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The geometry of hemi-slant submanifolds of a locally product Riemannian manifold

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Abstract: In the present paper, we study hemi-slant submanifolds of a locally product Riemannian manifold. We prove that the anti-invariant distribution involved in the definition of hemi-slant submanifold is integrable and give some applications of this result. We get a necessary and sufficient condition for a proper hemi-slant submanifold to be a hemi-slant product. We also study these types of submanifolds with parallel canonical structures. Moreover, we give two characterization theorems for the totally umbilical proper hemi-slant submanifolds. Finally, we obtain a basic inequality involving Ricci curvature and the squared mean curvature of a hemi-slant submanifold of a certain type of locally product Riemannian manifolds.

Key words: Locally product manifold, hemi-slant submanifold, slant distribution

1. Introduction

Study of slant submanifolds was initiated by Chen [8], as a generalization of both holomorphic and totally real submanifolds of a Kähler manifold. Slant submanifolds have been studied in different kind of structures: almost contact [13], neutral Kähler [4], Lorentzian Sasakian [2], and Sasakian [6] by several geometers. N. Papaghiuc [14] introduced semi-slant submanifolds of a Kähler manifold as a natural generalization of slant submanifold. Carriazo [7], introduced bi-slant submanifolds of an almost Hermitian manifold as a generalization of semi-slant submanifolds. One of the classes of bi-slant submanifolds is that of anti-slant submanifolds, which are studied by Carriazo [7]. However, Şahin [18] called these submanifolds hemi-slant submanifolds because the name anti-slant indicates it has no slant factor. We observe that a hemi-slant submanifold is a special case of generic submanifold introduced by Ronsse [16]. Since then many geometers have studied hemi-slant submanifolds in different kinds of structures: Kähler [3, 18], nearly Kähler [21], generalized complex space form [20], and almost Hermitian [19]. In some cases, we should note that hemi-slant submanifolds are also studied under the name pseudo-slant submanifolds; see [11] and [21]. Furthermore, the submanifolds of a locally product Riemannian manifold have been studied by many geometers. For example, Adati [1] defined and studied invariant and anti-invariant submanifolds, while Bejancu [5] and Pitis [15] studied semi-invariant submanifolds. Slant and semi-slant submanifolds of a locally product Riemannian manifold are examined by Şahin [17] and Li and Liu [12]. In this paper, we study the geometry of hemi-slant submanifolds of a locally product Riemannian manifold in detail.

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2. Preliminaries

This section is devoted to preliminaries. Actually, in subsection 2.1 we present the basic background needed for a locally product Riemannian manifold. Theory of submanifolds and distributions related to the study are given in subsection 2.2.

2.1. Locally product Riemannian manifolds

Let $\bar{M}$ be an m-dimensional manifold with a tensor field of type (1,1) such that

$$F^2 = I, (F \neq \pm I) \ , \tag{2.1}$$

where I is the identity morphism on the tangent bundle $T\bar{M}$ of $\bar{M}$. Then we say that $\bar{M}$ is an *almost product manifold* with almost product structure F. If an almost product manifold $(\bar{M}, F)$ admits a Riemannian metric g such that

$$g(F\bar{U}, F\bar{V}) = g(\bar{U}, \bar{V}) \tag{2.2}$$

for all $\bar{U}, \bar{V} \in T\bar{M}$, then $\bar{M}$ is called an *almost product Riemannian manifold*.

Next, we denote by $\overline{\nabla}$ the Riemannian connection with respect to g on $\bar{M}$. We say that $\bar{M}$ is a *locally product Riemannian manifold*, (briefly, *l.p.R. manifold*) if we have

$$(\overline{\nabla}_{\bar{U}} \ F)\bar{V} = 0 \,, \tag{2.3}$$

for all $\bar{U}, \bar{V} \in T\bar{M}$ [22].

2.2. Submanifolds

Let M be a submanifold of a l.p.R. manifold $(\bar{M}, g, F)$. Let $\overline{\nabla}, \nabla,$ and $\nabla^{\perp}$ be the Riemannian, induced Riemannian, and induced normal connection in $\bar{M}, M$ and the normal bundle $T^{\perp}M$ of M, respectively. Then for all $U, V \in TM$ and $\xi \in T^{\perp}M$ the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_U V = \nabla_U V + h(U, V) \tag{2.4}$$

and

$$\overline{\nabla}_U \xi = -A_{\xi} U + \nabla^{\perp}_U \xi \tag{2.5}$$

where h is the second fundamental form of M and A_{ξ} is the Weingarten endomorphism associated with ξ. The second fundamental form h and the shape operator A are related by

$$g(h(U, V), \xi) = g(A_{\xi} U, V) \ . \tag{2.6}$$

A submanifold M is said to be *totally geodesic* if its second fundamental form vanishes identically, that is, $h = 0$, or equivalently $A_{\xi} = 0$. We say that M is *totally umbilical* submanifold in $\overline{M}$ if for all $U, V \in TM$ we have

$$h(U, V) = g(U, V)H \ , \tag{2.7}$$

where H is the mean curvature vector field of M in $\bar{M}$. A normal vector field ξ is said to be parallel, if $\nabla^{\perp}_{U}\xi = 0$ for each vector field $U \in TM$.

