-strongly Gorenstein projective modules. We prove that the class of Ext-strongly Gorenstein projective modules is projective resolving. Moreover, we consider Ext-strongly Gorenstein projective precovers.

**Key words:** Strongly Gorenstein projective modules, precovers, Ext-strongly Gorenstein projective modules, projectively resolving

### 1. Introduction

Throughout this paper, all rings considered are associative with identity 1 unless otherwise specified and all modules will be unitary. By  $\mathcal{P}(R)$  we denote the classes of all projective  $R$ -modules. For an  $R$ -module  $M$ , we use  $pd_R(M)$  to denote the usual projective dimensions of  $M$ .

Recall that a class  $\mathcal{X}$  of  $R$ -modules is called resolving if  $\mathcal{P}(R) \subseteq \mathcal{X}$  and for every short exact sequence  $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$  with  $X'' \in \mathcal{X}$  the conditions  $X' \in \mathcal{X}$  and  $X \in \mathcal{X}$  are equivalent. The notion of a resolving class was introduced by Auslander and Bridger [1] in the studying of G- dimension zero modules in 1969. The G-dimension has strong parallels to the projective dimension. Enochs and Jenda [9,10] extended the ideas of Auslander and Bridger, and introduced Gorenstein projective dimensions, which have all been studied extensively by their founders and by Avramov, Christensen, Foxby, Frankild, Holm, Martsinkovsky, and Xu among others [2,7,8,11–13,15] over arbitrary associative rings. Bennis and Mahdou [4] studied a particular case of Gorenstein projective modules, which they call strongly Gorenstein projective (*SG*-projective for short) modules. They proved that every Gorenstein projective module is a direct summand of a strongly Gorenstein projective module in [4]. Using the characterizations of strongly Gorenstein projective modules they discussed the global Gorenstein projective dimension of a ring  $R$  in [5]. In general, the strongly Gorenstein projective  $R$ -modules are not closed under extensions and so the class of strongly Gorenstein projective  $R$ -modules is not projectively resolving. We know that  $\{\text{projective modules}\} \subseteq \{\text{strongly Gorenstein projective modules}\}$ . A natural question is whether or not there exists a projectively resolving class between the class of strongly Gorenstein projective modules and the class of projective modules? In this paper, we introduce and study an intermediate projectively resolving class, which is called the class of Ext-strongly Gorenstein (*Ext-SG* for short) projective modules. We obtain a sufficient and necessary condition under which *SG*-projective modules are closed under extensions relative to the *Ext-SG* projective modules. Moreover, we discuss Ext-strongly Gorenstein projective precovers.

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An  $R$ -module  $M$  is said to be Gorenstein projective if there exists an exact sequence of projective modules

$$\mathfrak{P} = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

such that  $M \cong \text{Im}(P_0 \rightarrow P^0)$  and such that  $\text{Hom}_R(\mathfrak{P}, Q)$  is exact whenever  $Q$  is a projective  $R$ -module. The exact sequence  $\mathfrak{P}$  is called a complete projective resolution. The class of all Gorenstein projective  $R$ -modules is denoted by  $\mathcal{GP}(R)$ . An  $R$ -module  $M$  is called strongly Gorenstein projective (SG-projective for short) if there exists a complete projective resolution of the form

$$\mathfrak{P} = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that  $M \cong \text{Ker}f$ . Every projective module is strongly Gorenstein projective and every strongly Gorenstein projective module is Gorenstein projective. The class of all strongly Gorenstein projective  $R$ -modules is denoted by  $\mathcal{SGP}(R)$ .

Recall that a class  $\mathcal{X}$  of  $R$ -modules is called projectively resolving if  $\mathcal{P}(R) \subseteq \mathcal{X}$  and for every short exact sequence  $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$  with  $X'' \in \mathcal{X}$  the conditions  $X' \in \mathcal{X}$  and  $X \in \mathcal{X}$  are equivalent. It was shown in [4, Theorem 2.7] that a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module. By [4, Example 2.13] and [6, Example 2.6],  $\{\text{SG-projective modules}\} \subsetneq \{\text{G-projective modules}\}$ . Hence direct summands of a strongly Gorenstein projective module need not be strongly Gorenstein projective and the class  $\mathcal{SGP}(R)$  is not projectively resolving (see [13, Proposition 1.4]). Let

$$\text{Ext}(M, N) = \{Y \mid \text{There is an exact sequence } 0 \rightarrow N \rightarrow Y \rightarrow M \rightarrow 0\}.$$

