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Rings with finite Ding homological dimensions

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Abstract: In this paper, we study Ding homological dimensions of complexes. Special attention is paid to the dimensions of homologically bounded complexes that have nice functorial descriptions. These results are applied to give new characterizations of rings R such that $\mathrm{l.Ggldim}(R) < \infty$ and quasi-Frobenius rings.

Key words: Ding projective (injective) modules, Ding projective (injective) dimension

1. Introduction

It is well known that among the commutative local Noetherian rings $(R, \mathfrak{m}, k)$, the regular rings are characterized by the condition $\mathrm{pd}_R k < \infty$. In [8, Theorem 2.1], Holm proved that the Gorenstein injective dimension $\mathrm{Gid}_R R$ of R measures Gorensteinness in the following sense:

An associative ring R with $\mathrm{Gid}_R R < \infty$ also has $\mathrm{id}_R R < \infty$ (and hence R is Gorenstein, provided that R is commutative and Noetherian).

For any R-module M, the Gorenstein injective dimension $\mathrm{Gid}_R M$ is a refinement of the injective dimension $\mathrm{id}_R M$, and if $\mathrm{id}_R M < \infty$, then there is an equality $\mathrm{Gid}_R M = \mathrm{id}_R M$ by [9, Proposition 2.27]. Also pursuing the themes described above, Christensen et al. in [3] studied finite Gorenstein homological dimensions of complexes to identify Gorenstein rings.

Note that Ding et al. and Mao and Ding in [5] and [12] considered 2 special cases of the Gorenstein projective and Gorenstein injective modules, which they called strongly Gorenstein flat and Gorenstein FP-injective modules, respectively. Since over a Ding–Chen ring the strongly Gorenstein flat and Gorenstein FP-injective modules have many nice properties analogous to Gorenstein projective and Gorenstein injective modules over a Gorenstein ring, Gillespie [7] renamed these modules as Ding projective and Ding injective modules, respectively.

Now it is only natural to ask: what do the Ding projective and Ding injective dimensions measure? The aim of this paper is to study the Ding homological dimensions of complexes; as applications, we get some new characterizations of rings R such that $\mathrm{l.Ggldim}(R) < \infty$ and quasi-Frobenius rings.

2. Preliminaries

In this paper, the ring R is assumed to be associative with identity, and modules are, unless otherwise explicitly stated, left R-modules. We denote the classes of projective, flat, injective, and FP-injective R-modules by $\mathcal{P}(R)$, $\mathcal{F}(R)$, $\mathcal{I}(R)$, and $\mathcal{FI}(R)$, respectively.

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An R-complex X is a sequence of R-modules X_l and R-linear maps ∂_l^X, $l \in \mathbb{Z}$,

$$X = \cdots \longrightarrow X_{l+1} \xrightarrow{\partial_{l+1}^X} X_l \xrightarrow{\partial_l^X} X_{l-1} \longrightarrow \cdots$$

such that $\partial_l^X \partial_{l+1}^X = 0$ for all $l \in \mathbb{Z}$. X_l and ∂_l^X are called the module in degree l and lth differential of X, respectively. For an R-complex X and any $i \in \mathbb{Z}$, let $Z_i^X = \mathrm{Ker}\partial_i^X$, $B_i^X = \mathrm{Im}\partial_{i+1}^X$, and $C_i^X = \mathrm{Coker}\partial_{i+1}^X$. The residue class module $\mathrm{H}_i(X) = Z_i^X / B_i^X$ is called the ith homology module of X. The homology complex $\mathrm{H}(X)$ is defined by setting $\mathrm{H}(X)_i = \mathrm{H}_i(X)$ and $\partial_i^{\mathrm{H}(X)} = 0$ for all $i \in \mathbb{Z}$. Furthermore, we set $\sup X = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}$ and $\inf X = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}$.

Let X be an R-complex and let u, v be integers. The hard left-truncation, $_{\sqsubset u}X$, of X at u and the hard right-truncation, $X_{v\sqsupset}$, of X at v are given by:

$$_{\sqsubset u}X = 0 \longrightarrow X_u \xrightarrow{\partial_u^X} X_{u-1} \xrightarrow{\partial_{u-1}^X} X_{u-2} \xrightarrow{\partial_{u-2}^X} \cdots \text{and}$$

$$X_{v\sqsupset} = \cdots \xrightarrow{\partial_{v+3}^X} X_{v+2} \xrightarrow{\partial_{v+2}^X} X_{v+1} \xrightarrow{\partial_{v+1}^X} X_v \longrightarrow 0.$$

The soft left-truncation, $_{\subset u}X$, of X at u and the soft right-truncation, $X_{v\supset}$, of X at v are given by:

$$_{\subset u}X = 0 \longrightarrow C_u^X \xrightarrow{\overline{\partial}_u^X} X_{u-1} \xrightarrow{\partial_{u-1}^X} X_{u-2} \xrightarrow{\partial_{u-2}^X} \cdots \text{and}$$

$$X_{v\supset} = \cdots \xrightarrow{\partial_{v+3}^X} X_{v+2} \xrightarrow{\partial_{v+2}^X} X_{v+1} \xrightarrow{\partial_{v+1}^X} Z_v^X \longrightarrow 0.$$

The differential $\overline{\partial}_u^X$ is the induced map on residue classes.

Given an R-module M, we denote by $\underline{M}$ the complex with M in the 0th place and 0 elsewhere, and identify M with $\underline{M}$ occasionally if there is no risk of ambiguity. Given an R-complex X and an integer n, $\Sigma^n X$ denotes the complex X shifted n degrees to the left, i.e. $(\Sigma^n X)_i = X_{i-n}$ and $\partial_i^{\Sigma^n X} = (-1)^n \partial_{i-n}^X$.