The Riemannian curvature tensor $\overline{R}$ of $\bar{M}$ is given by

$$\overline{R}(\bar{U}, \bar{V}) = \left[\overline{\nabla}_{\bar{U}}, \overline{\nabla}_{\bar{V}}\right] - \overline{\nabla}_{[\bar{U}, \bar{V}]}, \tag{2.8}$$

where $\bar{U}, \bar{V} \in T\bar{M}$.
Then the Codazzi equation is given by

$$\left(\overline{R}(U, V)W\right)^{\perp} = (\overline{\nabla}_{U}h)(V, W) - (\overline{\nabla}_{V}h)(U, W) \tag{2.9}$$

for all U V, $W \in TM$. Here, $\perp$ denotes the normal component and the covariant derivative of h, denoted by $\overline{\nabla}_{U}h$ is defined by

$$(\overline{\nabla}_{U}h)(V, W) = \nabla^{\perp}_{U}h(V, W) - h(\nabla_{U}V, W) - h(V, \nabla_{U}W). \tag{2.10}$$

Now, we write

$$FU = TU + NU \ , \tag{2.11}$$

for any $U \in TM$. Here TU is the tangential part of FU, and NU is the normal part of FU. Similarly, for any $\xi \in T^{\perp}M$, we put

$$F\xi = t\xi + \omega\xi \ , \tag{2.12}$$

where $t\xi$ is the tangential part of $F\xi$, and $\omega\xi$ is the normal part of $F\xi$.

A distribution $\mathcal{D}$ on a manifold $\bar{M}$ is called *autoparallel* if $\overline{\nabla}_{X}Y \in \mathcal{D}$ for any $X, Y \in \mathcal{D}$ and called *parallel* if $\overline{\nabla}_{U}X \in \mathcal{D}$ for any $X \in \mathcal{D}$ and $U \in T\bar{M}$. If a distribution $\mathcal{D}$ on $\bar{M}$ is autoparallel, then it is clearly integrable, and by Gauss formula $\mathcal{D}$ is totally geodesic in $\bar{M}$. If $\mathcal{D}$ is parallel then the orthogonal complementary distribution $\mathcal{D}^{\perp}$ is also parallel, which implies that $\mathcal{D}$ is parallel if and only if $\mathcal{D}^{\perp}$ is parallel. In this case $\bar{M}$ is locally product of the leaves of $\mathcal{D}$ and $\mathcal{D}^{\perp}$. Let M be a submanifold of $\bar{M}$. For two distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M, we say that M is $(\mathcal{D}_1, \mathcal{D}_2)$ mixed totally geodesic if for all $X \in \mathcal{D}_1$ and $Y \in \mathcal{D}_2$ we have $h(X, Y) = 0$, where h is the second fundamental form of M [20, 22].

3. Hemi-slant submanifolds of a locally product Riemannian manifold

In this section, we define the notion of hemi-slant submanifold and observe its effect on the tangent bundle of the submanifold and canonical projection operators and start to study hemi-slant submanifolds of a locally product Riemannian manifold.

Let $(\bar{M}, g, F)$ be a locally product Riemannian manifold and let M be a submanifold of $\bar{M}$. A distribution $\mathcal{D}$ on M is said to be a *slant distribution* if for $X \in \mathcal{D}_p$, the angle θ between FX and $\mathcal{D}_p$ is constant, i.e. independent of $p \in M$ and $X \in \mathcal{D}_p$. The constant angle θ is called the slant angle of the slant distribution $\mathcal{D}$. A submanifold M of $\bar{M}$ is said to be a *slant submanifold* if the tangent bundle TM of M is slant [12, 17]. Thus, the $F-$invariant and $F-$anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold that is neither $F-$invariant nor $F-$anti-invariant is called a *proper* slant submanifold.

Definition 3.1 *A* hemi-slant submanifold M *of a locally product Riemannian manifold* $\bar{M}$ *is a submanifold that admits two orthogonal complementary distributions* $\mathcal{D}^{\perp}$ *and* $\mathcal{D}^{\theta}$ *such that*

(a) TM admits the orthogonal direct decomposition $TM = \mathcal{D}^{\perp} \oplus \mathcal{D}^{\theta}$

(b) The distribution $\mathcal{D}^{\perp}$ is $F-$anti-invariant, i.e. $F\mathcal{D}^{\perp} \subseteq T^{\perp}M$.

(c) The distribution $\mathcal{D}^{\theta}$ is slant with slant angle θ.

In this case, we call θ the slant angle of M. Suppose the dimension of distribution $\mathcal{D}^{\perp}$ (resp. $\mathcal{D}^{\theta}$) is p (resp. q). Then we easily see the following particular cases.

(d) If $q = 0$, then M is an anti-invariant submanifold [1].

(e) If $p = 0$ and $\theta = 0$, then M is an invariant submanifold [1].

(f) If $p = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then M is a proper slant submanifold [17].

(g) If $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.

(h) If $p \neq 0$ and $\theta = 0$, then M is a semi-invariant submanifold [5].

We say that the hemi-slant submanifold M is *proper* if $p \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$.

Lemma 3.2 *Let* M *be a proper hemi-slant submanifold of a l.p.R. manifold* $\bar{M}$. *Then we have,*

$$F(\mathcal{D}^{\perp}) \perp N(\mathcal{D}^{\theta}) \ . \tag{3.1}$$

Proof For any $X \in \mathcal{D}^{\perp}$ and $Z \in \mathcal{D}^{\theta}$, using (2.2) and (2.11), we have
$g(FX, NZ) = g(FX, FZ) = g(X, Z) = 0$. This completes the proof. □

In view of Lemma 3.2, for a hemi-slant submanifold M of a l.p.R. manifold $\bar{M}$, the normal bundle $T^{\perp}M$ of M is decomposed as

$$T^{\perp}M = F(\mathcal{D}^{\perp}) \oplus N(\mathcal{D}^{\theta}) \oplus \mu \ , \tag{3.2}$$

where μ is the orthogonal complementary distribution of $F(\mathcal{D}^{\perp}) \oplus N(\mathcal{D}^{\theta})$ in $T^{\perp}M$ and it is the invariant subbundle of $T^{\perp}M$ with respect to F.

