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## Generalized Heineken–Mohamed type groups

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**Abstract:** We prove that a torsion group  $G$  with all subgroups subnormal is a nilpotent group or  $G = N(A_1 \times \cdots \times A_n)$  is a product of a normal nilpotent subgroup  $N$  and  $p_i$ -subgroups  $A_i$ , where  $A_i = A_1^{(i)} \cdots A_{m_i}^{(i)} \triangleleft G$ ,  $A_j^{(i)}$  is a Heineken–Mohamed type group, and  $p_1, \dots, p_n$  are pairwise distinct primes ( $n \geq 1$ ;  $i = 1, \dots, n$ ;  $j = 1, \dots, m_i$  and  $m_i$  are positive integers).

**Key words:** Nilpotent group, indecomposable group, Heineken–Mohamed type group

### 1. Introduction

The concept of an  $HM^*$ -group (i.e. a  $p$ -group  $G$  with the nilpotent derived subgroup  $G'$  and the divisible Černikov quotient group  $G/G'$ ) was introduced by Asar [2] in the class of infinite  $p$ -groups. Quasicyclic groups, Heineken–Mohamed type groups (i.e. nonnilpotent groups with nilpotent and subnormal proper subgroups) and Čarin groups (see [5, 10] and [22, Example 1.5.1]) are  $HM^*$ -groups. Recall that a group  $G$  is called *indecomposable* if any two proper subgroups generate a proper subgroup in  $G$  and is called *decomposable* otherwise. Heineken–Mohamed type groups are indecomposable  $p$ -groups with the normalizer condition (see examples in [15, 18, 24]). With indecomposable groups are connected barely transitive groups (see [21, Lemma 2.10]).

Casolo [11] proved that a torsion group  $G$  with all subgroups subnormal contains a normal nilpotent subgroup  $N$  such that  $G/N$  is an abelian divisible group of finite rank (see also [12]). We make this result more precise in the following:

**Theorem 1.1** *Let  $G$  be a torsion group with all subgroups subnormal. Then  $G$  is a nilpotent group or  $G = NA$  is a product of a normal nilpotent subgroup  $N$  and a nonnilpotent subgroup*

$$A = A_1 \times \cdots \times A_n \quad (n \geq 1),$$

where  $A_i$  is a  $p_i$ -group,  $A_i = A_1^{(i)} \cdots A_{m_i}^{(i)} \triangleleft G$ ,  $A_j^{(i)}$  is a Heineken–Mohamed type group, and  $p_1, \dots, p_n$  are pairwise distinct primes ( $i = 1, \dots, n$ ;  $j = 1, \dots, m_i$  and  $m_i$  are positive integers).

Recently Smith [28] established that a group  $G$  with all subgroups subnormal such that its torsion part is a  $\pi$ -group, where  $\pi$  is a finite set of primes, is nilpotent-by-(divisible Černikov).

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Throughout,  $p$  will always denote a prime,  $\mathbb{C}_{p^\infty}$  the quasicyclic  $p$ -group, and  $\mathbb{Z}$  the ring of integers. For a group  $G$ ,  $G', G'', \dots, G^{(n)}, \dots$  will indicate the terms of derived series of  $G$  and  $G^n$  the subgroup of  $G$  generated by the  $n$ th powers of all elements in  $G$ .

For notations see [19, 23, 27].

## 2. Preliminary results

In this part we collect some results concerning groups  $G$  with the locally nilpotent derived subgroup  $G'$  and the divisible Černikov quotient  $p$ -group  $G/G'$ .

**Lemma 2.1** *Let  $G$  be a group with the locally nilpotent derived subgroup  $G'$  and the quotient group  $G/G'$  be a divisible Černikov  $p$ -group. Then the following hold:*

- (i)  $G' = [G', G]$ ;
- (ii) each nontrivial homomorphic image of  $G$  has infinite exponent and, in particular,  $G = G^m$  for each positive integer  $m$ ;
- (iii)  $G'$  contains no proper  $G$ -invariant subgroup of finite index;
- (iv) if  $G$  is a hypercentral group, then it is a divisible Černikov  $p$ -group.