The category of R-complexes is denoted $\mathcal{C}(R)$. The full subcategories $\mathcal{C}_{\sqsubset}(R)$, $\mathcal{C}_{\sqsupset}(R)$, and $\mathcal{C}_{\square}(R)$ of $\mathcal{C}(R)$ consist of complexes X with $X_l = 0$ for, respectively, $l \gg 0$, $l \ll 0$, and $|l| \gg 0$. The corresponding complexes are called left-bounded, right-bounded, and bounded complexes in order.

The derived category is written $\mathcal{D}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$, and $\square$ to denote homological boundedness conditions. They are named homologically left-bounded, homologically right-bounded, and homologically bounded complexes, respectively.

A morphism $\alpha : X \to Y$ of R-complexes is called a quasi-isomorphism if $\mathrm{H}(\alpha)$ is an isomorphism. For a morphism $\alpha : X \to Y$, we denote by $\mathrm{Cone}(\alpha)$ the mapping cone of α. It is given by $\mathrm{Cone}(\alpha)_i = Y_i \oplus X_{i-1}$ and $\partial_i^{\mathrm{Cone}(\alpha)}(y_i, x_{i-1}) = (\partial_i^Y(y_i) + \alpha_{i-1}(x_{i-1}), -\partial_{i-1}^X(x_{i-1}))$.

The right derived functor of the homomorphism functor of R-complexes and the left derived functor of the tensor product of R-complexes are denoted by $\mathrm{RHom}_R(-,\,-)$ and $- \otimes^L -$. The symbol "$\simeq$" is used to designate isomorphisms in $\mathcal{D}(R)$ and quasi-isomorphisms in $\mathcal{C}(R)$.

An R-complex $X \in \mathcal{D}_{\square}(R)$ is said to be of finite projective (flat) dimension if $X \simeq U$, where U is a complex of projective (flat) modules and $U_l = 0$ for $|l| \gg 0$. An R-complex $Y \in \mathcal{D}_{\square}(R)$ is said to be of finite injective (FP-injective) dimension if $Y \simeq V$, where V is a complex of injective (FP-injective) modules and $V_l = 0$ for $|l| \gg 0$. By $\mathbf{P}(R)$, $\mathbf{I}(R)$, $\mathbf{F}(R)$, and $\mathbf{FI}(R)$, we denote the full subcategories of $\mathcal{D}_{\square}(R)$ whose objects are complexes of finite projective, injective, flat, and FP-injective dimension, respectively. For the other notations, we would like to refer to Christensen ([2]).

Definition 2.1 ([10,13]) *An R-module N is called FP-injective (or absolutely pure) if $\mathrm{Ext}_R^1(M, N) = 0$ for all finitely presented R-modules M. The FP-injective dimension of N, denoted by FP-id$_R N$, is defined to be the smallest nonnegative integer n such that $\mathrm{Ext}_R^{n+1}(M, N) = 0$ for every finitely presented R-module M. If no such n exists, set FP-id$_R N = \infty$.*

Definition 2.2 ([7]) *An R-module M is called Ding projective if there exists a $\mathrm{Hom}_R(-, \mathcal{F}(R))$-exact exact sequence of projective R-modules*

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M = \mathrm{Coker}(P_1 \to P_0)$. We denote the class of Ding projective modules by $\mathcal{DP}$.

An R-module N is called Ding injective if there exists a $\mathrm{Hom}_R(\mathcal{FI}(R), -)$-exact exact sequence of injective R-modules

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

with $N = \mathrm{Coker}(I_1 \to I_0)$. We denote the class of Ding injective modules by $\mathcal{DI}$.

Next we establish some results on preservation of quasi-isomorphisms. These will play an important part in the proof of the main Theorem 3.4, and the ideal is inspired by that of [3].

Lemma 2.3 *(1) If M is a Ding projective R-module, then $\mathrm{Ext}_R^{\geq 1}(M, W) = 0$ for all R-modules W of finite flat or finite injective dimension.*

(2) If N is a Ding injective R-module, then $\mathrm{Ext}_R^{\geq 1}(U, N) = 0$ for all R-modules U of finite projective dimension.

(3) Let R be a left coherent ring. If N is a Ding injective R-module, then $\mathrm{Ext}_R^{\geq 1}(U, N) = 0$ for all R-modules U of finite FP-injective dimension.

Proof (1) For an R-module W of finite flat dimension, $\mathrm{Ext}_R^{\geq 1}(M, W) = 0$ is an immediate consequence of [5, Lemma 2.4 (1)].

Assume that $\mathrm{id}_R W = m < \infty$. Since M is Ding projective, we have an exact sequence

$$0 \to M \to P^0 \to P^1 \to \cdots \to P^{1-m} \to C \to 0,$$

where all P^i are projective R-modules. Breaking this sequence into short exact ones, we see that $\mathrm{Ext}_R^i(M, W) \cong \mathrm{Ext}_R^{i+m}(C, W)$ for $i > 0$, so the Exts vanish as desired since $\mathrm{Ext}_R^{i+m}(C, W) = 0$ for $i > 0$.

(2) The proof is dual to (1).

(3) Note that [12, Theorem 2.4], the proof is similar to that of (1). □

Lemma 2.4 ([3, Lemmas 2.4, 2.5]) *(1) Assume that $X, Y \in \mathcal{C}(R)$ with either $X \in \mathcal{C}_{\sqsupset}(R)$ or $Y \in \mathcal{C}_{\sqsupset}(R)$. If $\mathrm{H}(Hom_R(X_l, Y)) = 0$ for all $l \in \mathbb{Z}$, then $\mathrm{H}(Hom_R(X, Y)) = 0$.*

(2) Assume that $X, Y \in \mathcal{C}(R)$ with either $X \in \mathcal{C}_{\sqsubset}(R)$ or $Y \in \mathcal{C}_{\sqsubset}(R)$. If $\mathrm{H}(Hom_R(X, Y_l)) = 0$ for all $l \in \mathbb{Z}$, then $\mathrm{H}(Hom_R(X, Y)) = 0$.