The following facts follow easily from (2.1), (2.11), and (2.12) and will be used later.

$$T^2 + tN = I, \tag{3.3a}$$

$$\omega^2 + Nt = I, \tag{3.3b}$$

$$NT + \omega N = 0, \tag{3.3c}$$

$$Tt + t\omega = 0. \tag{3.3d}$$

As in a slant submanifold [17], for a hemi-slant submanifold M of a l.p.R. manifold $\overline{M}$, we have

$$T^2 Z = \cos^2\theta Z \, , \tag{3.4}$$

$$g(TZ, TW) = \cos^2\theta g(Z, W) \tag{3.5}$$

and

$$g(NZ, NW) = \sin^2\theta g(Z, W) \ , \tag{3.6}$$

where $Z, W \in \mathcal{D}^\theta$.

Here, we omit the proofs of equations (3.4)–(3.6), because the proof of (3.4) is the same as the proof of Theorem 3.1 of [17] and the other proofs are also the same as the proofs of equations (3.3) and (3.4) in Lemma 3.1 of [17].

Lemma 3.3 *Let M be a proper hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then we have,*

$$T(\mathcal{D}^\perp) = \{0\}, \tag{3.7a}$$

$$T(\mathcal{D}^\theta) = \mathcal{D}^\theta. \tag{3.7b}$$

Proof Since $\mathcal{D}^\perp$ is anti-invariant with respect to F, (a) follows from (2.11). For any $Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^\perp$, using (2.1), (2.2), and (2.11), we have $g(TZ, X) = g(FZ, X) = g(Z, FX) = 0$. Hence, we conclude that $T(\mathcal{D}^\theta) \perp \mathcal{D}^\perp$. Since $T(\mathcal{D}^\theta) \subseteq TM$, it follows that $T(\mathcal{D}^\theta) \subseteq \mathcal{D}^\theta$. Let W be in $\mathcal{D}^\theta$. Then using (3.4), we have $W = \frac{1}{\cos^2\theta}(\cos^2\theta W) = \frac{1}{\cos^2\theta}T^2W = \frac{1}{\cos^2\theta}T(TW)$. Therefore, we find $W \in T(\mathcal{D}^\theta)$. It follows that $\mathcal{D}^\theta \subseteq T(\mathcal{D}^\theta)$. Thus, we get the assertion (b). □

Thanks to Theorem 3.1 [17], we characterize hemi-slant submanifolds of a l.p.R. manifold.

Theorem 3.4 *Let M be a submanifold of a l.p.R. manifold $\bar{M}$. Then M is a hemi-slant submanifold if and only if there exist a constant $\lambda \in [0, 1]$ and a distribution $\mathcal{D}$ on M such that*

- **(a)** $\mathcal{D} = \{U \in TM \mid T^2U = \lambda U\}$,
- **(b)** for any $X \in TM$ orthogonal to $\mathcal{D}$, $TX = 0$.

Moreover, in this case $\lambda = \cos^2\theta$, where θ is the slant angle of M.

Proof Let M be a hemi-slant submanifold of $\bar{M}$ with anti-invariant distribution $\mathcal{D}^\perp$ and slant distribution $\mathcal{D}^\theta$. Here, θ is the slant angle of M; in which case, we have $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta$. Then we can choose $\mathcal{D}$ as $\mathcal{D}^\theta$. Moreover, we have $\lambda = \cos^2\theta$ thanks to (3.4). Hence, (a) follows. (b) follows from Lemma 3.3. Conversely, (a) and (b) imply $TM = \mathcal{D}^\perp \oplus \mathcal{D}$. Since $T(\mathcal{D}) \subseteq \mathcal{D}$, we conclude that $\mathcal{D}^\perp$ is an anti-invariant distribution from (b). □

Example. Consider the Euclidean 6-space $\mathbb{R}^6$ with usual metric g. Define the almost product structure F on $(\mathbb{R}^6, g)$ by

$$F(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad F(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial x_i}, \quad i = 1, 2, 3,$$

where $(x_1, x_2, x_3, y_1, y_2, y_3)$ are natural coordinates of $\mathbb{R}^6$. Then $\bar{M} = (\mathbb{R}^6, g, F)$ is an almost product Riemannian manifold. Furthermore, it is easy to see that $\bar{M}$ is a l.p.R. manifold. Let M be a submanifold of $\bar{M}$ defined by

$$f(u, v, w) = \big(\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}, u + v, \frac{w}{\sqrt{2}}, \frac{w}{\sqrt{2}}, 0\big), \qquad u \neq 0.$$

Then a local frame of TM is given by

$$X = \frac{\partial}{\partial x_3},$$

$$Z = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3},$$

$$W = \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_2}.$$

By using the almost product structure F above, we see that FX is orthogonal to TM; thus $\mathcal{D}^{\perp} = \mathrm{span}\{\mathrm{X}\}$. Moreover, it is not difficult to see that $\mathcal{D}^{\theta} = span\{Z, W\}$ is a slant distribution with slant angle $\theta = \pi/4$. Thus, M is a proper hemi-slant submanifold of $\bar{M}$.

4. Integrability

In this section, we give a necessary and sufficient condition for the integrability of the slant distribution of the hemi- slant submanifold. After that we prove that the anti invariant distribution of the hemi-slant submanifold is always integrable and give some applications of this result.

Let M be a submanifold of a l.p.R. manifold $\bar{M}$. For any $U, V \in TM$, we have $\overline{\nabla}_U FV = F\overline{\nabla}_U V$ from (2.3). Then, using (2.4-2.5), (2.11-2.12) and identifying the components from TM and $T^{\perp}M$, we have the following.

Lemma 4.1 *Let M be a submanifold of a l.p.R. manifold $\bar{M}$. Then we have*

$$\nabla_U TV - A_{NV}U = T\nabla_U V + t\,h(U,V), \tag{4.1}$$

$$h(U,TV) + \nabla^{\perp}_U NV = N\nabla_U V + \omega\,h(U,V)\ . \tag{4.2}$$

for all $U, V \in TM$.