For a class  $\mathcal{X}$  of  $R$ -modules, define

$$\text{Ext}(M, \mathcal{X}) = \bigcup_{N \in \mathcal{X}} \text{Ext}(M, N) \quad (\text{Ext}(\mathcal{X}, N) = \bigcup_{M \in \mathcal{X}} \text{Ext}(M, N)).$$

**Definition 1** Let  $M \in \mathcal{SGP}(R)$ .  $M$  is called a *lSG* (resp., *rSG*)-projective  $R$ -module if  $\text{Ext}(M, \mathcal{SGP}(R)) \subseteq \mathcal{SGP}(R)$  (resp.,  $\text{Ext}(\mathcal{SGP}(R), M) \subseteq \mathcal{SGP}(R)$ ).

The class of all *lSG* (resp., *rSG*)-projective  $R$ -modules is denoted by  $\mathcal{lSGP}(R)$  (resp.,  $\mathcal{rSGP}(R)$ ). If  $M \in \mathcal{lSGP}(R)$  or  $M \in \mathcal{rSGP}(R)$ , then we say that  $M$  is an *Ext-strongly Gorenstein projective*  $R$ -module.

The following result can be found in [16], but now we give a simple proof.

**Lemma 2** ([16, Theorem 2.1]). Let  $0 \rightarrow K \rightarrow M \rightarrow P \rightarrow 0$  be an exact sequence of  $R$ -modules with  $P$  projective. Then  $K \in \mathcal{SGP}(R)$  if and only if  $M \in \mathcal{SGP}(R)$ .

**Proof** “ $\Rightarrow$ ” follows from the fact that  $\mathcal{SGP}(R)$  is closed under direct sums, because  $M \cong K \oplus P$ .

“ $\Leftarrow$ ” Suppose  $M \in \mathcal{SGP}(R)$ . The isomorphism  $M \cong K \oplus P$  implies that there exists an exact sequence  $0 \rightarrow K \oplus P \rightarrow Q \rightarrow K \oplus P \rightarrow 0$  with  $Q$  projective. Note that  $K \oplus P$  is Gorenstein projective; hence  $\text{Ext}^1(K \oplus P, P) = 0$  by [13, Theorem 2.20]. Therefore, we have the following  $3 \times 3$  commutative diagram with

all rows and all columns being exact:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & Q_0 & \longrightarrow & K \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K \oplus P & \longrightarrow & Q & \longrightarrow & K \oplus P \longrightarrow 0 \\
 & & \downarrow \pi_P & & \downarrow & & \downarrow \pi_P \\
 0 & \longrightarrow & P & \longrightarrow & P \oplus P & \longrightarrow & P \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

The middle column gives that  $Q_0$  is projective. From the top row, there is an exact sequence  $0 \rightarrow K \rightarrow Q_0 \rightarrow K \rightarrow 0$  with  $Q_0$  projective. Clearly,  $K$  is Gorenstein projective, then for each projective  $R$ -module  $T$ ,  $Ext^{i \geq 1}(K, T) = 0$  by [13, Theorem 2.20]. Thus  $K \in \mathcal{SGP}(R)$  by [4, Proposition 2.9].  $\square$

**Proposition 3** *Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$  be an exact sequence of  $R$ -modules with  $M \in \mathcal{SGP}(R)$ .*

- (1) *If  $M_2 \in \mathcal{LSGP}(R)$ , then  $M_1 \in \mathcal{SGP}(R)$ .*
- (2) *If  $M \in \mathcal{rSGP}(R)$  and  $M_2 \in \mathcal{SGP}(R)$ , then  $M_1 \in \mathcal{SGP}(R)$ .*

**Proof** (1) Since  $M \in \mathcal{SGP}(R)$ , there is an exact sequence  $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$  with  $P$  projective. Consider the pullback of  $M_2 \rightarrow M$  and  $P \rightarrow M$ :

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & M & \xlongequal{\quad} & M \\
 & & & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M_1 & \longrightarrow & T & \longrightarrow & P \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

By the middle column,  $T \in \mathcal{SGP}(R)$ . Using Lemma 2 in the middle row, we have  $M_1 \in \mathcal{SGP}(R)$ .