Proposition 2.5 *Let M, $N \in \mathcal{C}_{\sqsubset}(R)$.*

(1) Let $\alpha : D \to D'$ be a quasi-isomorphism between right-bounded complexes of modules in $\mathcal{DP}(R)$. If each M_i has finite flat dimension, then the morphism $Hom_R(\alpha, M) : \mathrm{Hom}_R(D', M) \to \mathrm{Hom}_R(D, M)$ is a quasi-isomorphism.

(2) Let $\alpha : D \to D'$ be a quasi-isomorphism between right-bounded complexes of modules in $\mathcal{DP}(R)$. If each N_i has finite injective dimension, then $Hom_R(\alpha, N) : \mathrm{Hom}_R(D', N) \to \mathrm{Hom}_R(D, N)$ is a quasi-isomorphism.

Proof (1) From Lemma 2.4 (2) and the next isomorphism,

$$\mathrm{Cone}(\mathrm{Hom}_R(\alpha, M)) \simeq \Sigma^1 \mathrm{Hom}_R(\mathrm{Cone}(\alpha), M),$$

it suffices to show that $\mathrm{Hom}_R(\mathrm{Cone}(\alpha), M_l)$ is exact for all $l \in \mathbb{Z}$. Note that $\mathrm{Cone}(\alpha)$ is an exact, right-bounded complex in $\mathcal{DP}(R)$. Set $X_j = \mathrm{Ker}(\partial_j^{\mathrm{Cone}(\alpha)})$ for each integer j, and note $X_{j-1} \in \mathcal{DP}(R)$ for $j \ll 0$. Consider the exact sequences

$$0 \to X_j \to \mathrm{Cone}(\alpha)_j \to X_{j-1} \to 0. \qquad (*_j)$$

From [11, Theorem 2.1] we know that $\mathcal{DP}(R)$ is closed under kernels of epimorphisms, so an induction argument on $(*_j)$ implies $X_j \in \mathcal{DP}(R)$ for all j. Thus, Lemma 2.3 (1) yields $\mathrm{Ext}_R^{\geq 1}(X_j, M_l) = 0$ and $\mathrm{Ext}_R^{\geq 1}(\mathrm{Cone}(\alpha)_j, M_l) = 0$ for all j and $l \in \mathbb{Z}$. The long exact sequence in $\mathrm{Ext}_R(-, M_l)$ shows that $(*_j)$ is $\mathrm{Hom}_R(-, M_l)$ exact. It follows that $\mathrm{Hom}_R(\mathrm{Cone}(\alpha), M_l)$ is exact.

(2) The proof is similar to that of (1). □

Corollary 2.6 *(1) If $X \simeq A \in \mathcal{C}_{\sqsupset}^{\mathcal{DP}}(R)$ and $U \simeq V \in \mathcal{C}_{\square}^{\mathcal{F}}(R)$, then $\mathrm{RHom}_R(X, U)$ is represented by $\mathrm{Hom}_R(A, V)$.*

(2) If $X \simeq A \in \mathcal{C}_{\sqsupset}^{\mathcal{DP}}(R)$ and $U \simeq I \in \mathcal{C}_{\square}^{\mathcal{I}}(R)$, then $\mathrm{RHom}_R(X, U)$ is represented by $\mathrm{Hom}_R(A, I)$.

Proof We prove part (1), and the proof of part (2) is similar.

Taking a projective resolution $P \in \mathcal{C}_{\sqsupset}^{\mathcal{P}}(R)$ of X, then $\mathrm{RHom}_R(X, U)$ is represented by $\mathrm{Hom}_R(P, V)$. Since $P \simeq X \simeq A$, there is by [2, (A. 3.6)] a quasi-isomorphism $\alpha : P \to A$, and hence the morphism

$$\mathrm{Hom}_R(\alpha, V) : \mathrm{Hom}_R(A, V) \to \mathrm{Hom}_R(P, V)$$

is a quasi-isomorphism by Proposition 2.5 (1). In particular, the 2 complexes $\mathrm{Hom}_R(A, V)$ and $\mathrm{Hom}_R\ (P, V)$ are equivalent, so $\mathrm{Hom}_R(A, V)$ also represents $\mathrm{RHom}_R\ (X, U)$. □

The next 2 results are parallel to Proposition 2.5 and Corollary 2.6.

Proposition 2.7 *Let M, $N \in \mathcal{C}_{\sqsupset}(R)$.*

(1) Let R be a left coherent ring and $\beta : H \to H'$ be a quasi-isomorphism between left-bounded complexes of modules in $\mathcal{DI}(R)$. If each M_i has finite FP-injective dimension, then the morphism $Hom_R(M, \beta)$: $\mathrm{Hom}_R(M, H) \to \mathrm{Hom}_R(M, H')$ is a quasi-isomorphism.

(2) Let $\beta : H \to H'$ be a quasi-isomorphism between left-bounded complexes of modules in $\mathcal{DI}(R)$. If each N_i has finite projective dimension, then the morphism $Hom_R(N, \beta) : \mathrm{Hom}_R(N, H) \to \mathrm{Hom}_R(N, H')$ is a quasi-isomorphism.

Corollary 2.8 *(1) Let R be a left coherent ring. If $Y \simeq B \in \mathcal{C}_{\sqsubset}^{\mathcal{DI}}(R)$ and $U \simeq V \in \mathcal{C}_{\square}^{\mathcal{FI}}(R)$, then* $\mathrm{RHom}_R(U,Y)$ *is represented by* $\mathrm{Hom}_R(V,B)$.