In a similar way, we have:

Lemma 4.2 *Let M be a submanifold of a l.p.R. manifold $\overline{M}$. Then we have*

$$\nabla_U\, t\,\xi - A_{\omega\xi}U = -TA_{\xi}U + t\nabla^{\perp}_U\,\xi\ , \tag{4.3}$$

$$h(U, t\,\xi) + \nabla^{\perp}_U\,\omega\,\xi = -NA_{\xi}U + \omega\nabla^{\perp}_U\,\xi \tag{4.4}$$

for any $U \in TM$ and $\xi \in T^{\perp}M$.

Theorem 4.3 *Let M be a hemi-slant manifold of a l.p.R. manifold $\overline{M}$. Then the slant distribution $\mathcal{D}^{\theta}$ is integrable if and only if*

$$A_{NZ}W - A_{NW}Z + \nabla_Z TW - \nabla_W TZ \in \mathcal{D}^{\theta} \tag{4.5}$$

for any $Z, W \in \mathcal{D}^{\theta}$.

Proof From (4.1), we have

$$\nabla_Z TW - A_{NW} Z = T\nabla_Z W + t\, h(Z, W) \tag{4.6}$$

and

$$\nabla_W TZ - A_{NZ} W = T\nabla_W Z + th(W, Z) \tag{4.7}$$

for any Z, $W \in \mathcal{D}^\theta$. By (4.6) and (4.7), we get

$$A_{NZ} W - A_{NW} Z + \nabla_Z TW - \nabla_W TZ = T[Z, W] \ . \tag{4.8}$$

Thus, our assertion follows from (3.7b) and (4.8). □

In the following we give an application of Theorem 4.3.

Theorem 4.4 *Let M be a hemi-slant manifold of a l.p.R. manifold $\bar{M}$. If M is $\mathcal{D}^\theta$-totally geodesic, then the slant distribution $\mathcal{D}^\theta$ is integrable.*

Proof Suppose that M is $\mathcal{D}^\theta$-totally geodesic, that is, for any Z, $W \in \mathcal{D}^\theta$ we have

$$h(Z, W) = 0. \tag{4.9}$$

By (4.1) and (4.9), we have

$$A_{NZ} W - \nabla_W TZ = -T\nabla_W Z \tag{4.10}$$

and similarly

$$A_{NW} Z - \nabla_Z TW = -T\nabla_Z W\,. \tag{4.11}$$

From (4.10) and (4.11), using Lemma 3.3, we get

$$g(A_{NZ} W - A_{NW} Z + \nabla_Z TW - \nabla_W TZ, X) = g(T[Z, W], X) = 0 \tag{4.12}$$

for any $X \in \mathcal{D}^\perp$. The last equation (4.12) says that

$$A_{NZ} W - A_{NW} Z + \nabla_Z TW - \nabla_W TZ \in \mathcal{D}^\theta$$

and by Theorem 4.3, we deduce that $\mathcal{D}^\theta$ is integrable. □

Lemma 4.5 *Let M be a hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then*

$$A_{NX} Y = -A_{NY} X \tag{4.13}$$

for any $X, Y \in \mathcal{D}^\perp$.

Proof For any $X \in \mathcal{D}^\perp$ and $U \in TM$, using (3.7a), we have

$$-T\nabla_U X = A_{NX} U + t\, h(U, X) \tag{4.14}$$

from (4.1). Let Y be in $\mathcal{D}^{\perp}$. Using (3.7b), we obtain

$$0 = -g(T\nabla_U X, Y) = g(A_{NX}U, Y) + g(th(U, X), Y) \tag{4.15}$$

from (4.14). On the other hand, using (2.2), (2.6), (2.11), and (2.12), we find

$$g(t\,h(U, X), Y) = g(A_{NY}U, X). \tag{4.16}$$

Thus, from (4.15) and (4.16), we deduce that

$$g(A_{NX}Y + A_{NY}X, U) = 0. \tag{4.17}$$

This equation gives (4.13). □

Theorem 4.6 *Let M be a hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the anti-invariant distribution $\mathcal{D}^{\perp}$ is integrable if and only if*

$$A_{NX}Y = A_{NY}X \tag{4.18}$$

for all X, $Y \in \mathcal{D}^{\perp}$.

Proof From (4.1), using (3.7a), we have

$$-A_{NY}X = T\nabla_X Y + t\,h(X, Y) \tag{4.19}$$

for all $X \in \mathcal{D}^{\perp}$. By interchanging X and Y in (4.19), then subtracting it from (4.19) we obtain

$$A_{NX}Y - A_{NY}X = T[X, Y]\ . \tag{4.20}$$

Because of (3.7a), we know that $\mathcal{D}^{\perp}$ is integrable if and only if $T[X, Y] = 0$ for all $X, Y \in \mathcal{D}^{\perp}$. Thus, our assertion comes from (4.20). □

By Lemma 4.5 and Theorem 4.6, we have the following result.

Corollary 4.7 *Let M be a hemi-slant submanifold of a l.p.R. manifold $\overline{M}$. Then the anti-invariant distribution $\mathcal{D}^{\perp}$ is integrable if and only if*

$$A_{NX}Y = 0 \tag{4.21}$$

for all X, $Y \in \mathcal{D}^{\perp}$.

Now, we give the main result of this section.

