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Abstract: In this paper, we study spherically symmetric Finsler metrics $F = |y|\phi(|x|, \frac{<x,y>}{|y|})$, where $x \in \mathbb{B}^n(r) \subset \mathbb{R}^n$, $y \in T_x\mathbb{B}^n(r)\backslash\{0\}$ and $\phi : [0,r) \times \mathbb{R} \to \mathbb{R}$. By investigating a PDE equivalent to these metrics being locally projectively flat, we manufacture projectively flat spherically symmetric Finsler metrics in terms of error functions and, using Shen's result, we give its flag curvature.

Key words: Spherically symmetric, locally projectively flat, scalar flag curvature, error function

1. Introduction

It is an important problem in Finsler geometry to study and characterize projectively flat Finlser metrics on an open domain in $\mathbb{R}^n$. This is Hilbert's fourth problem in the regular case. Beltrami's theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature [7]. However, the situation is much more complicated for Finsler metrics. In fact, there are many projectively flat Finsler metrics that are not of constant flag curvature [2, 6, 8]. Conversely, there are infinitely many nonprojectively flat Finsler metrics with constant flag curvature. The flag curvature is the most important Riemannian quantity in Finsler geometry because it is an analogue of sectional curvature in Riemannian geometry [1].

Recently, Huang and Mo [5] discussed a class of interesting Finsler metrics. They are of the form $F = |y|\phi(|x|, \frac{<x,y>}{|y|})$, and such metrics were said to spherically symmetric in [5]. Many known projectively flat Finsler metrics are spherically symmetric.

Below are 3 important spherically symmetric Finsler metrics:

(1) The Klein metric,

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x,y>^2)}}{1-|x|^2}, \tag{1.1}$$

is a spherically symmetric Finsler metric on $\mathbb{B}^n$, where $\phi = \dfrac{\sqrt{1-b^2+s^2}}{1-b^2}$ and $s = \dfrac{<x,y>}{|y|}$.

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(2) The Funk metric,

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x, y>^2)}}{1 - |x|^2} + \frac{<x, y>}{1 - |x|^2}, \tag{1.2}$$

is a spherically symmetric Finsler metric on $\mathbb{B}^n$, where $\phi = \dfrac{\sqrt{1 - b^2 + s^2} + s}{1 - b^2}$ and $s = \dfrac{<x, y>}{|y|}$.

(3) The Berwald metric,

$$F(x, y) = \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - <x, y>^2)} + <x, y>)^2}{(1 - |x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - <x, y>^2)}}, \tag{1.3}$$

is a spherically symmetric Finsler metric on $\mathbb{B}^n$, where $\phi = \dfrac{(\sqrt{1 - b^2 + s^2} + s)^2}{(1 - b^2)^2\sqrt{1 - b^2 + s^2}}$ and $s = \dfrac{<x, y>}{|y|}$.

Recall that general (α, β)-metrics are Finsler metrics of the form $F = \alpha\phi(\|\beta\|_\alpha, \frac{\beta}{\alpha})$, where α is a Riemann metric and β is a 1-form [10]. In particular, (1.1) – (1.3) show that such metrics are general (α, β)-metrics. In [5], the second differential equation was obtained for F to be projectively flat.

Lemma 1.1 [5] *Let $F(x, y) = |y|\phi(|x|, \frac{<x, y>}{|y|})$ be a spherically symmetric Finsler metric on $\mathbb{B}^n(r) \subset \mathbb{R}^n$. Then $F = F(x, y)$ is projectively flat if and only if $\phi = \phi(s)$ satisfies*

$$s\phi_{bs} + b\phi_{ss} - \phi_b = 0, \tag{1.4}$$

where $b := \|\beta\|_\alpha, s =: \dfrac{\beta}{\alpha}$.

Note that ϕ_b means derivation of ϕ with respect to the first variable b.