(2) If $Y \simeq B \in \mathcal{C}_{\sqsubset}^{\mathcal{DI}}(R)$ and $U \simeq P \in \mathcal{C}_{\square}^{\mathcal{P}}(R)$, then $\mathrm{RHom}_R(U,Y)$ *is represented by* $\mathrm{Hom}_R(P,B)$.

3. Ding homological dimensions of complexes

Obviously, by the definitions of Ding projective and Ding injective modules, we see that projective R-modules are Ding projective and injective R-modules are Ding injective. Thus, for every homologically right-bounded complex X, there exists a right-bounded complex A of Ding projective R-modules with $A \simeq X$ in $\mathcal{D}(R)$ (as one could take A to be a projective resolution of X). Every such A is called a Ding projective resolution of X.

Ding injective resolution of homologically left-bounded complexes is defined in a similar way, and it always exists.

Definition 3.1 *The Ding projective dimension,* $\mathrm{Dpd}_R X$, *of* $X \in \mathcal{D}_{\sqsupset}(R)$ *is defined as*

$$\mathrm{Dpd}_R X = \inf\{\sup\{l \in \mathbb{Z} \mid A_l \neq 0\} \mid X \simeq A \in \mathcal{C}_{\sqsupset}^{\mathcal{DP}}(R)\}.$$

The Ding injective dimension, $\mathrm{Did}_R Y$, *of* $Y \in \mathcal{D}_{\sqsubset}(R)$ *is defined as*

$$\mathrm{Did}_R Y = \inf\{\sup\{l \in \mathbb{Z} \mid B_{-l} \neq 0\} \mid Y \simeq B \in \mathcal{C}_{\sqsubset}^{\mathcal{DI}}(R)\}.$$

To prove the main results, we need the following 2 lemmas.

Lemma 3.2 *Let W be a flat R-module. If $X \in \mathcal{D}_{\square}(R)$ is equivalent to $A \in \mathcal{C}_{\sqsupset}^{\mathcal{DP}}(R)$ and $n \geq \sup X$, then*

$$\mathrm{Ext}_R^m(C_n^A, W) = \mathrm{H}_{-(m+n)}(\mathrm{RHom}_R(X,W))$$

for $m > 0$. In particular, there is an inequality:

$$\inf(\mathrm{RHom}_R(C_n^A, W)) \geq \inf(\mathrm{RHom}_R(X,W)) + n.$$

Proof Since $n \geq \sup X = \sup A$ we have $A_{n\sqsupset} \simeq \Sigma^n C_n^A$, cf. [2, (A.1.14.3)], and since W is flat it follows by Corollary 2.6 (1) that $\mathrm{RHom}_R(C_n^A, W)$ is represented by $\mathrm{Hom}_R(\Sigma^{-n} A_{n\sqsupset}, W)$. For $m > 0$ the isomorphism class $\mathrm{Ext}_R^m(C_n^A, W)$ is then represented by

$$\begin{aligned}
\mathrm{H}_{-m}(\mathrm{Hom}_R(\Sigma^{-n} A_{n\sqsupset}, W)) &= \mathrm{H}_{-m}(\Sigma^n \mathrm{Hom}_R(A_{n\sqsupset}, W)) \\
&= \mathrm{H}_{-(m+n)}(\mathrm{Hom}_R(A_{n\sqsupset}, W)) \\
&= \mathrm{H}_{-(m+n)}(\sqsubset_{-n} \mathrm{Hom}_R(A, W)) \\
&= \mathrm{H}_{-(m+n)}(\mathrm{Hom}_R(A, W)),
\end{aligned}$$

cf. [2, (A.2.1.3), (A.1.3.1), (A.1.20.2)]. It also follows from Corollary 2.6 (1) that the complex $\mathrm{Hom}_R(A,W)$ represents $\mathrm{RHom}_R(X,W)$, so

$$\mathrm{Ext}_R^m(C_n^A, W) = \mathrm{H}_{-(m+n)}(\mathrm{RHom}_R(X,W))$$

and the inequality of infima follows. $\square$

Lemma 3.3 *Let J be an FP-injective R-module. If $Y \in \mathcal{D}_{\square}(R)$ is equivalent to $B \in \mathcal{C}_{\sqsubset}^{\mathcal{DI}}(R)$ and $n \geq -\inf Y$, then*

$$\mathrm{Ext}_R^m(J, Z_{-n}^B) = \mathrm{H}_{-(m+n)}(\mathrm{RHom}_R(J,Y))$$

for $m > 0$. In particular, there is an inequality:

$$\inf(\mathrm{RHom}_R(J, Z_{-n}^B)) \geq \inf(\mathrm{RHom}_R(J,Y)) + n.$$

Proof Since $-n \leq \inf Y = \inf B$, we have ${}_{\sqsubset -n}B \simeq \Sigma^{-n} Z_{-n}^B$, cf. [2, (A.1.14.1)], and since J is FP-injective it follows by Corollary 2.8 (1) that $\mathrm{RHom}_R(J, Z_{-n}^B)$ is represented by $\mathrm{Hom}_R(J, \Sigma^n {}_{\sqsubset -n}B)$. For $m > 0$ the isomorphism class $\mathrm{Ext}_R^m(J, Z_{-n}^B)$ is now represented by

$$\begin{aligned}
\mathrm{H}_{-m}(\mathrm{Hom}_R(J, \Sigma^n {}_{\sqsubset -n}B)) &= \mathrm{H}_{-m}(\Sigma^n \mathrm{Hom}_R(J, {}_{\sqsubset -n}B)) \\
&= \mathrm{H}_{-(m+n)}(\mathrm{Hom}_R(J, {}_{\sqsubset -n}B)) \\
&= \mathrm{H}_{-(m+n)}({}_{\sqsubset -n}\mathrm{Hom}_R(J,B)) \\
&= \mathrm{H}_{-(m+n)}(\mathrm{Hom}_R(J,B)),
\end{aligned}$$

cf. [2, (A.2.1.1), (A.1.3.1), (A.1.20.1)]. It also follows from Corollary 2.8 (1) that the complex $\mathrm{Hom}_R(J,B)$ represents $\mathrm{RHom}_R(J,Y)$, so

$$\mathrm{Ext}_R^m(J, Z_{-n}^B) = \mathrm{H}_{-(m+n)}(\mathrm{RHom}_R(J,Y))$$

and the inequality of infima follows. □

We are now in a position to prove the following:

Theorem 3.4 *Let $X \in \mathcal{D}_{\sqsupset}(R)$ be a complex of finite Ding projective dimension. For $n \in \mathbb{Z}$, the following are equivalent:*

(1) $\mathrm{Dpd}_R X \leq n$.