Theorem 4.8 *Let M be a hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the anti-invariant distribution $\mathcal{D}^{\perp}$ is always integrable.*

Proof Let $\bar{M}$ be a l.p.R. manifold with Riemannian metric g and almost product structure F. Define the symmetric (0,2)-type tensor field Ω by $\Omega(\bar{U}, \bar{V}) = g(F\bar{U}, \bar{V})$ on the tangent bundle $T\bar{M}$. It is not difficult to see that $(\bar{\nabla}_{\bar{U}}\Omega)(\bar{V}, \bar{W}) = g((\bar{\nabla}_{\bar{U}}F)\bar{V}, \bar{W})$ on $T\bar{M}$. Thus, because of (2.3), we deduce that

$$3\,d\Omega(\bar{V}, \bar{W}, \bar{U}) = \mathcal{G}(\bar{\nabla}_{\bar{U}}\Omega)(\bar{V}, \bar{W}) = 0$$

for all $\bar{U}, \bar{V}, \bar{W} \in T\bar{M}$, that is, $d\Omega \equiv 0$, where $\mathcal{G}$ denotes the cyclic sum over $\bar{U}, \bar{V}, \bar{W} \in T\bar{M}$. Next, for any X, $Y \in \mathcal{D}^{\perp}$ and $U \in TM$ we have

$$\begin{aligned} 0 = 3\, d\Omega(U, X, Y) &= U\,\Omega(X, Y) + X\,\Omega(Y, U) + Y\,\Omega(U, X) \\ &\quad -\Omega([U, X], Y) - \Omega([X, Y], U) - \Omega([Y, U], X) \\ &= g(T[Y, X], U]) . \end{aligned}$$

It follows that $T[X, Y] = 0$ and because of (3.7a), $[Y, X] \in \mathcal{D}^{\perp}$. □

Corollary 4.9 *Let M be a hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the following facts hold:*

$$A_{ND^{\perp}} D^{\perp} = 0 \tag{4.22}$$

$$A_{NX} Z \in D^{\theta}, \quad \textit{i.e.,}\ A_{ND^{\perp}} D^{\theta} \subseteq D^{\theta} \tag{4.23}$$

and

$$g(h(TM, \mathcal{D}^{\perp}), N\mathcal{D}^{\perp}) = 0\,, \tag{4.24}$$

where $X \in \mathcal{D}^{\perp}$ and $Z \in \mathcal{D}^{\theta}$.

Proof (4.22) follows from Corollary 4.7 and Theorem 4.8. (4.23) follows from (4.22). Finally, using (2.6), (4.22) gives (4.24). □

Next, we give another application of Theorem 4.8.

Theorem 4.10 *Let M be a proper hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. The anti-invariant distribution $\mathcal{D}^{\perp}$ defines a totally geodesic foliation on M if and only if $h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}) \perp N\mathcal{D}^{\theta}$.*

Proof For $X, Y \in \mathcal{D}^{\perp}$, we put $\nabla_X Y = {}^{\perp}\nabla_X Y + {}^{\theta}\nabla_X Y$, where ${}^{\perp}\nabla_X Y$ (resp. ${}^{\theta}\nabla_X Y$) denotes the anti-invariant (resp. slant) part of $\nabla_X Y$. Then using Lemma 3.3 and (3.5), for any $Z \in \mathcal{D}^{\theta}$ we have

$$g(\nabla_X Y, Z) = g({}^{\theta}\nabla_X Y, Z) = \tfrac{1}{\cos^2\theta}\, g(T\,{}^{\theta}\nabla_X Y, TZ) = \tfrac{1}{\cos^2\theta}\, g(T\nabla_X Y, TZ)\,. \tag{4.25}$$

On the other hand, from (4.1), we have

$$T\nabla_X Y + t\,h(X, Y) = -A_{NY} X = 0\ , \tag{4.26}$$

since the distribution $\mathcal{D}^{\perp}$ is integrable. Therefore, using (4.26), from (4.25), we get

$$g(\nabla_X Y, Z) = -\tfrac{1}{\cos^2\theta}\, g(t\,h(X, Y), TZ) = -\tfrac{1}{\cos^2\theta}\, g(Fh(X, Y), TZ)\,. \tag{4.27}$$

Here, using (2.2), (2.11), and (3.4), we find

$$g(F\,h(X, Y), TZ) = g(h(X, Y), NTZ). \tag{4.28}$$

From (4.27) and (4.28), we get

$$g(\nabla_X Y, Z) = -\tfrac{1}{\cos^2\theta}\, g(h(X, Y), NTZ)\,. \tag{4.29}$$

Since $TZ \in \mathcal{D}^{\theta}$, our assertion comes from (4.29). □

5. Hemi-slant product

In this section, we give a necessary and sufficient condition for a proper hemi-slant submanifold to be a hemi-slant product.

Definition 5.1 *A proper hemi-slant submanifold M of a l.p.R. manifold $\bar{M}$ is called a hemi-slant product if it is locally product Riemannian of an anti-invariant submanifold $M_\perp$ and a proper slant submanifold M_θ of $\bar{M}$.*

Now, we are going to examine the problem when a proper hemi-slant submanifold of a l.p.R. manifold is a hemi-slant product.

We first give a result that is equivalent to Theorem 4.10.

Theorem 5.2 *Let M be a proper hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the anti-invariant $\mathcal{D}^\perp$ defines a totally geodesic foliation on M if and only if*

$$g(A_{NY}Z, X) = -g(A_{NZ}Y, X), \tag{5.1}$$

where X, $Y \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$.

Proof For any X, $Y \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$, using (2.4), (2.2), and (2.3), we have

$$g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, Z) = g(\overline{\nabla}_X FY, FZ).$$

Hence, using (2.11), (2.4), (2.5), and (2.2), we obtain

$$g(\nabla_X Y, Z) = -g(A_{NY}X, TZ) + g(\nabla_X Y, FNZ) + g(h(X,Y), FNZ).$$

Here, using (3.3c), (3.3a), (2.12), and (3.4), we have

$FNZ = tNZ - NTZ$ and $tNZ = Z - T^2 Z = \sin^2\theta Z$. Thus, with the help of (2.6), we get

$$g(\nabla_X Y, Z) = -g(A_{NY}X, TZ) + \sin^2\theta g(\nabla_X Y, Z) - g(A_{NTZ}Y, X).$$

After some calculations, we find

$$\cos^2\theta g(\nabla_X Y, Z) = -g(A_{NY}TZ, X) - g(A_{NTZ}Y, X).$$

It follows that the distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation on M if and only if

$$g(A_{NY}TZ, X) = -g(A_{NTZ}Y, X). \tag{5.2}$$

Putting $Z = TZ$ in (5.2), we obtain (5.1) and vice versa. □

Theorem 5.3 *Let M be a proper hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then the distribution $\mathcal{D}^\theta$ defines a totally geodesic foliation on M if and only if*

$$g(A_{NX}W, Z) = -g(A_{NW}X, Z), \tag{5.3}$$

where $X \in \mathcal{D}^\perp$ and $Z, W \in \mathcal{D}^\theta$.

