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A note on infinite groups whose subgroups are close to be normal-by-finite

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Abstract: A group G is said to have the CF-property if the index $|X : X_G|$ is finite for every subgroup X of G. Extending previous results by Buckley, Lennox, Neumann, Smith, and Wiegold, it is proven here that if G is a locally graded group whose proper subgroups have the CF-property, then G is abelian-by-finite, provided that all its periodic sections are locally finite. Groups in which all proper subgroups of infinite rank have the CF-property are also studied.

Key words: Normal-by-finite subgroup, CF-group, group of infinite rank

1. Introduction

A subgroup X of a group G is said to be *normal-by-finite* if the core X_G of X in G has finite index in X. Finite subgroups and normal subgroups of an arbitrary group are obvious examples of normal-by-finite subgroups. A group G is called a *CF-group* if all its subgroups are normal-by-finite. Clearly, Tarski groups (i.e. infinite simple groups whose proper nontrivial subgroups have prime order) have the CF-property, but Buckley et al. [3] proved that any locally finite CF-group contains an abelian subgroup of finite index. A generalization of this result can be found in [7].

Recall that a group G is *locally graded* if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index. Although it is an open question whether an arbitrary locally graded CF-group G is abelian-by-finite, it was shown in [11] that this is actually the case if all periodic homomorphic images of G are locally finite. Groups with this property form a large class, containing in particular every group whose finitely generated subgroups are generalized radical (a group is called *generalized radical* if it has an ascending series whose factors are either locally nilpotent or locally finite).

The class of CF-groups is obviously subgroup-closed, and it is easy to show that there exists a periodic metabelian group that is not a CF-group but in which all proper subgroups have the CF-property. On the other hand, the aim of this paper is to show that the above quoted result can be extended to the case of groups whose proper subgroups have the CF-property. Of course, a group G satisfies this condition if and only if the index $|X : X_Y|$ is finite, whenever $X \leq Y < G$, which means that the subgroup X is close to be normal-by-finite in G. Our first main result is the following.

Theorem A *Let G be a locally graded group whose periodic sections are locally finite. If every proper subgroup of G has the CF-property, then G is abelian-by-finite.*

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It was also proved in [11] that if G is a locally graded group for which there exists a positive integer k such that $|X : X_G| \leq k$ for every subgroup X, then G is abelian-by-finite. Groups with such property are called *BCF-groups*. In particular, all periodic sections of a locally graded BCF-group are locally finite, and hence our main theorem has the following consequence.

Corollary *Let G be a locally graded group whose proper subgroups have the BCF-property. Then G is abelian-by-finite.*

Recall that a group G is said to have *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this property. In recent years a series of papers have been published on the behavior of groups of infinite rank in which every proper subgroup of infinite rank satisfies a suitable property (see, for instance, [4, 5, 6, 8]). Our second main result provides a further contribution to this topic.

Theorem B *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank have the CF-property. Then G is abelian-by-finite.*

It is an open question whether an arbitrary locally graded group of infinite rank must contain a proper subgroup of infinite rank, and this problem seems to be the main obstacle in order to extend Theorem B to the case of locally graded groups.

Most of our notation is standard and can be found in [10].

2. Proofs

If $\mathfrak{X}$ is a class of groups, recall that a group G is *minimal non-$\mathfrak{X}$* if it is not an $\mathfrak{X}$-group but all proper subgroups of G belong to $\mathfrak{X}$. Let A be a group of type p^∞ for some prime number p, and consider the standard wreath product $G = A \wr \langle x \rangle$ of A by a group $\langle x \rangle$ of order 2. Since A is not normal in G, the group G does not have the CF-property. On the other hand, every proper subgroup of G is either finite or abelian or a finite extension of a group of type p^∞, and hence it is a CF-group. Therefore the periodic metabelian group G is minimal non-CF.

Our first lemma describes a general property of CF-groups, which is of independent interest.