(2) X is equivalent to a complex $A \in \mathcal{C}_{\square}^{\mathcal{DP}}(R)$ concentrated in degrees of at most n, and A can be chosen with $A_l = 0$ for $l < \inf X$.

(3) $n \geq \inf U - \inf \mathrm{RHom}_R(X,U)$ for all $U \not\simeq 0$ with $U \in \mathcal{C}_{\square}^{\mathcal{F}}(R)$ or $U \in \mathbf{I}(R)$.

(4) $n \geq -\inf \mathrm{RHom}_R(X,W)$ for all flat R-modules W.

(5) $n \geq \sup X$ and the module $C_n^A = \mathrm{Coker}(A_{n+1} \to A_n)$ is Ding projective whenever $X \simeq A \in \mathcal{C}_{\sqsupset}^{\mathcal{DP}}(R)$.

Moreover, the following hold:

$$\begin{aligned}
\mathrm{Dpd}_R X &= \sup\{\inf U - \inf \mathrm{RHom}_R(X,U) \mid 0 \not\simeq U \in \mathcal{C}_{\square}^{\mathcal{F}}(R)\} \\
&= \sup\{-\inf \mathrm{RHom}_R(X,W) \mid W \text{ is flat}\}.
\end{aligned}$$

Proof It is immediate by Definition 3.1 that (2) implies (1), and that (3) implies (4) is obvious.

(1) $\Rightarrow$ (3) Choose a complex $A \in \mathcal{C}_{\square}^{\mathcal{DP}}(R)$, such that $A \simeq X$ and $A_l = 0$ for $l > n$. First, let $0 \not\simeq U \in \mathcal{C}_{\square}^{\mathcal{F}}(R)$. Set $i = \inf U$ and note that $i \in \mathbb{Z}$ as $U \in \mathcal{D}_{\square}(R)$ with $\mathrm{H}(U) \neq 0$. By Corollary 2.6 (1) the complex $\mathrm{Hom}_R(A,U)$ represents $\mathrm{RHom}_R(X,U)$; in particular, $\inf \mathrm{RHom}_R(X,U) = \inf \mathrm{Hom}_R(A,U)$. For $l < i - n$ and $p \in \mathbb{Z}$, either $p > n$ or $p + l \leq n + l < i$, so the module

$$\mathrm{Hom}_R(A,U)_l = \Pi_{p \in \mathbb{Z}} \mathrm{Hom}_R(A_p, U_{p+l})$$

vanishes. Hence, $\mathrm{H}_l(\mathrm{Hom}_R(A,U)) = 0$ for $l < i-n$, and $\inf \mathrm{RHom}_R(X,U) \geq i-n = \inf U - n$ as desired.

Next, let $U \in \mathbf{I}(R)$ and choose a complex $I \in \mathcal{C}^{\mathcal{I}}_{\square}(R)$ such that $U \simeq I$. Set $i = \inf U$ and consider the soft truncation $V = I_{i\supset}$. The modules in V have finite injective dimension and $U \simeq V$, and hence $\mathrm{Hom}_R(A,V) \simeq \mathrm{RHom}_R(X,U)$ by Corollary 2.6 (2) and the proof continues as above.

(4) $\Rightarrow$ (5) To see that $n \geq \sup X$, it is sufficient to show that

$$\sup\{-\inf \mathrm{RHom}_R(X,W) \mid W \text{ is flat}\} \geq \sup X. \tag{$*$}$$

By assumption, $g = \mathrm{Dpd}_R X$ is finite, i.e. $X \simeq A$ for some complex

$$A = 0 \to A_g \xrightarrow{\partial^A_g} A_{g-1} \to \cdots \to A_i \to 0,$$

and it is clear that $g \geq \sup X$ since $X \simeq A$. For any flat R-module W, the complex $\mathrm{Hom}_R(A,W)$ is concentrated in degrees $-i$ to $-g$,

$$0 \to \mathrm{Hom}_R(A_i,W) \to \cdots \to \mathrm{Hom}_R(A_{g-1},W) \xrightarrow{\mathrm{Hom}_R(\partial^A_g,W)} \mathrm{Hom}_R(A_g,W) \to 0,$$

and isomorphic to $\mathrm{RHom}_R(X,W)$ in $\mathcal{D}(R)$, cf. Corollary 2.6 (1). First, consider the case $g = \sup X$. The differential $\partial^A_g : A_g \to A_{g-1}$ is not injective, as A has homology in degree $g = \sup X = \sup A$. By the definition of Ding projective R-modules, there exists a flat R-module Q and an injective homomorphism $\varphi : A_g \to Q$. Because ∂^A_g is not injective, $\varphi \in \mathrm{Hom}_R(A_g,Q)$ cannot have the form $\mathrm{Hom}_R(\partial^A_g,Q)(\psi) = \psi\partial^A_g$ for some $\psi \in \mathrm{Hom}_R(A_{g-1},Q)$. That is, the differential $\mathrm{Hom}_R(\partial^A_g,Q)$ is not surjective; hence, $\mathrm{Hom}_R(A,Q)$ has nonzero homology in degree $-g = -\sup X$, and $(*)$ follows.