Proof Using (2.4), (2.2), and (2.3), we have $g(\nabla_Z W, X) = g(\overline{\nabla}_Z FW, FX)$ for any $Z, W \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^\perp$. Next, using (2.11) and (3.1), we obtain $g(\nabla_Z W, X) = -g(TW, \overline{\nabla}_Z NX) - g(NW, \overline{\nabla}_Z FX)$. Hence, using (2.5) and (2.1), we get $g(\nabla_Z W, X) = g(TW, A_{NX}Z) - g(FNW, \overline{\nabla}_Z X)$. With the help of (2.12), (3.3a), (3.3c), and (2.4), we arrive at

$$g(\nabla_Z W, X) = g(A_{NX}Z, TW) - \sin^2\theta\, g(\nabla_Z X, W) + g(h(X, Z), NTW).$$

Upon direct calculation, we find

$$\cos^2\theta \; g(\nabla_Z W, X) = g(A_{NX}TW, Z) + g(A_{NTW}X, Z)$$

Therefore, we deduce that the slant distribution $\mathcal{D}^\theta$ defines a totally geodesic foliation if and only if

$$g(A_{NX}TW, Z) = -g(A_{NTW}X, Z), \tag{5.4}$$

By putting $W = TW$, we see that the last equation is equivalent to the equation (5.3). □

Thus, from Theorems 5.2 and 5.3, we obtain the expected result.

Corollary 5.4 *Let M be a proper hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. Then M is a hemi-slant product manifold $M = M_\perp \times M_\theta$ if and only if*

$$A_{NX}Z = -A_{NZ}X, \tag{5.5}$$

where $X \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$.

6. Hemi-slant submanifolds with parallel canonical structures

In this section, we get several results for the hemi-slant submanifolds with parallel canonical structures using the previous results.

Let M be any submanifold of a l.p.R. manifold $\bar{M}$ with the endomorphism T and the normal bundle valued 1-form N defined by (2.11). We put

$$(\overline{\nabla}_U T)V = \nabla_U TV - T\nabla_U V \tag{6.1}$$

and

$$(\overline{\nabla}_U N)V = \nabla^\perp_U NV - N\nabla_U V \tag{6.2}$$

for any $U, V \in TM$. Then the endomorphism T (resp.1-form N) is parallel if $\overline{\nabla} T \equiv 0$ (resp. $\overline{\nabla} N \equiv 0$). From (4.1) and (4.2) we have

$$(\overline{\nabla}_U T)V = A_{NV}U + t\, h(U, V) \tag{6.3}$$

and

$$(\overline{\nabla}_U N)V = \omega\, h(U, V) - h(U, TV), \tag{6.4}$$

respectively.

Theorem 6.1 *Let M be any submanifold of a l.p.R. manifold $\bar{M}$. Then T is parallel, i.e. $\overline{\nabla} T \equiv 0$ if and only if*

$$A_{NV}U = 0 \tag{6.5}$$

for all $U, V \in TM$.

Proof For any $U, V, W \in TM$ from (6.3), we have

$$g\big((\overline{\nabla}_W T)V, U\big) = g(A_{NV}W, U) + g(t\,h(W,V), U)\,.$$

Hence, using (2.12), (2.2), and (2.11), we obtain

$$g((\overline{\nabla}_W T)V, U) = g(A_{NV}W, U) + g(h(W,V), NU)\,.$$

Since A is self-adjoint, with the help of (2.6), we get

$$g((\overline{\nabla}_W T)V, U) = g(A_{NV}U, W) + g(A_{NU}V, W). \tag{6.6}$$

Now let T be parallel; then from (6.6) it follows that

$$A_{NV}U = -A_{NU}V \tag{6.7}$$

for all $U, V \in TM$. On the other hand, from (6.3), we have

$$A_{NV}U = A_{NU}V\ \ , \tag{6.8}$$

since h is a symmetric tensor field. Thus, (6.5) follows from (6.7) and (6.8).

□

From Corollary 5.4 and Theorem 6.1, we have the following result.

Corollary 6.2 *Let M be a proper hemi-slant submanifold of a l.p.R. manifold $\bar{M}$. If T is parallel, then M is a hemi-slant product.*

Theorem 6.3 *Let M be a proper hemi-slant submanifold of $\overline{M}$. If N is parallel, then*

$$(a)\quad A_\mu \mathcal{D}^\perp = 0\,, \qquad\qquad (b)\quad A_{N\mathcal{D}^\perp}\mathcal{D}^\theta = 0\,,$$

$$(c)\quad M \textit{ is } (\mathcal{D}^\perp, \mathcal{D}^\theta)\textit{-mixed totally geodesic.}$$

