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KHALID MAHMOOD AWAN

JOSIP PECARIC

MIHAELA RIBICIC PENAVA

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Companion inequalities to Ostrowski–Grüss type inequality and applications

Khalid Mahmood AWAN[1,*], **Josip PEČARIĆ**[2], **Mihaela Ribičić PENAVA**[3]
[1]Department of Mathematics, University of Sargodha, Sargodha, Pakistan
[2]Faculty of Textile Technology, University of Zagreb, Zagreb, Croatia
[3]Department of Mathematics, University of Osijek, Osijek, Croatia

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Abstract: The aim of this paper is to give some companion inequalities to the Ostrowski-Grüss type inequality for n-time differentiable absolutely continuous functions by using recently obtained bounds for the Chebyshev functional.

Key words: Chebyshev functional, Ostrowski–Grüss type inequality

1. Introduction

For two functions $f, g : [a, b] \to \mathbb{R}$ such that $f, g, f \cdot g \in L^1 [a, b]$, the Chebyshev functional [8] is defined by

$$T(f, g) = \frac{1}{b-a} \int_a^b f(s) g(s) \mathrm{d}s - \frac{1}{b-a} \int_a^b f(s) \mathrm{d}s \cdot \frac{1}{b-a} \int_a^b g(s) \mathrm{d}s.$$

The following integral inequality is well known as the Grüss inequality [8]:

$$\left| \frac{1}{b-a} \int_a^b f(s) g(s) \mathrm{d}s - \frac{1}{b-a} \int_a^b f(s) \mathrm{d}s \cdot \frac{1}{b-a} \int_a^b g(s) \mathrm{d}s \right| \leq \frac{1}{4} (\Gamma - \gamma)(\Phi - \phi),$$

where $f, g : [a, b] \to \mathbb{R}$ are two integrable functions such that $\gamma \leq f(s) \leq \Gamma$, and $\phi \leq g(s) \leq \Phi$, for all $s \in [a, b]$, and $\gamma, \Gamma, \phi, \Phi$ are real constants. The result related to the Chebyshev functional is the Ostrowski inequality [9]:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(s) \mathrm{d}s \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) M,$$

where $f : [a, b] \to \mathbf{R}$ is a differentiable function such that $|f'(x)| \leq M$ for all $x \in [a, b]$. More about Ostrowski type inequalities and companion inequalities to the Ostrowski type inequality can be found in papers [4, 5] and in monographs [1, 6].

If f is a differentiable function, f' is integrable, and $\gamma \leq f'(s) \leq \Gamma$, for all $s \in [a, b]$, then the following Ostrowski–Grüss inequality can be stated:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(s) \mathrm{d}s - \left(x - \frac{a+b}{2}\right) \frac{f(b) - f(a)}{b-a} \right| \leq \frac{1}{4\sqrt{3}} (b-a)(\Gamma - \gamma), \tag{1.1}$$

*Correspondence: khalid819@uos.edu.pk
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for all $x \in [a,b]$. Inequality (1.1) was proved in [7].

Cerone and Dragomir [3] proved the following result:

Lemma 1 *If $\phi : [a,b] \to \mathbf{R}$ is an absolutely continuous function with*

$$(\cdot - a)(b - \cdot)(\phi')^2 \in L^1[a,b],$$

then the following inequality holds:

$$T(\phi,\phi) \le \frac{1}{2(b-a)} \int_a^b (s-a)(b-s)[\phi'(s)]^2 \,\mathrm{d}s. \tag{1.2}$$

The constant $1/2$ is best possible.

Let us recall that the nonnegative normalized weighted function $w : [a,b] \to [0,\infty\rangle$ is an integrable function satisfying $\int_a^b w(s)\,ds = 1$ and $W(s) = \int_a^s w(u)\,\mathrm{d}u$ for $s \in [a,b]$, $W(s) = 0$ for $s < a$, and $W(s) = 1$ for $s > b$.

We shall use the usual convention $f^{(0)} = f$, $0! = 1$ and $\sum_{i=0}^{-1} \cdot = 0$.

In [2] Aljinović et al. proved the following identity using some generalizations of the Montgomery identity:

Theorem 1 *Let I be an open interval in $\mathbf{R}$, $[a,b] \subset I$, and let $f : I \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 1$. Let $w : [a,b] \to [0,\infty\rangle$ be some nonnegative normalized weighted function. For each $x \in \left[a, \frac{a+b}{2}\right]$, the following identity then holds:*

$$\begin{aligned}\int_a^b w(s)f(s)\mathrm{d}s =& \frac{1}{2}\left[\sum_{i=0}^{n-1} \frac{f^{(i)}(x)}{i!} \int_a^b w(s)(s-x)^i \mathrm{d}s\right.\\ &\left. + \sum_{i=0}^{n-1} \frac{f^{(i)}(a+b-x)}{i!} \int_a^b w(s)(s-a-b+x)^i \mathrm{d}s\right]\\ &+ \frac{1}{(n-1)!} \int_a^b S_{w,n}(x,s) f^{(n)}(s)\mathrm{d}s,\end{aligned} \tag{1.3}$$

where

$$S_{w,n}(x,s) = \begin{cases} -\int_a^s w(u)(u-s)^{n-1}\,du, & a \le s \le x, \\ -\frac{1}{2}\left[\int_a^s w(u)(u-s)^{n-1}\,du - \int_s^b w(u)(u-s)^{n-1}\,du\right], & x < s \le a+b-x, \\ \int_s^b w(u)(u-s)^{n-1}\,du, & a+b-x < s \le b. \end{cases} \tag{1.4}$$

Remark 1 *In the nonweighted case ($w(s) = \frac{1}{b-a}$, $s \in [a,b]$) equality (1.3) reduces to*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s =& \frac{1}{2}\sum_{i=0}^{n-1} \left[f^{(i)}(x) + (-1)^i f^{(i)}(a+b-x)\right] \frac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!\,(b-a)}\\ &+ \frac{1}{n!}\int_a^b S_n(x,s) f^{(n)}(s)\mathrm{d}s,\end{aligned} \tag{1.5}$$

where

$$S_n(x,s) = \begin{cases} \frac{(a-s)^n}{b-a}, & a \le s \le x, \\ \frac{(a-s)^n + (b-s)^n}{2(b-a)}, & x < s \le a+b-x, \\ \frac{(b-s)^n}{b-a}, & a+b-x < s \le b. \end{cases}$$

In this paper we will obtain some companion inequalities to the Ostrowski–Grüss type inequality by using Lemma 1 and identities (1.3) and (1.5).

2. Main results

Here and hereafter the symbol $\left[f^{(n)}; a, b\right]$ denotes the divided difference of function $f^{(n)}$:

$$\left[f^{(n)}; a, b\right] = \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a}.$$

Theorem 2 *Let I be an open interval in $\mathbf{R}$, $[a,b] \subset I$, and let $w : [a,b] \to [0,\infty\rangle$ be some nonnegative normalized weighted function. Let $f : I \to \mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous. For each $x \in \left[a, \frac{a+b}{2}\right]$, the following representation then holds:*

$$\int_a^b w(s)f(s)\mathrm{d}s = \frac{1}{2}\left[\sum_{i=0}^{n-1} \frac{f^{(i)}(x)}{i!} \int_a^b w(s)(s-x)^i \mathrm{d}s + \sum_{i=0}^{n-1} \frac{f^{(i)}(a+b-x)}{i!} \int_a^b w(s)(s-a-b+x)^i \mathrm{d}s\right]$$
$$+ \frac{1}{(n-1)!} \int_a^b S_{w,n}(x,s)\mathrm{d}s \left[f^{(n-1)}; a, b\right] + R_{w,n}(f,x). \tag{2.1}$$

The remainder $R_{w,n}(f,x)$ satisfies the estimation

$$|R_{w,n}(f,x)| \le \frac{\sqrt{2(b-a)}}{2(n-1)!} \left[T\left(S_{w,n}(x,\cdot), S_{w,n}(x,\cdot)\right)\right]^{1/2}$$
$$\cdot \left[\int_a^b (s-a)(b-s)\left(f^{(n+1)}(s)\right)^2 \mathrm{d}s\right]^{1/2}, \tag{2.2}$$

where $S_{w,n}(x,\cdot)$ is defined by (1.4).

Proof Identity (1.3) can be rewritten as

$$\int_a^b w(s)f(s)\mathrm{d}s = \frac{1}{2}\left[\sum_{i=0}^{n-1} \frac{f^{(i)}(x)}{i!} \int_a^b w(s)(s-x)^i \mathrm{d}s\right.$$
$$\left. + \sum_{i=0}^{n-1} \frac{f^{(i)}(a+b-x)}{i!} \int_a^b w(s)(s-a-b+x)^i \mathrm{d}s\right]$$
$$+ \frac{1}{(n-1)!\,(b-a)} \int_a^b S_{w,n}(x,s)\mathrm{d}s \int_a^b f^{(n)}(s)\mathrm{d}s + R_{w,n}(f,x),$$

where

$$R_{w,n}(f,x) = \frac{1}{(n-1)!}\int_a^b S_{w,n}(x,s)f^{(n)}(s)\mathrm{d}s - \frac{1}{(n-1)!}\cdot\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b S_{w,n}(x,s)\mathrm{d}s$$

because

$$\int_a^b f^{(n)}(s)\,\mathrm{d}s = f^{(n-1)}(b) - f^{(n-1)}(a).$$

Using Cauchy–Schwartz inequality for double integrals and the inequality between arithmetic and quadratic integral means [8], we obtain:

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b S_{w,n}(x,s)f^{(n)}(s)\mathrm{d}s - \frac{1}{b-a}\int_a^b S_{w,n}(x,s)\mathrm{d}s\cdot\frac{1}{b-a}\int_a^b f^{(n)}(s)\mathrm{d}s\right|\\&\leq \left[T\left(S_{w,n}(x,\cdot),S_{w,n}(x,\cdot)\right)\right]^{1/2}\cdot\left[T\left(f^{(n)},f^{(n)}\right)\right]^{1/2}.\end{aligned}\tag{2.3}$$

If we apply Lemma 1 with $f^{(n)}$ in place of ϕ, we get

$$\begin{aligned}&\left[T\left(S_{w,n}(x,\cdot),S_{w,n}(x,\cdot)\right)\right]^{1/2}\cdot\left[T\left(f^{(n)},f^{(n)}\right)\right]^{1/2}\\&\leq\frac{1}{\sqrt{2(b-a)}}\left[T\left(S_{w,n}(x,\cdot),S_{w,n}(x,\cdot)\right)\right]^{1/2}\cdot\left[\int_a^b(s-a)(b-s)\left(f^{(n+1)}(s)\right)^2\mathrm{d}s\right]^{1/2},\end{aligned}$$

which completes the proof. □

In the next theorem we will use the beta function and the incomplete beta function of Euler type defined by

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\,\mathrm{d}t,\qquad B_r(u,v)=\int_0^r t^{u-1}(1-t)^{v-1}\,\mathrm{d}t,\qquad u,v>0.$$

Theorem 3 *Let I be an open interval in $\mathbf{R}$, $[a,b]\subset I$, and let $f:I\to\mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous. For each $x\in\left[a,\frac{a+b}{2}\right]$, the following identity then holds:*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s =& \frac{1}{2}\sum_{i=0}^{n-1}\left[f^{(i)}(x)+(-1)^i f^{(i)}(a+b-x)\right]\frac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!\,(b-a)}\\&+\frac{(-1)^n+1}{2}\cdot\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)!\,(b-a)}\left[f^{(n-1)};a,b\right]+R_n(f,x).\end{aligned}\tag{2.4}$$

The remainder $R_n(f,x)$ satisfies the estimation

$$|R_n(f,x)|\leq\frac{\sqrt{2(b-a)}}{2n!}\left[T\left(S_n(x,\cdot),S_n(x,\cdot)\right)\right]^{1/2}\cdot\left[\int_a^b(s-a)(b-s)\left(f^{(n+1)}(s)\right)^2\mathrm{d}s\right]^{1/2},\tag{2.5}$$

where

$$T\left(S_n\left(x,\cdot\right),S_n\left(x,\cdot\right)\right)=\frac{3\left(x-a\right)^{2n+1}+\left(b-x\right)^{2n+1}}{2\left(2n+1\right)\left(b-a\right)^3}+\frac{\left(-1\right)^n\left(b-a\right)^{2n-2}}{2}\left[B_{\frac{b-x}{b-a}}\left(n+1,n+1\right)-B_{\frac{x-a}{b-a}}\left(n+1,n+1\right)\right]-\left[\frac{\left(-1\right)^n+1}{2}\cdot\frac{\left(x-a\right)^{n+1}+\left(-1\right)^n\left(b-x\right)^{n+1}}{\left(n+1\right)\left(b-a\right)^2}\right]^2, \tag{2.6}$$

Proof Identity (1.5) can be written as

$$\frac{1}{n!}\int_a^b S_n(x,s)f^{(n)}(s)\mathrm{d}s=\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-\frac{1}{2}\sum_{i=0}^{n-1}\left[f^{(i)}\left(x\right)+\left(-1\right)^i f^{(i)}\left(a+b-x\right)\right]\frac{\left(b-x\right)^{i+1}-\left(a-x\right)^{i+1}}{\left(i+1\right)!\left(b-a\right)}.$$

By elementary calculations we get

$$\int_a^b S_n\left(x,s\right)\mathrm{d}s=\int_a^x\frac{\left(a-s\right)^n}{b-a}\mathrm{d}s+\int_x^{a+b-x}\frac{\left(a-s\right)^n+\left(b-s\right)^n}{2\left(b-a\right)}\mathrm{d}s+\int_{a+b-x}^b\frac{\left(b-s\right)^n}{b-a}\mathrm{d}s=\frac{\left(-1\right)^n+1}{2}\cdot\frac{\left(x-a\right)^{n+1}+\left(-1\right)^n\left(b-x\right)^{n+1}}{\left(n+1\right)\left(b-a\right)}.$$