**Proof** For the proof see, e.g., [2, 3]. □

**Lemma 2.2** [20, Chapter XV, § 63] *Let  $G$  be a group with the normalizer condition. If  $H$  is a normal hypercentral subgroup of  $G$  and the quotient group  $G/H$  is cyclic, then  $G$  is a hypercentral group.*

**Lemma 2.3** [13, Lemma 1.18] *If a hypercentral group  $G = AB$  is a product of an abelian normal subgroup  $A$  and a divisible abelian subgroup  $B$ , then  $G$  is abelian.*

**Lemma 2.4** [13, Proposition 1.7] *If a  $p$ -group  $G$  contains an abelian subgroup of finite index, then it is hypercentral.*

**Lemma 2.5** *Let  $G$  be a nonhypercentral group with the normalizer condition. If  $G$  contains a normal hypercentral soluble subgroup  $A$  with the quasicyclic quotient group  $G/A$ , then every nonhypercentral subgroup is subnormal in  $G$ .*

**Proof** If  $H$  is a proper nonhypercentral subgroup of  $G$ , then, in view of Lemma 2.2,  $G = AH$  and, by [17, Lemma],

$$G/A'(A \cap H)$$

is a hypercentral group. Therefore, it is abelian by Lemma 2.3 and  $A'H \triangleleft G$ . Since  $A'H/A''(A' \cap H)$  is hypercentral, by the same argument we obtain that  $A''H \triangleleft A'H$ . By induction,  $H$  is subnormal in  $G$ . □

**Lemma 2.6** *The following properties hold:*

- (i) Any indecomposable soluble torsion group  $G$  is a locally finite  $p$ -group.
- (ii) Let  $G$  be a group with the hypercentral derived subgroup  $G'$  and the quotient group  $G/G'$  be a divisible Černikov  $p$ -group. If  $G$  is indecomposable, then it is a locally finite  $p$ -group.
- (iii) Any indecomposable locally nilpotent group is a  $p$ -group.

**Proof** (i) If  $G$  is abelian and indecomposable, then it is a quasicyclic or cyclic  $p$ -group. Therefore, we suppose that  $G$  is nonabelian. Then  $G/G'$  is a divisible Černikov  $p$ -group. Assume that there is an integer  $k$  such that  $G/G^{(k)}$  is a  $p$ -group and  $G^{(k)}$  is not a  $p$ -group, where  $1 \leq k \leq n-1$  and  $n$  is the derived length of  $G$ .

The group  $G$  is a soluble torsion group and so it is a locally finite group. Since  $G$  is soluble and indecomposable, it is countable. Then  $G$  has a local system consisting of its finite subgroups. Since the group  $G$  contains an element of order  $p$  and, by Hall's Theorem, it has a Hall  $p$ -subgroup and a Hall  $p'$ -subgroup,  $G$  is generated by them. Then  $G$  being indecomposable implies that  $G$  is a  $p$ -group.

(ii) We denote the derived subgroup  $G'$  by  $H$ . If  $H$  is not torsion, then the torsion part  $\tau H$  is normal in  $H$  and the quotient group  $H_1 = H/\tau H$  is torsion-free hypercentral. Then  $H_1/H'_1$  is not torsion (see, e.g., [27, Exercises 12.2.6]) and therefore without loss of generality we can assume that  $H$  is a torsion-free abelian group. Let  $q$  be a prime different from  $p$ . By [9, Lemma 2.3], there exists a proper submodule  $N$  of a  $\mathbb{Z}[G/H]$ -module  $H$  such that  $H/N$  is torsion as a group and contains an element of order  $q$ . Then  $G/N$  is a decomposable group, a contradiction. Hence,  $G$  is a torsion group and this yields that  $G$  is a locally finite  $p$ -group.

(iii) It follows by the same argument as in (ii). □

If  $G$  is a barely transitive group with the locally nilpotent derived subgroup  $G'$  and the quotient group  $G/G'$  is a divisible Černikov  $p$ -group, then  $G$  is an indecomposable locally finite  $p$ -group by Lemma 2.3 and [7, Proposition 2.4]. In view of Lemma 2.6, it is a very interesting open question whether nontorsion (and therefore non-“barely transitive”) indecomposable soluble groups exist.

**Lemma 2.7** *Let  $G$  be a group with the hypercentral derived subgroup  $G'$  and the divisible Černikov quotient  $p$ -group  $G/G'$ . Then the following hold:*

- (1) if the derived subgroup  $G'$  has no proper supplements in  $G$ , then  $G$  is a  $p$ -group;
- (2) if  $G$  is an indecomposable group, then every proper subgroup of it is hypercentral;
- (3) if  $G$  is a soluble group with the derived subgroup  $G'$  of finite exponent, then every finite subgroup is subnormal in  $G$ ;
- (4) if  $G$  is a soluble group with the normalizer condition, then
  - (i)  $G'$  is not supplemented nontrivially in  $G$ ;
  - (ii)  $G/G' \cong \mathbb{C}_{p^\infty}$  if and only if  $G$  is an indecomposable group;
  - (iii) if  $G'$  is nilpotent of finite exponent, then all subgroups are subnormal in  $G$ .