Proof Let N be parallel, it follows from (6.4) that

$$h(U, TV) = \omega h(U, V) \tag{6.9}$$

for any $U, V \in TM$. Then, for any $X \in \mathcal{D}^\perp$, we have

$$\omega h(U, X) = 0 \tag{6.10}$$

from (6.9). For any $\xi \in \mu$, using (2.11), (2.2), and (2.6), we have

$$g(\omega h(U, X), \xi) = g(h(U, X), F\xi) = g(A_{F\xi}X, U)\,.$$

Thus, using (6.10) we get

$$g(A_{F\xi}X, U) = 0\,. \tag{6.11}$$

Since μ is invariant with respect to F, the assertion (a) comes from (6.11). On the other hand, for any $X \in \mathcal{D}^{\perp}$, using (2.2), (2.11), (2.12), and (6.9), we have

$$\begin{aligned} g(h(U,TZ),NX) &= g(h(U,TZ),FX) = g(\omega h(U,Z),FX) \\ &= g(Fh(U,Z),FX) = g(h(U,Z),X) = 0, \end{aligned}$$

that is, $g(h(U,TZ),NX) = 0$. Putting $Z = TZ$ in last equation, we obtain

$$\cos^2\theta\, g(h(U,Z),NX) = \cos^2\theta\, g(A_{NX}Z,U) = 0\,.$$

Since $\theta \neq \frac{\pi}{2}$, the assertion (b) follows. Lastly, using (3.4), from (6.9), we have

$\omega^2 h(X,Z) = \omega h(X,TZ) = h(X,T^2Z) = \cos^2\theta h(X,Z).$

On the other hand, using (3.7a), we have

$\omega^2 h(X,Z) = \omega^2 h(Z,X) = \omega h(Z,TX) = 0.$

Thus, we get

$\cos^2\theta\, h(X,Z) = 0.$

Since $\theta \neq \frac{\pi}{2}$, we deduce that $h(X,Z) = 0$, which proves the last assertion. □

7. Totally umbilical hemi-slant submanifolds

In this section we shall give two characterization theorems for the totally umbilical proper hemi-slant submanifolds of a l.p.R. manifold. First we prove

Theorem 7.1 *If M is a totally umbilical proper hemi-slant submanifold of a l.p.R. manifold $\bar{M}$, then either the anti-invariant distribution $\mathcal{D}^{\perp}$ is 1-dimensional or the mean curvature vector field H of M is perpendicular to $F(\mathcal{D}^{\perp})$. Moreover, if M is a hemi-slant product, then $H \in \mu$.*

Proof Since M is a totally umbilical proper hemi-slant submanifold either $Dim(\mathcal{D}^{\perp}) = 1$ or $Dim(\mathcal{D}^{\perp}) > 1$. If $Dim(\mathcal{D}^{\perp}) = 1$, it is obvious. If $Dim(\mathcal{D}^{\perp}) > 1$, then we can choose $X, Y \in \mathcal{D}^{\perp}$ such that $\{X, Y\}$ is orthonormal. By using (2.11), (2.7), (2.6), and (4.22), we have

$$g(H,FY) = g(h(X,X),NY) = g(A_{NY}X,X) = 0 \tag{7.1}$$

It means that

$$H \perp F(\mathcal{D}^{\perp}). \tag{7.2}$$

Moreover, if M is a hemi-slant product, for any $Z \in \mathcal{D}^\theta$, using (5.5) and (2.7), we have

$$g(H, NZ) = g(h(X, X), NZ) = g(A_{NZ}X, X) = -g(A_{NX}Z, X)$$
$$= -g(h(Z, X), NX) = 0.$$

Hence, it follows that

$$H \perp N(\mathcal{D}^\theta). \tag{7.3}$$

Thus, using (7.2) and (7.3) from (3.2), we get $H \in \mu$. □

Before giving the second result of this section, recall the following fact about locally product Riemannian manifolds.

Let $M_1(c_1)$ (resp. $M_2(c_2)$) be a real space form with sectional curvature c_1 (resp. c_2). Then the Riemannian curvature tensor $\overline{R}$ of the locally product Riemannian manifold $\overline{M} = M_1(c_1) \times M_2(c_2)$ has the form

$$\overline{R}(\bar{U},\bar{V})\bar{W} = \frac{(c_1+c_2)}{4}\Big\{g(\bar{V},\bar{W})\bar{U} - g(\bar{U},\bar{W})\bar{V} + g(F\bar{V},\bar{W})F\bar{U} - g(F\bar{U},\bar{W})F\bar{V}\Big\} \tag{7.4}$$
$$+ \frac{(c_1-c_2)}{4}\Big\{g(F\bar{V},\bar{W})\bar{U} - g(F\bar{U},\bar{W})\bar{V} + g(\bar{V},\bar{W})F\bar{U} - g(\bar{U},\bar{W})F\bar{V}\Big\},$$

where $\bar{U}, \bar{V}, \bar{W} \in T\bar{M}$ [22].

Theorem 7.2 *Let M be a totally umbilical hemi-slant submanifold with parallel mean curvature vector field H of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$ with $c_1 \neq c_2$. Then M cannot be proper.*

Proof Let $X \in \mathcal{D}^\perp$ and $Z \in \mathcal{D}^\theta$ be two unit vector fields. Since H is parallel, using (2.10) and (2.7) from the Codazzi equation (2.9), we have

$$(\overline{R}(X, Z)X)^\perp = -\nabla^\perp_Z H = 0. \tag{7.5}$$

On the other hand, equation (7.4) gives

$$\overline{R}(X, Z)X = -\frac{1}{4}\Big\{(c_1 + c_2)Z + (c_1 - c_2)FZ\Big\}. \tag{7.6}$$

Taking the normal component of (7.6), we get

$$(\overline{R}(X, Z)X)^\perp = -\frac{1}{4}(c_1 - c_2)NZ, \tag{7.7}$$

which contradicts (7.5). □

We have immediately from Theorem 7.2 that:

Corollary 7.3 *There exists no totally geodesic proper hemi-slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$ with $c_1 \neq c_2$.*

8. Ricci curvature of hemi-slant submanifolds

In this section we obtain a Chen-type inequality for hemi-slant submanifolds of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. We first present the following fundamental facts about this topic.