Next, assume that $g > \sup X = s$ and consider the exact sequence

$$A = 0 \to A_g \to \cdots \to A_{s+1} \to A_s \to C^A_s \to 0. \tag{$\triangle$}$$

It shows that $\mathrm{Dpd}_R C^A_s \leq g-s$, and it is easy to check that equality must hold, as otherwise we would have $\mathrm{Dpd}_R X < g$. A straightforward computation based on Corollary 2.6 (1) and Lemma 3.2 shows that for all $m > 0$, all $n \geq \sup X$, and all flat R-modules W one has

$$\mathrm{Ext}^m_R(C^A_n,W) = \mathrm{H}_{-(m+n)}(\mathrm{RHom}_R(X,W)). \tag{$\sharp$}$$

By [11, Theorem 2.4] we have $\mathrm{Ext}^{g-s}_R(C^A_n,Q) \neq 0$ for some flat R-module Q , from which $\mathrm{H}_{-g}(\mathrm{RHom}_R(X,Q)) \neq 0$ by $(\sharp)$ and $(*)$ follows. We conclude that $n \geq \sup X$.

It remains to be proven that C^A_n is Ding projective for any right-bounded complex $A \simeq X$ of Ding projective R-modules. By assumption, $\mathrm{Dpd}_R X$ is finite, so a bounded complex $\widetilde{A} \simeq X$ of Ding projective R-modules does exist. Consider the cokernel $C^{\widetilde{A}}_n$. Since $n \geq \sup X = \sup \widetilde{A}$, it fits in an exact sequence $0 \to \widetilde{A}_t \to \cdots \to \widetilde{A}_{n+1} \to \widetilde{A}_n \to C^{\widetilde{A}}_n \to 0$, where all the $\widetilde{A}_l$s are Ding projective. By $(\sharp)$ and [11, Theorem 2.4] it now also follows that $C^{\widetilde{A}}_n$ is Ding projective. With this, it is sufficient to prove the following:

If $P, A \in \mathcal{C}_{\sqsupset}(R)$ are complexes of, respectively, projective and Ding projective modules, and $P \simeq X \simeq A$, then the cokernel C^P_n is Ding projective if and only if C^A_n is so.

Let A and P be 2 such complexes. As P consists of projective modules there is a quasi-isomorphism $\pi : P \xrightarrow{\simeq} A$, which induces a quasi-isomorphism between the truncated complexes, ${}_{\subset n}\pi : {}_{\subset n}P \xrightarrow{\simeq} {}_{\subset n}A$. The mapping cone

$$\mathrm{Cone}({}_{\subset n}\pi) = 0 \to C_n^P \to P_{n-1} \oplus C_n^A \to P_{n-2} \oplus A_{n-1} \to \cdots$$

is a bounded exact complex, in which all modules but the 2 left-most ones are known to be Ding projective. It follows by the projectively resolving properties of Ding projective modules, cf. [11, Theorem 2.1], that C_n^P is Ding projective if and only if $P_{n-1} \oplus C_n^A$ is so, which implies that C_n^A is Ding projective.

(5) $\Rightarrow$ (2) Choose a projective resolution $A \in \mathcal{C}_{\sqsupset}^{\mathcal{P}}(R) \subseteq \mathcal{C}_{\sqsupset}^{\mathcal{DP}}(R)$ of X with $A_l = 0$ for $l < \inf X$ by [2, (A.3.2)]. Since $n \geq \sup X = \sup A$ it follows by [2, (A.1.14.2)] that $X \simeq {}_{\subset n}A$, and ${}_{\subset n}A \in \mathcal{C}_{\square}^{\mathcal{DP}}(R)$ as C_n^A is Ding projective.

To show the last claim, we still assume that $\mathrm{Dpd}_R X$ is finite. The 2 equalities are immediate consequences of the equivalence of (1), (3), and (4). $\square$

In the following, we treat Ding projective dimension for modules. The Ding projective resolution of an R-module M was defined in the usual way by Ding et al. [5, (3.1)]. All modules have a projective resolution and, hence, a Ding projective one.

Lemma 3.5 *Let M be an R-module. If $\underline{M} \simeq A \in \mathcal{C}_{\sqsupset}^{\mathcal{DP}}(R)$, then the truncated complex*

$$A_{0\supset} = \cdots \to A_l \to \cdots \to A_2 \to A_1 \to Z_0^A \to 0$$

is a Ding projective resolution of M.

Proof Suppose $\underline{M} \simeq A \in \mathcal{C}_{\sqsupset}^{\mathcal{DP}}(R)$; then $\inf A$=0, so $A_{0\supset} \simeq A \simeq M$ by [2, (A. 1.14.4)], and we have an exact sequence of modules

$$\cdots \to A_l \to \cdots \to A_2 \to A_1 \to Z_0^A \to M \to 0. \qquad (+)$$

Setting $v = \inf\{l \in \mathbb{Z} \mid A_l \neq 0\}$, then also the sequence

$$0 \to Z_0^A \to A_0 \to \cdots \to A_{v+1} \to A_v \to 0$$

is exact. All the modules $A_0, \cdots, A_v$ are Ding projective, so it follows by the projectively resolving properties of Ding projective modules, cf. [11, Theorem 2.1], that Z_0^A is Ding projective, and therefore $A_{0\supset}$ is a Ding projective resolution of M, cf. (+). $\square$

Now, we compare our notion of Ding projective dimension for complexes with earlier concepts, which are restricted to modules. For an R-module M, Ding et al. defined in [5, (3.1)] the Ding projective dimension of M to be

$\mathrm{Dpd}_R M = \inf\{n |$there exists an exact sequence $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ of R -modules, where each A_i is Ding projective$\}$.