- (2) The proof is similar to that of (1).  $\square$

**Theorem 4** *Given a ring  $R$ ,  $\mathcal{LSGP}(R)$  and  $\mathcal{rSGP}(R)$  are projectively resolving.*

**Proof** We only show that  $lSGP(R)$  is projectively resolving. Using the same method, one can prove that  $rSGP(R)$  is also projectively resolving. By Lemma 2,  $\mathcal{P}(R) \subseteq lSGP(R)$ . Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$  be an exact sequence of  $R$ -modules with  $M \in lSGP(R)$ . Next we shall prove that  $M_1 \in lSGP(R)$  if and only if  $M_2 \in lSGP(R)$ . Assume  $M_1 \in lSGP(R)$ . Then  $M_2 \in SGP(R)$  as  $M \in lSGP(R)$ . Let  $0 \rightarrow K \rightarrow W \rightarrow M_2 \rightarrow 0$  be any exact sequence with  $K \in SGP(R)$ . Consider the pullback of  $W \rightarrow M_2$  and  $M_1 \rightarrow M_2$ :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & K & \xlongequal{\quad} & K & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & T & \longrightarrow & W & \longrightarrow & M \longrightarrow 0. \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

By the first column,  $T \in SGP(R)$  since  $M_1 \in lSGP(R)$  and  $K \in SGP(R)$ . From the middle row, we have  $W \in SGP(R)$  as  $M \in lSGP(R)$  and  $T \in SGP(R)$ . Therefore  $M_2 \in lSGP(R)$ .

Conversely, suppose  $M_2 \in lSGP(R)$ . Since  $M \in lSGP(R)$ , there is an exact sequence  $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$  with  $P$  projective. Consider the pullback of  $M_2 \rightarrow M$  and  $P \rightarrow M$ :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & M & \xlongequal{\quad} & M & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M_1 & \longrightarrow & G & \longrightarrow & P \longrightarrow 0. \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Since  $M, M_2 \in lSGP(R)$ , by the above proof,  $G \in lSGP(R)$ . From the middle row, we obtain the exact sequence  $0 \rightarrow M_1 \rightarrow G \rightarrow P \rightarrow 0$  with  $P$  projective. By Lemma 2,  $M_1 \in SGP(R)$ . For any exact sequence  $0 \rightarrow M'_1 \rightarrow L \rightarrow M_1 \rightarrow 0$  with  $M'_1 \in SGP(R)$ , we have the following commutative diagram with exact rows and exact columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & M'_1 & \xlongequal{\quad} & M'_1 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & L & \longrightarrow & L \oplus P & \longrightarrow & P \longrightarrow 0. \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & M_1 & \longrightarrow & G & \longrightarrow & P \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

From the middle column,  $L \oplus P \in \mathcal{SGP}(R)$  since  $G \in \mathcal{LSGP}(R)$  and  $M'_1 \in \mathcal{SGP}(R)$ . The middle row gives  $L \in \mathcal{SGP}(R)$  by Lemma 2; i.e.  $\text{Ext}(M_1, \mathcal{SGP}(R)) \subseteq \mathcal{SGP}(R)$ . Therefore  $M_1 \in \mathcal{LSGP}(R)$ . Thus  $\mathcal{LSGP}(R)$  is projectively resolving.  $\square$

**Proposition 5** *Given an exact sequence of  $R$ -modules  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$  with  $M \in \mathcal{GP}(R)$ , then*

- (1) *If  $M_1 \in \mathcal{LSGP}(R)$  and  $M_2 \in \mathcal{SGP}(R)$  (or  $M_1 \in \mathcal{SGP}(R)$  and  $M_2 \in \mathcal{rSGP}(R)$ ), then  $M \in \mathcal{SGP}(R)$ .*
- (2) *If  $M_1, M_2 \in \mathcal{LSGP}(R)$  (resp.,  $\mathcal{rSGP}(R)$ ), then  $M \in \mathcal{LSGP}(R)$  (resp.,  $\mathcal{rSGP}(R)$ ).*

**Proof** (1) Suppose that  $M_1 \in \mathcal{LSGP}(R)$ ,  $M_2 \in \mathcal{SGP}(R)$ . Then there is an exact sequence  $0 \rightarrow M_1 \rightarrow P \rightarrow M_1 \rightarrow 0$  with  $P$  projective. Consider the pushout of  $M_1 \rightarrow M_2$  and  $M_1 \rightarrow P$ :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & P & \longrightarrow & T & \longrightarrow & M \longrightarrow 0. \tag{\dagger} \\
 & & \downarrow & & \downarrow & & \\
 & & M_1 & \xlongequal{\quad} & M_1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

By the middle column,  $T \in \mathcal{SGP}(R)$ . Since  $M$  is Gorenstein projective,  $\text{Ext}_R^1(M, P) = 0$ . Then the middle row is split, i.e.  $T \cong M \oplus P$ . Using Lemma 2, we have  $M \in \mathcal{SGP}(R)$ .