Let $\bar{M}$ be a n-dimensional Riemannian manifold equipped with a Riemannian metric g and $\{e_1, ..., e_n\}$ be an orthonormal basis for $T_p\bar{M}$, $p \in \bar{M}$. Then the *Ricci tensor* $\overline{S}$ is defined by

$$\overline{S}(U, V) = \sum_{i=1}^{n} \overline{R}(e_i, U, V, e_i), \tag{8.1}$$

where $U, V \in T_p\bar{M}$. For a fixed $i \in \{1, ..., n\}$, the *Ricci curvature* of e_i, denoted by $\overline{R}ic(e_i)$, is given by

$$\overline{R}ic(e_i) = \sum_{i \neq j}^{n} \overline{K}_{ij}, \tag{8.2}$$

where $\overline{K}_{ij} = g(\overline{R}(e_i, e_j)e_j, e_i)$ is the *sectional curvature* of the plane spanned by e_i and e_j at $p \in \bar{M}$. Let Π_k be a k-plane of $T_p\bar{M}$ and $\{e_1, ..., e_k\}$ any orthonormal basis of Π_k. For a fixed $i \in \{1, ..., k\}$, the k-*Ricci curvature* [9] of Π_k at e_i, denoted by $\overline{R}ic_{\Pi_k}(e_i)$, is defined by

$$\overline{R}ic_{\Pi_k}(e_i) = \sum_{i \neq j}^{k} \overline{K}_{ij}. \tag{8.3}$$

It is easy to see that $\overline{R}ic_{(T_p\bar{M})}(e_i) = \overline{R}ic(e_i)$ for $1 \leq i \leq n$, since $\Pi_n = T_p\bar{M}$.

We now recall the following basic inequality [10, Theorem 3.1] involving Ricci curvature and the squared mean curvature of a submanifold of a Riemannian manifold.

Theorem 8.1 *([10, Theorem 3.1]) Let M be an m-dimensional submanifold of a Riemannian manifold $\bar{M}$. Then, for any unit vector $X \in T_pM$, we have*

$$Ric(X) \leq \frac{1}{4}m^2\|H\|^2 + \overline{R}ic_{(T_pM)}(X) \tag{8.4}$$

where $Ric(X)$ is the Ricci curvature of X.

Of course, the equality case of (8.4) was also discussed in [10], but we will not deal with the equality case in this paper.

Now, we are ready to state the main result of this section.

Theorem 8.2 *Let M be an m-dimensional hemi-slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. Then, for unit vector $V \in T_pM$, we have*

$$4Ric(V) \leq m^2\|H\|^2 + (c_1 + c_2)\Big\{(m-1) + \sum_{i=2}^{m} g(Te_i, e_i)g(TV, V) \tag{8.5}$$

$$-\|TV\|^2 + g^2(TV, V)\Big\} + (c_1 - c_2)\Big\{\sum_{i=2}^{m} g(Te_i, e_i) + (m-1)g(TV, V)\Big\}$$

where $\{V, e_2, ..., e_m\}$ is an orthonormal basis for T_pM.

Proof Since M is an m-dimensional hemi-slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$, then for any unit vector $V \in T_pM$, using (7.4) and (2.11) from (8.3) we have

$$4\overline{R}ic_{(T_pM)}(V) = (c_1+c_2)\Big\{(m-1) + \sum_{i=2}^{m} g(Te_i, e_i)g(TV, V) \tag{8.6}$$

$$-\|TV\|^2 + g^2(TV, V)\Big\} + (c_1 - c_2)\Big\{\sum_{i=2}^{m} g(Te_i, e_i) + (m-1)g(TV, V)\Big\}$$

Thus, using (8.6) in (8.4) we get (8.5). □

Remark 8.3 *In general, $g(F\overline{V}, \overline{V}) \neq 0$ for any unit vector $\overline{V} \in T_p\bar{M}$ in a l.p.R. manifold $\bar{M}$, contrary to almost Hermitian $(g(J\overline{V}, \overline{V}) = 0)$ and almost contact $((g(\varphi\overline{V}, \overline{V}) = 0)$ manifolds. However, we can establish that the almost product structure F in a l.p.R. manifold $\bar{M}$ such that $g(F\overline{V}, \overline{V}) = 0$, for all $\overline{V} \in T_p\bar{M}$. In fact, if $\bar{M}$ is an even dimensional l.p.R. manifold with an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2n}\}$, then we can define F by*

$$F(e_i) = e_{n+i}, \quad F(e_{n+i}) = e_i, \quad i \in \{1, 2, ..., n\}.$$

Hence, we observe easily that the almost product structure F satisfies

$$g(Fe_i, e_i) = 0. \tag{8.7}$$

For example, the almost product structure F in the example of section 3 satisfies the condition (8.7). On the other hand, because of Lemma 3.3 and equation (3.5), we have $TV = 0$, if $V \in \mathcal{D}^{\perp}$ and $\|TV\|^2 = \cos^2\theta$, if $V \in \mathcal{D}^{\theta}$ and $\|V\| = 1$, respectively. Thus, by Theorem 8.2 we get the following two results.

Corollary 8.4 *Let M be an m-dimensional anti-invariant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. If the almost product structure F of $\bar{M}$ satisfies the condition (8.7), then we have*

$$4Ric(V) \leq m^2\|H\|^2 + (c_1 + c_2)(m-1),$$

where $V \in T_pM$ is any unit vector.

Corollary 8.5 *Let M be an m-dimensional slant submanifold of a l.p.R. manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$. If the almost product structure F of $\bar{M}$ satisfies the condition (8.7), then we have*

$$4Ric(Z) \leq m^2\|H\|^2 + (c_1 + c_2)\{(m-1) - \cos^2\theta\},$$

where $Z \in T_pM$ is any unit vector.

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