The next corollary shows that our definition of Ding projective dimension for complexes coincides with that of Ding and Mao. Additionally, the following corollary extends a result of Mahdou and Tamekkante [11, Theorem 2.4].

Corollary 3.6 *Let M be an R-module of finite Ding projective dimension and $n \in \mathbb{N}_0$. The following are equivalent:*

(1) M has a Ding projective resolution of length of at most n. That is, there is an exact sequence of modules $0 \to A_n \to A_{n-1} \to \cdots \to A_0 \to M \to 0$, where $A_0, A_1, \cdots, A_n$ are Ding projective.

(2) $\mathrm{Dpd}_R M \leq n$.

(3) $\mathrm{Ext}^i_R(M, W) = 0$ for all $i > n$ and all R-modules W with $\mathrm{fd}_R W < \infty$.

(4) $\mathrm{Ext}^i_R(M, F) = 0$ for all $i > n$ and all flat R-modules F.

(5) In any Ding projective resolution of M

$$\cdots \to A_l \to A_{l-1} \to \cdots \to A_0 \to M \to 0,$$

the kernel $K_n = \mathrm{Ker}(A_{n-1} \to A_{n-2})$ is a Ding projective R-module.

Proof If the sequence $\cdots \to A_l \to A_{l-1} \to \cdots \to A_0 \to M \to 0$ is exact, then $\underline{M}$ is equivalent to $A = \cdots \to A_l \to A_{l-1} \to \cdots \to A_0 \to 0$. The complex A belongs to $\mathcal{C}^{\mathcal{DP}}_{\sqsupset}(R)$, and it has $C^A_0 \cong M$, $C^A_1 \cong \mathrm{Ker}(A_0 \to M)$, and $C^A_l \cong Z^A_{l-1} = \mathrm{Ker}(A_{l-1} \to A_{l-2})$ for $l \geq 2$. In view of the Lemma 3.5, the equivalence of the 5 conditions now follows from Theorem 3.4. □

Next, we turn to the Ding injective dimension. The proof of Theorem 3.7 below relies on Corollary 2.8 and Lemma 3.3 instead of Corollary 2.6 and Lemma 3.2 but is otherwise similar to that of Theorem 3.4; hence, it is omitted.

Theorem 3.7 *Let R be a left coherent ring and $Y \in \mathcal{D}_{\sqsubset}(R)$ be a complex of finite Ding injective dimension. For $n \in \mathbb{Z}$, the following are equivalent:*

(1) $\mathrm{Did}_R Y \leq n$.

(2) Y is equivalent to a complex $B \in \mathcal{C}^{\mathcal{DI}}_{\square}(R)$ concentrated in degrees of at least $-n$, and B can be chosen with $B_l = 0$ for $l > \mathrm{sup}Y$.

(3) $n \geq -\mathrm{sup}U - \mathrm{infRHom}_R(U, Y)$ for all $U \not\simeq 0$ with $U \in \mathcal{C}^{\mathcal{FI}}_{\square}(R)$ or $U \in \mathbf{P}(R)$.

(4) $n \geq -\mathrm{infRHom}_R(J, Y)$ for all FP-injective R-modules J.

(5) $n \geq -\mathrm{inf}Y$ and for any left-bounded complex $B \simeq Y$ of Ding injective modules, the kernel $Z^B_{-n} = \mathrm{Ker}(B_{-n} \to B_{-(n+1)})$ is Ding injective.

Moreover, the following hold:

$$\begin{aligned} \mathrm{Did}_R Y &= \sup\{-\mathrm{sup}U - \mathrm{infRHom}_R(U, Y) \mid 0 \not\simeq U \in \mathcal{C}^{\mathcal{FI}}_{\square}(R)\} \\ &= \sup\{-\mathrm{infRHom}_R(J, Y) \mid J \text{ is } FP\text{-injective}\}. \end{aligned}$$

Lemma 3.8 *Let N be an R-module. If $N \simeq B \in \mathcal{C}^{\mathcal{DI}}_{\sqsubset}(R)$, then the truncated complex*

$$_{\subset 0}B = 0 \to C^B_0 \to B_{-1} \to \cdots \to B_l \to \cdots$$

is a Ding injective coresolution of N.

Proof Suppose $N \simeq B \in \mathcal{C}^{\mathcal{DI}}_{\sqsubset}(R)$; then $\sup B$=0, so $_{\subset 0}B \simeq B \simeq N$ by [2, (A. 1.14.2)], and we have an exact sequence of modules

$$0 \to N \to C^B_0 \to B_{-1} \to \cdots \to B_l \to \cdots. \qquad (+)$$

Setting $u = \sup\{l \in \mathbb{Z} \mid B_l \neq 0\}$, then also the sequence

$$0 \to B_u \to B_{u-1} \to \cdots \to B_0 \to C^B_0 \to 0$$

is exact. All the modules $B_u, \cdots, B_0$ are Ding injective, so it follows by the injectively resolving properties of Ding injective modules, cf. [15, Theorem 2.8], that C_0^B is Ding injective, and therefore ${}_{\subset 0}B$ is a Ding injective coresolution of N, cf. (+). □

Corollary 3.9 *Let R be a left coherent ring and N an R-module of finite Ding injective dimension and $n \in \mathbb{N}_0$. The following are equivalent:*

(1) N has a Ding injective coresolution of length of at most n. That is, there is an exact sequence of modules $0 \to N \to B_0 \to B_{-1} \to \cdots \to B_{-n} \to 0$, where $B_0, B_{-1}, \cdots, B_{-n}$ are Ding injective.

(2) $\mathrm{Did}_R N \leq n$.

(3) $\mathrm{Ext}_R^i(L, N) = 0$ for all $i > n$ and all R-modules L with $\mathrm{FP\text{-}id}_R(L) < \infty$.