In this paper, by investigating the PDE (1.4), we manufacture projectively flat spherically symmetric Finsler metrics in terms of error functions. We have the following:

Theorem 1.1 *Let $F = |y|\phi(|x|, \frac{<x, y>}{|y|})$ be a spherically symmetric Finlser metric and $\phi(b, s)$ be a function defined by*

$$\phi(b, s) = e^{\lambda b^2}[C_1 s - C_2(e^{-\lambda s^2} + s\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)], \tag{1.5}$$

where $\lambda > 0$, C_1, C_2 are arbitrary constants. We then have the following properties:

(1) The spherically symmetric Finsler metric given in (1.5) is projectively flat on $\mathbb{B}^n(r) \subset \mathbb{R}^n$ and its projective factor P is given by

$$P = \lambda <x, y> + \frac{1}{2}H. \tag{1.6}$$

(2) F is of scalar flag curvature and its flag cuevature is given by

$$K = \frac{\lambda^2 s^2}{f^2(b)g^2(s)} - \frac{2C_2\lambda e^{-\lambda s^2}}{f^2(b)g^3(s)} + \frac{3H^2}{4|y|^2 f^2(b)g^2(s)}, \tag{1.7}$$

where

$$f(b) = e^{\lambda b^2}, \tag{1.8}$$

$$g(s) = C_1 s - C_2(e^{-\lambda s^2} + s\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)), \tag{1.9}$$

$$H = \frac{C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)}{g(s)}. \tag{1.10}$$

2. Preliminaries

A Finsler metric on a manifold is a family of Minkowski norms on the tangent spaces. By definition, a Minkowski norm on a vector space V is a nonnegative function $F : V \to [0, +\infty)$ with the following properties:

(1) F is a positive y-homogeneous of degree one, for any $y \in V$ and any $\lambda > 0$,

$$F(\lambda y) = \lambda F(y). \tag{2.1}$$

(2) F is $\mathcal{C}^\infty$ on $V\backslash\{0\}$ and any tangent vector $y \in V\backslash\{0\}$, and the following bilinear symmetric form $\mathbf{g}_y : V \times V \to \mathbb{R}$ is positive definite:

$$\mathbf{g}_{(u,v)} := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y + s\mu + t\nu)]_{s=t=0}. \tag{2.2}$$

Let M be a manifold. Let $TM = \cup_{x\in M} T_x M$ be the tangent bundle of M, where $T_x M$ is the tangent space at $x \in M$. We set $TM_0 := TM\backslash\{0\}$, where $\{0\}$ stands for $\{(x, 0)|x \in M, 0 \in T_x M\}$. A Finsler metric on M is a function $F : TM \to [0, \infty)$ with the following properties:

(1) F is $\mathcal{C}^\infty$ on TM_0.

(2) At each point $x \in M$, the restriction $F_x := F|_{T_x M}$ is a Minkowski norm on $T_x M$.

Riemannian metrics are a special case of Finsler metrics: they are Finsler metrics with the quadratic restriction [3].

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate in which the geodesics are straight lines as point sets. It is known that every locally projectively flat Finsler metrics is of scalar curvature [3]. Similar results on projectively flat Finsler metric were discussed by Chern and Shen in [3].

A Finsler metric $F = F(x, y)$ on an open domain $\mathcal{U} \subset \mathbb{R}^n$ is said to be projectively flat in $\mathcal{U}$ if all geodesics are straight lines. Let G^i denote the spray coefficients of F, which are given by

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\}, \tag{2.3}$$

where $(g^{ij}) = (\frac{1}{2}[F^2]_{y^i y^j})$.

In this case, the flat curvature K is a scalar function on $T\mathcal{U}$ given by

$$K = \frac{P^2 - P_{x^m} y^m}{F^2}, \tag{2.4}$$

where

$$P = \frac{F_{x^k} y^k}{2F} \tag{2.5}$$

is said to be the projective factor [4].

3. The solutions of the PDE

In order to solve the following linear PDE,

$$s\phi_{bs} + b\phi_{ss} - \phi_b = 0, \tag{3.1}$$

where $\phi = \phi(b, s)$. First, we would like to point out that this partial differential equation is of mixed type. When $s \neq 0$, equation (3.1) is hyperbolic, and when $s = 0$, equation (3.1) is parabolic. This equation is interesting, similar to the famous Tricomi equation in gas dynamics.