**Proof** (1) We can prove by the same argument as in the proof of Lemma 2.6(ii).

(2) Since  $G/G'$  is a quasicyclic group and  $G'H$  is proper in  $G$  for any proper subgroup of  $G$ , we conclude that  $G'H = G'\langle a \rangle$  for some element  $a \in G$ . Hence,  $G'H$  is a hypercentral group.

(3) If  $F$  is a finite subgroup of  $G$ , then  $G'F$  is a proper subgroup of  $G$  and we deduce that  $\overline{G'F}$  is a nilpotent subgroup of  $\overline{G} = G/G''$  by [4, Lemma 3.8]. As a consequence,  $G''F$  is subnormal in  $G$ . Repeating this process, we obtain that  $G^{(n)}F$  is subnormal in  $G$  for any positive integer  $n$ .

(4) Let  $G$  be a soluble group with the normalizer condition. Since  $G'$  is hypercentral and  $G/G'$  is a  $p$ -group,  $G$  is a  $p$ -group. By [17, Lemma],  $G = G'S$  for some proper subgroup  $S$  and it follows that the quotient group

$$\overline{G} = G/G''(G' \cap S)$$

is hypercentral, which leads to a contradiction. If, moreover,  $G'$  is of finite exponent, then every subgroup is subnormal in  $G$  by condition (3) and [2, Lemma 3.6]. □

**Lemma 2.8** *Let  $G$  be a nonhypercentral group with the hypercentral derived subgroup  $G'$  and the divisible Černikov quotient  $p$ -group  $G/G'$ . Then the following hold:*

(1) *if all subgroups of  $G$  are subnormal, then*

$$G = G_1 \cdots G_n \cdot D \quad (n \geq 1),$$

*where  $D$  is a nilpotent group,  $G_i$  is a Heineken–Mohamed type group,  $D' \leq G' = G'_1 \cdots G'_n \leq D$  and  $G'_i = [G'_i, G_i] = [G', G_i] = [G'_i, G]$ ;*

(2) *if  $G/G'$  is a quasicyclic  $p$ -group, then*

- (a)  *$G$  satisfies the normalizer condition if and only if it is a minimal nonhypercentral group;*
- (b) *every proper subgroup is subnormal in  $G$  if and only if  $G$  is a Heineken–Mohamed type group.*

**Proof** (1) By definition,

$$\overline{G} = G/G' = \overline{T}_1 \times \cdots \times \overline{T}_s$$

is a direct product of quasicyclic  $p$ -subgroups  $\overline{T}_i$  ( $i = 1, \dots, s$ ). If  $T_i$  is an inverse image of  $\overline{T}_i$  in  $G$ , then

$$G' \leq T_i \triangleleft G.$$

Since  $G$  is nonhypercentral, there exists an integer  $l$  ( $1 \leq l \leq s$ ) such that  $T_1, \dots, T_l$  are nonhypercentral, while  $T_{l+1}, \dots, T_s$  are hypercentral groups. We denote the product  $T_{l+1} \cdots T_s$  by  $D$ . Then  $D$  is hypercentral and, by [26, Theorem 2.7], it is a nilpotent group.

Let  $T = T_i$  ( $i = 1, \dots, l$ ). Obviously,

$$[G', T] \leq T' \leq G'.$$

Moreover, the quotient group  $T/[G', T]$  (and respectively  $T/[T', T]$ ) is nilpotent (and consequently abelian), and therefore we obtain that

$$T' = [T', T] = [T', G] = [G', T].$$

Since  $G/(T'_1 \cdots T'_l)$  is a hypercentral group, we conclude that the derived subgroup  $G' = T'_1 \cdots T'_l$ .

(2) Let  $G/G'$  be a quasicyclic  $p$ -group.

(a) Suppose that  $G$  satisfies the normalizer condition. If it is decomposable, then  $G = G'S$  for some  $S < G$ , and so the quotient group

$$G/G''(G' \cap S)$$

is abelian by [17, Lemma], a contradiction. Hence,  $G$  is an indecomposable group. By Lemma 2.7(2),  $G$  is a minimal nonhypercentral group. The converse is an immediate consequence of the fact that  $G'K$  is proper in  $G$  for every proper subgroup  $K$  of  $G$  and therefore  $K$  is not equal to its normalizer  $N_G(K)$ .