(2) Since  $\mathcal{LSGP}(R) \subseteq \mathcal{SGP}(R)$ , we also obtain the diagram  $(\dagger)$  and  $T \cong M \oplus P$  by the proof of (1). Then there is an exact sequence :

$$0 \rightarrow M \rightarrow T \rightarrow P \rightarrow 0. \tag{*}$$

From the middle column of the diagram  $(\dagger)$ , we obtain that  $T \in \mathcal{LSGP}(R)$  since  $\mathcal{LSGP}(R)$  is projectively resolving by Theorem 4. Note that  $P$  is a projective  $R$ -module; hence, applying Theorem 4 to the exact sequence

(\*), we have  $M \in lSGP(R)$ . □

**Corollary 6** *Let  $M \in \mathcal{GP}(R)$ . If there exists an exact sequence*

$$0 \rightarrow L_n \rightarrow L_{n-1} \rightarrow \cdots \rightarrow L_1 \rightarrow L_0 \rightarrow M \rightarrow 0$$

*with each  $L_i \in lSGP(R)$  (resp.,  $rSGP(R)$ ) ( $0 \leq i \leq n$ ), then  $M \in lSGP(R)$  (resp.,  $rSGP(R)$ ).*

**Proof** Let  $K_i = \ker(L_i \rightarrow L_{i-1})$  where  $0 \leq i \leq n - 2$  and  $L_{-1} = M$ . By [13, Theorem 2.5], each  $K_i$  ( $0 \leq i \leq n - 2$ ) is Gorenstein projective. Applying Proposition 5 to the following short exact sequences

$$0 \rightarrow L_n \rightarrow L_{n-1} \rightarrow K_{n-2} \rightarrow 0,$$

$$0 \rightarrow K_{n-2} \rightarrow L_{n-2} \rightarrow K_{n-3} \rightarrow 0,$$

$$\cdots \cdots \cdots,$$

$$0 \rightarrow K_1 \rightarrow L_1 \rightarrow K_0 \rightarrow 0,$$

we have that each  $K_i$  ( $0 \leq i \leq n - 2$ )  $\in lSGP(R)$  (resp.,  $rSGP(R)$ ). Applying Proposition 5 to the short exact sequence  $0 \rightarrow K_0 \rightarrow L_0 \rightarrow M \rightarrow 0$  again, we also have  $M \in lSGP(R)$  (resp.,  $rSGP(R)$ ). □

For any ring  $R$ , it is clear that there are the following inclusions of classes:

$$\mathcal{P}(R) \subseteq lSGP(R) \subseteq SG\mathcal{P}(R) \subseteq \mathcal{GP}(R).$$

The reverse inclusions are not true in general. [3, Corollaries 3.9 and 3.10] gave examples of rings over which the classes  $SG\mathcal{P}(R)$  and  $\mathcal{GP}(R)$  are the same or different, respectively.

**Example 7** *Recall that a commutative ring  $R$  is called quasi-Frobenius if it is noetherian and self-injective. Consider the quasi-Frobenius local ring  $R = k[X]/(X^2)$  where  $k$  is a field, and denote by  $\overline{X}$  the residue class in  $R$  of  $X$ .*

(1) *The  $R\overline{X}$  is strongly Gorenstein projective and strongly Gorenstein injective, but it is neither projective nor injective.*

(2) *For each  $M \in SG\mathcal{P}(R)$ ,  $Ext_R^1(R\overline{X}, M) = 0$ .*

(3)  *$R\overline{X} \in lSGP(R)$ .*

**Proof** (1) This is [4, Example 2.5].

(2) Let  $M \in SG\mathcal{P}(R)$ . Then there is an exact sequence  $0 \rightarrow M \rightarrow P \xrightarrow{\varphi} M \rightarrow 0$  with  $P$  projective. Since  $R\overline{X}$  is strongly Gorenstein projective, it is also Gorenstein projective and so  $Ext_R^1(R\overline{X}, P) = 0$  by [12, Proposition 2.3]. Using the functor  $Hom_R(R\overline{X}, -)$  to the above exact sequence, we obtain another exact sequence

$$0 \rightarrow Hom_R(R\overline{X}, M) \rightarrow Hom_R(R\overline{X}, P) \xrightarrow{\varphi_*} Hom_R(R\overline{X}, M) \rightarrow Ext_R^1(R\overline{X}, M) \rightarrow 0.$$