Then

$$R_n(f,x)=\frac{1}{n!}\int_a^b S_n(x,s)f^{(n)}(s)\mathrm{d}s-\frac{1}{n!}\cdot\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b S_n(x,s)\mathrm{d}s. \tag{2.7}$$

From Cauchy–Schwartz inequality for double integrals and the inequality between arithmetic and quadratic integral means [8], we obtain

$$\left|\frac{1}{b-a}\int_a^b S_n(x,s)f^{(n)}(s)\mathrm{d}s-\frac{1}{b-a}\int_a^b S_n(x,s)\mathrm{d}s\cdot\frac{1}{b-a}\int_a^b f^{(n)}(s)\mathrm{d}s\right|\le\left[T\left(S_n\left(x,\cdot\right),S_n\left(x,\cdot\right)\right)\right]^{1/2}\cdot\left[T\left(f^{(n)},f^{(n)}\right)\right]^{1/2}. \tag{2.8}$$

If we apply Lemma 1 with $f^{(n)}$ in place of ϕ, we obtain

$$\left[T\left(S_n\left(x,\cdot\right),S_n\left(x,\cdot\right)\right)\right]^{1/2}\cdot\left[T\left(f^{(n)},f^{(n)}\right)\right]^{1/2}\le\frac{1}{\sqrt{2\left(b-a\right)}}\left[T\left(S_n\left(x,\cdot\right),S_n\left(x,\cdot\right)\right)\right]^{1/2}\cdot\left[\int_a^b\left(s-a\right)\left(b-s\right)\left(f^{(n+1)}\left(s\right)\right)^2\mathrm{d}s\right]^{1/2}.$$

Using the incomplete beta function of Euler type we get the following identity:

$$\int_a^b S_n^2\left(x,s\right)\mathrm{d}s=\frac{3\left(x-a\right)^{2n+1}+\left(b-x\right)^{2n+1}}{2\left(2n+1\right)\left(b-a\right)^2}+\frac{\left(-1\right)^n\left(b-a\right)^{2n-1}}{2}\left[B_{\frac{b-x}{b-a}}\left(n+1,n+1\right)-B_{\frac{x-a}{b-a}}\left(n+1,n+1\right)\right].$$

Finally, we have

$$T\left(S_n\left(x,\cdot\right),S_n\left(x,\cdot\right)\right)=\frac{1}{b-a}\int_a^b S_n^2\left(x,s\right)\mathrm{d}s-\frac{1}{\left(b-a\right)^2}\left(\int_a^b S_n\left(x,s\right)\mathrm{d}s\right)^2=\frac{3\left(x-a\right)^{2n+1}+\left(b-x\right)^{2n+1}}{2\left(2n+1\right)\left(b-a\right)^3}$$
$$+\frac{\left(-1\right)^n\left(b-a\right)^{2n-2}}{2}\left[B_{\frac{b-x}{b-a}}\left(n+1,n+1\right)-B_{\frac{x-a}{b-a}}\left(n+1,n+1\right)\right]$$
$$-\left[\frac{\left(-1\right)^n+1}{2}\cdot\frac{\left(x-a\right)^{n+1}+\left(-1\right)^n\left(b-x\right)^{n+1}}{\left(n+1\right)\left(b-a\right)^2}\right]^2.$$

□

Corollary 1 *Let I be an open interval in $\mathbf{R}$, $[a,b]\subset I$, and let $f:I\to\mathbf{R}$ be such that f' is absolutely continuous. For each $x\in\left[a,\frac{a+b}{2}\right]$, the following inequality then holds:*

$$\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-\frac{f\left(x\right)+f\left(a+b-x\right)}{2}\right|\le K\left(x\right)\cdot\left[\int_a^b\left(s-a\right)\left(b-s\right)\left(f''\left(s\right)\right)^2\mathrm{d}s\right]^{1/2},\tag{2.9}$$

where

$$K\left(x\right)=\frac{1}{b-a}\left[\frac{8\left(x-a\right)^3+\left(a+b-2x\right)^3}{24}\right]^{1/2}.$$

Proof Applying Theorem 3 with $n=1$, we get above inequality. □

Remark 2 *For $x=a$ inequality (2.9) reduces to the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-\frac{f\left(a\right)+f\left(b\right)}{2}\right|\le\frac{\sqrt{b-a}}{2\sqrt{6}}\cdot\left[\int_a^b\left(s-a\right)\left(b-s\right)\left(f''\left(s\right)\right)^2\mathrm{d}s\right]^{1/2}.$$

For $x=\frac{a+b}{2}$ inequality (2.9) reduces to the following midpoint inequality:

$$\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-f\left(\frac{a+b}{2}\right)\right|\le\frac{\sqrt{b-a}}{2\sqrt{6}}\cdot\left[\int_a^b\left(s-a\right)\left(b-s\right)\left(f''\left(s\right)\right)^2\mathrm{d}s\right]^{1/2}$$

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