Lemma 1 *Let G be a CF-group whose periodic homomorphic images are locally finite. Then the subgroup G'' is locally finite.*

Proof Let a be any element of infinite order of G. As the subgroup $\langle a \rangle$ is normal-by-finite, there exists a positive integer $n(a)$ (depending of course on a) such that the cyclic subgroup $\langle a^{n(a)} \rangle$ is normal in G, and hence

$$[G', a^{n(a)}] = \{1\}.$$

Let A be the subgroup of G generated by the powers $a^{n(a)}$, where a runs over all elements of infinite order of G. Clearly, A is an abelian normal subgroup of G, and the factor group G/A is periodic, and so even locally finite. Moreover, $A \cap G'$ is contained in the center of G', so that $G'/Z(G')$ is locally finite, and it follows from a classical result of Schur that also the subgroup G'' is locally finite (see, for instance, [10] Part 1, Theorem 4.12). □

The structure of groups whose proper subgroups are abelian-by-finite was studied by Bruno and Phillips (see [1, 2]); in particular, they proved that every imperfect minimal non-(abelian-by-finite) group is periodic. For our purposes, we collect the main results of Bruno and Phillips in the next 2 lemmas.

Lemma 2 *Let G be a group whose proper subgroups are abelian-by-finite. If $G \neq G'$, then either G is abelian-by-finite or G/G' is a group of type p^∞ and G' is an abelian q-group, where p and q are prime numbers. Moreover, in the latter case, if $p \neq q$ there exists in G a subgroup P of type p^∞ such that $G = PG'$, and G' is a minimal normal subgroup of G.*

Lemma 3 *Let G be a locally finite p-group whose proper subgroups are abelian-by-finite. Then either G is abelian-by-finite or it is minimal nonhypercentral.*

Proof of Theorem A Assume for a contradiction that the group G is not abelian-by-finite. It follows from the result of Smith and Wiegold [11] that every proper subgroup of G is abelian-by-finite, so that G cannot contain proper subgroups of finite index, and in particular it is not finitely generated. Then every finitely generated subgroup E of G has the CF-property, and so E'' is locally finite by Lemma 1. Therefore, the subgroup G'' is likewise locally finite. On the other hand, an application of Lemma 2 yields that G is either perfect or locally finite, and hence G must be locally finite. It follows that the commutator subgroup G' is properly contained in G (see [1], Proposition 3). By Lemma 2, we have that G/G' is a group of type p^∞ and G' is a q-group, where p and q are prime numbers.

Suppose first that $p \neq q$, so that again Lemma 2 yields that G' is an (infinite abelian) minimal normal subgroup of G, and there exists in G a subgroup P of type p^∞ such that $G = PG'$. Then $Z(G) = C_P(G')$ is finite, and the factor group $G/Z(G)$ has a trivial center. Clearly, $G/Z(G)$ is likewise a counterexample, and hence replacing G by $G/Z(G)$ it can be assumed without loss of generality that $Z(G) = \{1\}$. Since G' is a minimal normal subgroup of G, there exist elements a of G' and y of P such that $\langle a\rangle^y \neq \langle a\rangle$. Let b be any element of G' such that $\langle b\rangle^y = \langle b\rangle$, so that $b^y = b^i$ for some positive integer $i < q$. The set

$$K = \{c \in G' \mid c^y = c^i\}$$

is a proper G-invariant subgroup of G', so that $K = \{1\}$. Thus, $b = 1$, and hence y moves all cyclic nontrivial subgroups of G', which means that

$$\langle u\rangle \cap \langle u\rangle^y = \{1\}$$

for every element u of G'. Application of Dietzmann's lemma and Maschke's theorem yields that there exists in G' an infinite collection $(E_n)_{n\in\mathbb{N}}$ of finite nontrivial $\langle y\rangle$-invariant subgroups such that

$$E = \langle E_n \mid n \in \mathbb{N}\rangle = \operatorname*{Dr}_{n\in\mathbb{N}} E_n.$$

For each positive integer n, let x_n be a nontrivial element of E_n, and consider the infinite subgroup

$$X = \langle x_n \mid n \in \mathbb{N}\rangle = \operatorname*{Dr}_{n\in\mathbb{N}} \langle x_n\rangle.$$

Then $X \cap X^y = \{1\}$, and so in particular the core of X in $\langle y, E\rangle$ is trivial. Therefore, the CF-property does not hold for the proper subgroup $\langle y, E\rangle$ of G, and this contradiction shows that $p = q$, so that G is a p-group.