Next we shall show that  $\varphi_*$  is surjective. In fact, for any

$$g \in Hom_R(R\overline{X}, M),$$

there exists  $m \in M$  such that  $g(\bar{X}) = m$ . Note that  $\varphi: P \rightarrow M$  is surjective; hence there is  $y \in P$  such that  $\varphi(y) = m$ . Define

$$f: R\bar{X} \rightarrow P \text{ by } r\bar{X} \mapsto ry.$$

for all  $r \in R$ . Clearly,  $f \in \text{Hom}_R(R\bar{X}, P)$  and  $\varphi_*(f) = g$ . Therefore  $\varphi_*$  is surjective. Thus  $\text{Ext}_R^1(R\bar{X}, M) = 0$ .

(3) Let  $M \in \mathcal{SGP}(R)$ . If  $0 \rightarrow M \rightarrow Y \rightarrow R\bar{X} \rightarrow 0$  is any extension of  $R\bar{X}$  by  $M$ , then this exact sequence is split by (2). Therefore  $Y \cong M \oplus R\bar{X}$ . Hence  $Y \in \mathcal{SGP}(R)$  by [4, Proposition 2.2]. Thus  $R\bar{X} \in \mathcal{lSGP}(R)$ .  $\square$

**Example 8** Let  $D$  be a principal ideal domain and  $P$  a nonzero prime ideal of  $D$ . Consider the rings  $R = D/P^2$  and  $S = D/P^3$ . Then  $\mathcal{SGP}(R) = \mathcal{GP}(R)$  and  $\mathcal{SGP}(S) \subsetneq \mathcal{GP}(S)$  (see [14, Example 2.2]).

Moreover, one will see  $\mathcal{lSGP}(S) \subsetneq \mathcal{SGP}(S)$  by the following Theorem 9.

In [14], the authors discussed the equality of the classes  $\mathcal{SGP}(R)$  and  $\mathcal{GP}(R)$  for a ring  $R$ . They proved that  $\mathcal{SGP}(R) = \mathcal{GP}(R)$  if and only if for every  $R$ -module  $M$  with  $\text{Gpd}(M) \leq 1$  ( $\text{Gpd}(M)$  denotes the Gorenstein projective dimension of  $M$ ) there exists a short exact sequence  $0 \rightarrow M \rightarrow Q \rightarrow M \rightarrow 0$  with  $\text{pd}(Q) \leq 1$ . Clearly, if the global dimension of  $R$  is finite then  $\mathcal{SGP}(R) = \mathcal{GP}(R) = \mathcal{P}(R)$ .

**Theorem 9** Let  $R$  be a ring. Then the following statements are equivalent:

- (1)  $\mathcal{lSGP}(R) = \mathcal{SGP}(R)$ .
- (2)  $\mathcal{SGP}(R) = \mathcal{GP}(R)$ .
- (3)  $r\mathcal{SGP}(R) = \mathcal{SGP}(R)$ .
- (4)  $\mathcal{SGP}(R)$  is closed under extensions.
- (5)  $\mathcal{lSGP}(R) \cap r\mathcal{SGP}(R) = \mathcal{SGP}(R)$ .

**Proof** (1)  $\Rightarrow$  (2) Suppose that  $\mathcal{lSGP}(R) = \mathcal{SGP}(R)$ . Then  $\mathcal{SGP}(R)$  is projectively resolving by Theorem 4. From [4, Proposition 2.2] and [13, Proposition 1.4],  $\mathcal{SGP}(R)$  is closed under countable direct sums and direct summands. [4, Theorem 2.7] gives that any Gorenstein projective module is a direct summand of a strongly Gorenstein projective module. Therefore each Gorenstein projective module is strongly Gorenstein projective. Thus  $\mathcal{SGP}(R) = \mathcal{GP}(R)$ .

(2)  $\Rightarrow$  (1) For any  $M \in \mathcal{SGP}(R)$ , since  $\mathcal{GP}(R)$  is projectively resolving,  $\text{Ext}(M, \mathcal{SGP}(R)) = \text{Ext}(M, \mathcal{GP}(R)) \subseteq \mathcal{GP}(R) = \mathcal{SGP}(R)$ . Then  $M \in \mathcal{lSGP}(R)$  by Definition 1 and so  $\mathcal{lSGP}(R) = \mathcal{SGP}(R)$ .

(2)  $\Rightarrow$  (3) The proof is similar to that of (2)  $\Rightarrow$  (1).