(4) $\mathrm{Ext}_R^i(J, N) = 0$ for all $i > n$ and all FP-injective R-modules J.

(5) In any Ding injective coresolution of N

$$0 \to N \to B_0 \to B_{-1} \to \cdots \to B_l \to \cdots,$$

the cokernel $W_{-n} = \mathrm{Coker}(B_{-n+2} \to B_{-n+1})$ is a Ding injective R-module.

Proof If the sequence $0 \to N \to B_0 \to B_{-1} \to \cdots \to B_l \to \cdots$ is exact, then N is equivalent to $B = 0 \to B_0 \to B_{-1} \to \cdots \to B_l \to \cdots$. The complex B belongs to $\mathcal{C}_{\sqsubset}^{\mathcal{DI}}(R)$, and it has $Z_0^B \cong N$, $Z_{-1}^B \cong \mathrm{Coker}(N \to B_0)$, and $Z_{-l}^B \cong C_{-l+1}^B = \mathrm{Coker}(B_{-l+2} \to B_{-l+1})$ for $l \geq 2$. In view of the Lemma 3.8, the equivalence of the 5 conditions now follows from Theorem 3.7. □

Recall that an R-module M is called Gorenstein flat if there exists an exact complex F of flat modules such that M is isomorphic to a cokernel of F, and $\mathrm{H}(E \otimes_R F) = 0$ for all injective right R-modules E. Denote the class of Gorenstein flat modules by $\mathcal{GF}$. By [3], the Gorenstein flat dimension, $\mathrm{Gfd}_R X$, for a homologically bounded-below complex X is defined by $\mathrm{Gfd}_R X = \inf\{\sup\{l \in \mathbb{Z} \mid A_l \neq 0\} \mid X \simeq A \in \mathcal{C}_{\sqsupset}^{\mathcal{GF}}(R)\}$.

Next, we give the connection between the Gorenstein flat dimension and Ding injective dimension for a homologically bounded-below complex X over a left coherent ring. For an R-complex X we use the notation $X^+ = \mathrm{RHom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$.

Proposition 3.10 *Let R be a left coherent ring and $X \in \mathcal{D}_{\sqsupset}(R)$. Then*

$$\mathrm{Gfd}_R X = \mathrm{Did}_R X^+.$$

Proof By Theorem 3.7, the adjoint isomorphism, and [6, Proposition 4.15], we have

$$\begin{aligned}
\mathrm{Did}_R X^+ &= \sup\{-\inf \mathrm{RHom}_R(J, X^+) \mid J \text{ is } FP\text{-injective}\} \\
&= \sup\{-\inf (X \otimes_R^L J)^+ \mid J \text{ is } FP\text{-injective}\} \\
&= \sup\{\sup (X \otimes_R^L J)^+ \mid J \text{ is } FP\text{-injective}\} \\
&= \mathrm{Gfd}_R X.
\end{aligned}$$

□

4. Applications

In this section, we focus on applying our previous results to new characterizations of some well-known rings. For a ring R, Mahdou and Tamekkante in [11, Theorem 3.1] proved $\mathrm{l.Ggldim}(R) = \sup\{\mathrm{Dpd}_R M | M \in R\text{-Mod}\}$, where $\mathrm{l.Ggldim}(R)$ denotes the left Gorenstein global dimension of R, which was defined in [1]. An immediate application of Theorem 3.4 is to derive the following characterizations of these rings.

Theorem 4.1 *The following conditions are equivalent for a ring R and a nonnegative integer n:*

(1) $\mathrm{l.Ggldim}(R) \leq n$.

(2) For any complex $X \in \mathcal{D}_{\sqsupset}(R)$, $\mathrm{Dpd}_R X \leq n + \sup X$.

Proof (1)⇒(2) Assume $\sup X = s$, and choose a complex $A \in \mathcal{C}_{\sqsupset}^{\mathcal{DP}}(R)$ equivalent to X. By Theorem 3.4 (5), it suffices to prove that $C_{n+s}^A = \mathrm{Coker}(A_{n+s+1} \to A_{n+s})$ is Ding projective.

As $\sup X = s$, there is an exact sequence of R-modules

$$0 \to C_{n+s}^A \to A_{n+s-1} \to \cdots \to A_{s+1} \to A_s \to C_s^A \to 0.$$

Since $\mathrm{Dpd}_R C_s^A \leq n$ by assumption, and the As are Ding projective, C_{n+s}^A is Ding projective as desired.

(2)⇒(1) is obvious. □

Since a 0-Gorenstein ring is indeed quasi-Frobenius, we have the following:

Corollary 4.2 *The following conditions are equivalent:*

(1) R is quasi-Frobenius.

(2) For every complex $X \in \mathcal{D}_{\sqsupset}(R)$, $\mathrm{Dpd}_R X = \sup X$.

Recall that a ring R is called an n-FC ring [4] if R is a left and right coherent ring with $\text{FP-id}_R R \leq n$ and $\text{FP-id} R_R \leq n$. Yang in [14] proved that $\sup\{\mathrm{Dpd}_R M | M \in R\text{-Mod}\} = \sup\{\mathrm{Did}_R M | M \in R\text{-Mod}\}$ when R is n-FC or commutative coherent. Using this result and a similar proof of Theorem 4.1, we have the following:

Proposition 4.3 *Let R be an n-FC or commutative coherent ring. Then the following conditions are equivalent:*

(1) $\mathrm{l.Ggldim}(R) \leq n$.

(2) For any complex $X \in \mathcal{D}_{\sqsubset}(R)$, $\mathrm{Did}_R X \leq n - \inf X$.

Corollary 4.4 *Let R be an FC or commutative coherent ring. Then the following conditions are equivalent:*

(1) R is quasi-Frobenius.

(2) For any complex $X \in \mathcal{D}_{\sqsubset}(R)$, $\mathrm{Did}_R X = \inf X$.

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