Second, equation (3.1) obviously has the trivial solution

$$\phi(b, s) = C_1 b + C_2 s + C_3. \tag{3.2}$$

Substituting (3.2) into (3.1) gives

$$C_1 = 0.$$

Thus, equation (3.1) has the trivial solution

$$\phi(b, s) = s + C_1'. \tag{3.3}$$

Third, we can find some solution by the method of variable separation. Let

$$\phi(b, s) = f(b)g(s). \tag{3.4}$$

Substituting (3.4) into (3.1) gives

$$sf'(b)g'(s) + bf(b)g''(s) - f'(b)g(s) = 0, \tag{3.5}$$

where $'$ denotes the differential with respect to x. Then it follows from (3.5) that

$$\frac{g''(s)}{g(s) - sg'(s)} = \frac{f(b)}{bf(b)} = \frac{\lambda}{2}, \tag{3.6}$$

where λ is a position constant.

Equation (3.6) is equivalent to

$$f(b) = \frac{\lambda}{2} bf(b), \tag{3.7}$$

and

$$g''(s) = \frac{\lambda}{2}(g(s) - sg'(s)). \tag{3.8}$$

The general solution to (3.7) is given by

$$f(b) = C_1 e^{\lambda b^2}, \tag{3.9}$$

and equation (3.8) can be solved as

$$g(s) = C_2 s - C_3[e^{-\lambda s^2} + s\sqrt{\pi\lambda} erf(\sqrt{\lambda} s)] \tag{3.10}$$

where erf(,) denotes the error function and is defined by

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \tag{3.11}$$

Thus, combining (3.9) and (3.10) leads to the following solution to (3.1):

$$\phi(b,s) = e^{\lambda b^2}[C_1 s - C_2(e^{-\lambda s^2} + s\sqrt{\pi\lambda} erf(\sqrt{\lambda} s)], \tag{3.12}$$

where C_1, C_2 are arbitrary constants.

4. Proof

Now we will manufacture a class of projectively flat Finsler metric by (3.12), and we will get its scalar flag curvature.

Let

$$F = \alpha\phi(b,s) : T\Omega \to [0,+\infty),$$

$$\alpha = |y| \quad , \quad \beta =< x, y >,$$

$$\phi(b,s) = e^{\lambda b^2}[C_1 s - C_2(e^{-\lambda s^2} + s\sqrt{\pi\lambda} erf(\sqrt{\lambda} s)]. \tag{4.1}$$

By Lemma (2.1), we know that

$$F = \alpha e^{\lambda b^2}[C_1 s - C_2(e^{-\lambda s^2} + s\sqrt{\pi\lambda} erf(\sqrt{\lambda} s)] \tag{4.2}$$

is a projectively flat spherically symmetric Finsler metric.

By a simple calculation,we get

$$\alpha_{x^k} y^k = 0, \quad s_{x^k} y^k = |y|, \quad [s^2]_{x^k} y^k = 2 < x, y >,$$

$$b_{x^k} y^k = \frac{< x, y >}{|y|}, \quad [b^2]_{x^k} y^k = 2 < x, y > . \tag{4.3}$$

$$\begin{aligned} F_{x^k} y^k \quad &= \alpha e^{\lambda b^2} \cdot \left[C_1 s_{x^k} y^k + C_2 e^{-\lambda s^2} \cdot \lambda [s^2]_{x^k} y^k - C_2\sqrt{\pi\lambda} erf(\sqrt{\lambda} s) s_{x^k} y^k - C_2 s\sqrt{\pi\lambda} \cdot 2\sqrt{\tfrac{\lambda}{\pi}} s_{x^k} y^k \right] \\ &+ \alpha e^{\lambda b^2} \cdot \lambda [b^2]_{x^k} y^k [C_1 s - C_2(e^{-\lambda s^2} + s\sqrt{\pi\lambda} erf(\sqrt{\lambda} s)] \\ &= 2\lambda < x, y > \alpha f(b) g(s) + \alpha f(b)[C_1|y| - C_2|y|\sqrt{\pi\lambda} erf(\sqrt{\lambda} s)], \end{aligned} \tag{4.4}$$