(3)  $\Rightarrow$  (2) The proof is similar to that of (1)  $\Rightarrow$  (2).

(1)  $\Rightarrow$  (4) By Theorem 4,  $\mathcal{lSGP}(R)$  is projectively resolving and so  $\mathcal{lSGP}(R)$  is closed under extensions. Thus  $\mathcal{SGP}(R)$  is closed under extensions by (1).

(4)  $\Rightarrow$  (1) by Definition 1.

(5)  $\Rightarrow$  (1) Suppose that  $\mathcal{lSGP}(R) \cap r\mathcal{SGP}(R) = \mathcal{SGP}(R)$ . Then  $\mathcal{SGP}(R) = \mathcal{lSGP}(R) \cap r\mathcal{SGP}(R) \subseteq \mathcal{lSGP}(R) \subseteq \mathcal{SGP}(R)$ . Therefore (1) holds.

(4)  $\Rightarrow$  (5) Suppose that (4) holds. Then  $\mathcal{lSGP}(R) = \mathcal{SGP}(R)$  and  $r\mathcal{SGP}(R) = \mathcal{SGP}(R)$ . Therefore  $\mathcal{lSGP}(R) \cap r\mathcal{SGP}(R) = \mathcal{SGP}(R)$ .  $\square$



Let  $\mathcal{X}$  be a class of  $R$ -modules. We denote by  $\text{res.dim}_{\mathcal{X}}M$  the  $\mathcal{X}$ -resolution dimension of an  $R$ -module  $M$ , which is defined as the minimal number  $n$  such that there exists an exact sequence  $0 \rightarrow X_n \rightarrow \dots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$  where  $X_i \in \mathcal{X}$  and  $i = 1, 2, \dots, n$ . If no such number  $n$  exists, then we set  $\text{res.dim}_{\mathcal{X}}M = \infty$ .

Let  $\mathcal{X}$  be any class of  $R$ -modules, and let  $M$  be an  $R$ -module. An  $\mathcal{X}$ -precover of  $M$  is an  $R$ -homomorphism  $\psi: X \rightarrow M$ , where  $X \in \mathcal{X}$ , and such that the sequence,

$$\text{Hom}_R(X', X) \xrightarrow{\psi_*} \text{Hom}_R(X', M) \longrightarrow 0$$

is exact for every  $X' \in \mathcal{X}$  ( $\mathcal{X}$ -preenvelopes of  $M$  are defined dually). For more details about precovers (and preenvelopes), the reader may consult [11, Chapters 5 and 6] or [15, Chapter 1].

**Theorem 10** *Let  $M$  be an  $R$ -module with  $M \notin \text{lSGP}(R)$ . Then the following statements are equivalent for each integer  $n \geq 1$ :*

- (1)  $\text{res.dim}_{\text{lSGP}(R)}M \leq n$ .
- (2)  $M$  admits a surjective  $\text{lSGP}(R)$ -precover,  $\psi: L \rightarrow M$ , where  $K = \ker\psi$  satisfies  $\text{pd}_R(K) = n - 1$ .
- (3) there exist 2 exact sequences  $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$  and  $0 \rightarrow M \rightarrow H \rightarrow L \rightarrow 0$  with  $L \in \text{lSGP}(R)$ ,  $\text{pd}_R(K) = n - 1$  and  $\text{pd}_R(H) = n$ .

**Proof** (1)  $\Rightarrow$  (2) By Theorem 4 and [17, Proposition 3.1 (1)],  $M$  admits a surjective  $\text{lSGP}(R)$ -precover,  $\psi: L \rightarrow M$ , where  $K = \ker\psi$  satisfies  $\text{pd}_R(K) = n - 1$ .

(2)  $\Rightarrow$  (3) By (2), there exists an exact sequence  $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$  with  $L \in \text{lSGP}(R)$ ,  $\text{pd}_R(K) = n - 1$ . Since  $L \in \text{lSGP}(R)$ , there is a projective  $R$ -module  $P$  such that the sequence  $0 \rightarrow L \rightarrow P \rightarrow L \rightarrow 0$  is exact. Consider the pushout of  $L \rightarrow P$  and  $L \rightarrow M$ :

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & M \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & H \longrightarrow 0. \\ & & & & \downarrow & & \downarrow \\ & & & & L & \longlongequal{\quad} & L \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

By the second row of the above diagram, we have  $\text{pd}_R(H) = n$ . Therefore the right column of the diagram gives the required exact sequence.

(3)  $\Rightarrow$  (1) It is clear. □

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