(b) Assume that every proper subgroup is subnormal in  $G$ . Then  $G$  satisfies the normalizer condition. By [11, Theorem 1], the derived subgroup  $G'$  is nilpotent. If  $G$  is a hypercentral group, then, by [26, Theorem 2.7], it is nilpotent and, consequently,  $G \cong \mathbb{C}_{p^\infty}$ . Therefore, we assume that  $G$  is nonhypercentral and, by (a), it is a minimal nonhypercentral group. If we assume that  $G$  is decomposable (and so  $G = G'S$  for some  $S < G$ ), then, by [17, Lemma],

$$\bar{G} = G/(G' \cap S) = \bar{G}' \times \bar{S}$$

is a hypercentral group. However, then it is abelian, a contradiction. Hence,  $G$  is an indecomposable group and for any  $K < G$  there exists  $a \in G$  such that  $G'K = G'\langle a \rangle < G$  and, by [1, Lemma 2.4],  $G'K$  is a nilpotent group. Thus,  $G$  is a Heineken–Mohamed type group.

The converse is clear. □

**Corollary 2.9** *Let  $G$  be a nonabelian indecomposable soluble group. If  $G'$  is a nilpotent group of finite (respectively infinite) exponent, then  $G$  is a Heineken–Mohamed type group (respectively a minimal nonhypercentral group).*

**Proof** By Lemma 2.6,  $G$  is a  $p$ -group for some prime  $p$ . Let  $B$  be a proper subgroup of  $G/G'$  and  $E(G', B)$  be a subgroup of  $G$ , which is an extension of  $G'$  by  $B$ . If  $G'$  is a nilpotent group of finite (respectively infinite) exponent, then [4, Lemma 3.8] (respectively Lemma 2.4) yields that the quotient group  $E(G', B)/G''$  is nilpotent (respectively hypercentral). By the Hall Theorem [14, Theorem 7] (respectively the Betten Theorem [8]),  $E(G', B)$  is a nilpotent (respectively hypercentral) group. Hence,  $G$  is a minimal nonnilpotent (respectively nonhypercentral) group. Moreover, if the derived subgroup  $G'$  is of finite exponent, then, by Lemma 2.7,  $G$  is a Heineken–Mohamed type group. □

Belyaev and Kuzucuoglu [6, Corollary 1] proved that a Heineken–Mohamed type group has a faithful barely transitive representation. Then we prove the following:

**Corollary 2.10** *A locally finite barely transitive group  $G$  with almost locally  $p$ -soluble point stabilizer is a Heineken–Mohamed type group or  $G \cong \mathbb{C}_{p^\infty}$ .*

**Proof** By [21, Lemma 2.10],  $G$  is indecomposable and, by Proposition 3 of [16], the derived subgroup  $G'$  is nilpotent of finite exponent. If  $G$  is nonabelian, then it is a Heineken–Mohamed type group by Corollary 2.9. □

### 3. Proof of Theorem 1.1

Suppose that  $G$  is a nonnilpotent  $p$ -group. As it was proved in [11], a torsion group  $G$  with subnormal proper subgroups contains a nilpotent normal subgroup  $N$  such that  $G/N$  is a divisible abelian Černikov  $p$ -group. If

$N^p = N$ , then, by [25, Lemma 1], a subgroup  $N\langle a \rangle$  is nilpotent (and so abelian) for any  $a \in G$ , a contradiction. Hence,  $N^p \neq N$ .

Let  $k, l$  be nonnegative integers. By [2, Lemma 3.3],

$$\overline{G} = G/N^{p^k} = \overline{T} \overline{N}$$

is a product of a maximal normal  $HM^*$ -subgroup  $\overline{T}$  and a normal nilpotent subgroup  $\overline{N}$ . If we denote the inverse image of  $\overline{T}$  by  $T$  and  $M = T \cap N$ , then, by [2, Lemma 3.3],

$$T/M^{p^k} = \overline{T}_0 \overline{M},$$

where  $\overline{T}_0$  is a maximal normal  $HM^*$ -subgroup of  $T/M^{p^k}$ . Let  $T_0$  be an inverse image of  $\overline{T}_0$  in a group  $T$ . If  $l < k$ , then

$$T_0/M^{p^k}T'_0 \text{ and } T_0/M^{p^l}T'_0$$

are divisible Černikov groups,  $M^{p^k}T'_0 \leq M^{p^l}T'_0$ , and so the section

$$M^{p^l}T'_0/M^{p^k}T'_0$$

is a finite group. There exists a subgroup  $D$  of  $T_0$  such that  $D/T'_0$  is the divisible part of  $T_0/T'_0$ . Then  $T = T_0M = DM$  and  $D' = T'_0$ . This yields that  $D$  is an  $HM^*$ -group. The conclusion follows from Lemma 2.8.

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