where

$$f(b) = e^{\lambda b^2}, \tag{4.5}$$

$$g(s) = C_1 s - C_2(e^{-\lambda s^2} + s\sqrt{\pi\lambda} erf(\sqrt{\lambda} s)). \tag{4.6}$$

$$\begin{aligned} P &= \tfrac{F_{x^k} y^k}{2F} = \lambda < x, y > + \tfrac{C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)}{2g(s)} \\ &= \lambda < x, y > + \tfrac{1}{2}H, \end{aligned} \tag{4.7}$$

where

$$H = \frac{C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)}{g(s)} \tag{4.8}$$

$$\begin{aligned} H_{x^k} y^k &= \frac{-C_2|y|\sqrt{\lambda\pi} \cdot 2\sqrt{\frac{\lambda}{\pi}}e^{-\lambda s^2} \cdot |y|g(s) - [C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)]^2}{g^2(s)} \\ &= -\frac{2C_2|y|^2\lambda e^{-\lambda s^2}}{g(s)} - \frac{[C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)]^2}{g^2(s)}. \end{aligned} \tag{4.9}$$

$$\begin{aligned} P^2 &= \lambda^2 < x, y >^2 + \lambda < x, y > \cdot \left[\frac{C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)}{g(s)}\right] + \frac{[C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)]^2}{4g^2(s)} \\ &= \lambda^2 < x, y >^2 + \lambda|y|^2 \cdot \left[\frac{C_1 s - C_2 s\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)}{g(s)}\right] + \frac{[C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)]^2}{4g^2(s)}. \end{aligned} \tag{4.10}$$

By (4.10),we get

$$\begin{aligned} P_{x^k} y^k &= \lambda|y|^2 + \tfrac{1}{2}H_{x^k} y^k \\ &= \lambda|y|^2 - \lambda|y|^2 \cdot \left[\tfrac{C_2 e^{-\lambda s^2}}{g(s)}\right] - \tfrac{[C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)]^2}{2g^2(s)}. \end{aligned} \tag{4.11}$$

$$\begin{aligned} P^2 - P_{x^k} y^k &= \lambda|y|^2 \cdot \left[\tfrac{C_1 s - C_2 s\sqrt{\pi\lambda}erf(\sqrt{\lambda}s) - C_2 e^{-\lambda s^2} + 2C_2 e^{-\lambda s^2}}{g(s)}\right] + \tfrac{3[C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)]^2}{4g^2(s)} \\ &+ \lambda^2 < x, y >^2 - \lambda|y|^2 \\ &= \lambda^2 < x, y >^2 + \lambda|y|^2 \cdot \left[\tfrac{2C_2 e^{-\lambda s^2}}{g(s)}\right] + \tfrac{3[C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)]^2}{4g^2(s)}. \end{aligned} \tag{4.12}$$

By (2.4) and (4.12), we get

$$\begin{aligned} K &= \tfrac{P^2 - P_{x^k} y^k}{F^2} \\ &= \left[\lambda^2 < x, y >^2 + \lambda|y|^2 \cdot \tfrac{2C_2 e^{-\lambda s^2}}{g(s)} + \tfrac{3[C_1|y| - C_2|y|\sqrt{\pi\lambda}erf(\sqrt{\lambda}s)]^2}{4g^2(s)}\right] \\ &/ \left[|y|^2 f^2(b) g^2(s)\right] \\ &= \tfrac{\lambda^2 s^2}{f^2(b) g^2(s)} - \tfrac{2C_2\lambda e^{-\lambda s^2}}{f^2(b) g^3(s)} + \tfrac{3H^2}{4|y|^2 f^2(b) g^2(s)}. \end{aligned} \tag{4.13}$$

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