By Lemma 3, the group G cannot be hypercentral, so that its hypercenter $\bar{Z}(G)$ is a proper subgroup and the factor group $G/\bar{Z}(G)$ is likewise a counterexample. Thus, it can be assumed without loss of generality that $Z(G) = \{1\}$. Suppose that G' contains a subgroup Q of type p^∞, and let H/G' be any proper subgroup of G/G'. Since H has the CF-property, Q is normal in H, and so its unique subgroup Q_1 of order p is normalized by H. On the other hand, G/G' is the join of its proper subgroups, and hence the subgroup Q_1 is normal in G, contradicting the assumption $Z(G) = \{1\}$. Therefore, G' is reduced, and so $G'/(G')^p$ is infinite. Clearly, the factor group $G/(G')^p$ is also a counterexample, and replacing G by $G/(G')^p$ we may also suppose that G' has exponent p. Consider again any proper subgroup H/G' of G/G'. Then H is nilpotent (see [10] Part 2, Lemma 6.34), and hence $Z(H) \cap G'$ is a nontrivial normal subgroup of G. Let V be a subgroup of G' such that

$$G' = \big(Z(H) \cap G'\big) \times V.$$

Since H is a CF-group, the index $|V : V_H|$ is finite. On the other hand, V_H has obviously trivial intersection with $Z(H)$, so that $V_H = \{1\}$ and V is finite. It follows that $Z(H) \cap G'$ has finite index in G', so that also the index $|H : Z(H)|$ is finite and hence the commutator subgroup H' of H is finite by the already quoted theorem of Schur. However, G has no finite nontrivial normal subgroups, so that $H' = \{1\}$ and H is abelian. As

$$G = \bigcup_{G' \leq H < G} H,$$

it follows that G itself is abelian, and this last contradiction completes the proof of the theorem. □

Groups in which all proper subgroups of infinite rank are abelian-by-finite were considered in [8], where the following result was proved, which will be used in the proof of Theorem B.

Lemma 4 *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are abelian-by-finite. Then all proper subgroups of G are abelian-by-finite.*

Finally, we need the following lemma.

Lemma 5 *Let G be a locally (soluble-by-finite) group of infinite rank. If all proper subgroups of infinite rank of G have the CF-property, then G is either abelian-by-finite or periodic.*

Proof Assume for a contradiction that the group G neither is abelian-by-finite nor periodic. Since any locally (soluble-by-finite) CF-group is abelian-by-finite, it follows from Lemma 4 that all proper subgroups of G are abelian-by-finite. Thus, $G = G'$ by Lemma 2. It is also clear that G has no proper subgroups of finite index, so that in particular every finitely generated subgroup of G is abelian-by-finite (and so has finite rank). Therefore, G contains an abelian normal subgroup A such that G/A has finite rank (see [9], Theorem 8). Let E be any finitely generated subgroup of G. As G/A cannot be finitely generated, the product EA is a proper subgroup of infinite rank of G, and hence it has the CF-property. Then $(EA)''$ is locally finite by Lemma 1, and so E'' is finite. Therefore $G = G''$ is locally finite, and this contradiction completes the proof. □

Proof of Theorem B Assume for a contradiction that the group G is not abelian-by-finite. Then G is periodic by Lemma 5, and a contradiction can be obtained arguing as in the proof of Theorem A. To this aim, it is enough to observe that in the second part of that proof the hypotheses are applied only to subgroups of infinite rank. □

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