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Almost analytic forms with respect to a quadratic endomorphism and their cohomology

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Abstract: The goal of this paper is to consider the notion of almost analytic form in a unifying setting for both almost complex and almost paracomplex geometries. We use a global formalism, which yields, in addition to generalizations of the main results of the previously known almost complex case, a relationship with the Frölicher–Nijenhuis theory. A cohomology of almost analytic forms is also introduced and studied as well as deformations of almost analytic forms with pairs of almost analytic functions.

Key words: Quadratic endomorphism, almost F -analytic form, F -symmetric form, almost (para)complex structure, cohomology

1. Introduction

The notion of almost analytic form was introduced a long time ago in the almost complex geometry and hence it was treated in local coordinates, especially by Japanese geometers [15, 16, 17, 18]. A global approach appeared in [14], unfortunately only in Romanian. Some of these global techniques were used in [9] and [13]; for example, in the former paper a differential is introduced in the algebra of pairs of almost analytic forms and a corresponding Poincaré type lemma is proved.

The present work aims to consider almost analytic forms in a unifying setting, which adds the almost paracomplex geometry. This type of even dimensional geometry is now in the mainstream of research as the surveys [1] and [4] and their several citations prove. In this way, we reveal the common parts of these geometries with respect to differential forms and present the techniques of [14] to a larger audience. An important feature of the global approach is that it yields a relationship with the Frölicher–Nijenhuis theory, widely used now for several important topics. Namely, we prove that for an almost F -analytic form its closeness with respect to the Frölicher–Nijenhuis derivative d_F is characterized by the usual (i.e. exterior derivative) closeness.

The content of the paper is as follows. In the first subsection of Section 2 we consider only 1-forms in order to offer a detailed picture of the techniques used herein. In the next subsection we consider the general case of r -forms with r less than or equal to $n =$ half of the dimension of the underlying manifold. A d_F -cohomology of almost analytic forms is introduced and studied and also some deformations of almost analytic forms with pairs of almost analytic functions are considered. In Section 3 we restrict ourselves to the Hermitian and para-Norden framework and reobtain the characterization of almost analyticity for n -forms in terms of

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harmonicity. Considering again the case of 1-forms, a local computation in the case of integrability of given endomorphism F gives an usual characterization of coefficients in terms of (para)Cauchy–Riemann equations.

2. Almost analytic forms with respect to a quadratic endomorphism

2.1. Almost analytic 1-forms

Fix a triple (M, F, ω) with M a smooth m -dimensional manifold, F a tensor field of $(1, 1)$ -type on M , and ω a differentiable 1-form, i.e. $\omega \in \Omega^1(M)$.

Definition 2.1 *i) F is a quadratic endomorphism if there exists $\varepsilon \in \mathbb{R}^*$ such that:*

$$F^2 = \varepsilon I. \tag{2.1}$$

ii) The F -conjugate of ω is the 1-form:

$$\bar{\omega} = \omega_F := \omega \circ F^{-1} = \frac{1}{\varepsilon} \omega \circ F. \tag{2.2}$$

It follows that:

$$\bar{\bar{\omega}} = \frac{1}{\varepsilon} \bar{\omega} \circ F = \frac{1}{\varepsilon} \omega. \tag{2.3}$$

To the pair (F, ω) we associate a 2-form defined by:

$$\Omega_{F, \omega}(X, Y) := d\omega(FX, Y) - \varepsilon d\bar{\omega}(X, Y), \tag{2.4}$$

which yields the main notion of this subsection:

Definition 2.2 *The 1-form ω is called almost F -analytic if $\Omega_{F, \omega} = 0$. Let $\Omega^1(M, F)$ be the set of almost F -analytic 1-forms.*

In the following we use the identity:

$$2d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]). \tag{2.5}$$

A first result shows that this property is invariant under F -conjugation:

Proposition 2.1 *The 1-form ω is almost F -analytic if and only if its F -conjugate $\bar{\omega}$ is almost F -analytic. If ω is almost F -analytic then ω is closed if and only if $\bar{\omega}$ is closed.*

Proof Using (2.1), (2.3), and (2.4) we get:

$$\Omega_{F, \bar{\omega}}(X, Y) = -\frac{1}{\varepsilon} \Omega_{F, \omega}(FX, Y), \quad \Omega_{F, \omega}(X, Y) = -\Omega_{F, \bar{\omega}}(FX, Y) \tag{2.6}$$

and the conclusion follows directly from (2.6). □

Recall now the Nijenhuis tensor field of F :

$$N_F(X, Y) := [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y], \tag{2.7}$$

which for our case (2.1) becomes $N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + \varepsilon[X, Y]$. We have the following skew-symmetries:

$$N_F(FX, Y) = -FN_F(X, Y) = N_F(X, FY), \quad N_F(FX, FY) = \varepsilon N_F(X, Y) \tag{2.8}$$

which yields a second property of almost F -analytic forms:

Proposition 2.2 *If ω is almost F -analytic then:*

$$\omega \circ N_F = \bar{\omega} \circ N_F = 0. \tag{2.9}$$

Proof Let ω be almost F -analytic. Using (2.5), $\omega \circ F = \varepsilon\bar{\omega}$, and $\bar{\omega} \circ F = \omega$, from $\Omega_{F,\omega}(X, Y) = d\omega(FX, Y) - \varepsilon d\bar{\omega}(X, Y) = 0$ we easily obtain:

$$\omega(F[X, Y]) = \varepsilon X\bar{\omega}(Y) - FX\omega(Y) + \omega([FX, Y]). \tag{2.10}$$

Putting $X \mapsto FX$ and $Y \mapsto FY$ in (2.10), by direct calculus we obtain:

$$\begin{aligned} (\omega \circ N_F)(X, Y) &= \omega([FX, FY]) - \omega(F[FX, Y]) - \omega(F[X, FY]) + \varepsilon\omega([X, Y]) \\ &= \omega([FX, FY]) - \varepsilon FX(\bar{\omega}(Y)) + \varepsilon X\omega(Y) - \varepsilon\omega([X, Y]) \\ &\quad - \varepsilon X\omega(Y) + \varepsilon FX\bar{\omega}(Y) - \omega([FX, FY]) + \varepsilon\omega([X, Y]) = 0. \end{aligned}$$

By Proposition 2.1 $\bar{\omega}$ is also almost F -analytic and the relation $(\bar{\omega} \circ N_F)(X, Y) = 0$ follows in a similar manner starting from $\Omega_{F,\bar{\omega}}(X, Y) = d\bar{\omega}(FX, Y) - d\omega(X, Y) = 0$. □

Another tool in our study is provided by the Obata operators associated to F , namely the maps $O_F, O_F^* : \Omega^2(M) \rightarrow \Omega^2(M)$:

$$\begin{cases} O_F(\rho)(X, Y) := \frac{1}{2}[\rho(X, Y) - \rho(FX, FY)] \\ O_F^*(\rho)(X, Y) := \frac{1}{2}[\rho(X, Y) + \rho(FX, FY)], \end{cases} \tag{2.11}$$

which give a classification of 2-forms with respect to F :

Definition 2.3 *The 2-form ρ is called F -pure if $O_F^*(\rho) = 0$ and respectively F -hybrid if $O_F(\rho) = 0$.*

Proposition 2.3 *i) If F is an almost complex structure ($\varepsilon = -1$) and ω is almost F -analytic form then the 2-forms $d\omega, d\bar{\omega}$ are F -pure.*

ii) If F is an almost product structure ($\varepsilon = 1$) and ω is almost F -analytic form then the 2-forms $d\omega, d\bar{\omega}$ are F -hybrid.

Proof i) Let $\varepsilon = -1$. From the characterization of almost F -analyticity, setting $X \mapsto FX$ in (2.10) we have:

$$X(\omega(Y)) + FX(\omega(FY)) = \omega([X, Y]) + \omega(F[FX, Y]), \tag{2.12}$$

and now $X \rightarrow Y$ in (2.12):

$$Y(\omega(X)) + FY(\omega(FX)) = -\omega([X, Y]) - \omega(F[X, FY]). \tag{2.13}$$

From (2.13) minus (2.12) we get:

$$2d\omega + \omega([X, Y]) + 2d\bar{\omega}(FX, FY) + \omega([FX, FY]) = 2\omega([X, Y]) + \omega \circ F([FX, Y] + [X, FY]),$$

which means:

$$4O_F^*(d\omega) = -\omega \circ N_F = 0.$$

By analogy:

$$4O_F^*(d\bar{\omega}) = -\bar{\omega} \circ N_F = 0.$$

ii) Let $\varepsilon = 1$. Again, with $X \rightarrow FX$ in relation (2.10) we have:

$$X(\omega(Y)) - FX(\omega(FY)) = \omega([X, Y]) - \omega(F[FX, Y]) \tag{2.14}$$

and $X \leftrightarrow Y$ in this equality gives:

$$Y(\omega(X)) - FY(\omega(FX)) = -\omega([X, Y]) + \omega(F[X, FY]). \tag{2.15}$$

With (2.14) minus (2.15) we obtain:

$$2d\omega(X, Y) + \omega([X, Y]) - 2d\bar{\omega}([FX, FY]) - \omega([FX, FY]) = 2\omega([X, Y]) - \omega \circ F([FX, Y] + [X, FY]),$$

which means: $4O_F(d\omega) = \omega \circ N_F = 0$. Also: $4O_F(d\bar{\omega}) = \bar{\omega} \circ N_F = 0$ and the assertion is proved. □

An important consequence of this result is the following:

Corollary 2.1 *If $\varepsilon \in \{-1, +1\}$ then definition (2.4) and hence the definition of almost F -analyticity do not depend on the place of F .*

Proof From Proposition 2.3 we have that the almost F -analyticity implies:

$$d\omega(X, Y) = \varepsilon d\omega(FX, FY), \tag{2.16}$$

and then the right-hand side of (2.4) is:

$$d\omega(FX, Y) - \varepsilon d\bar{\omega}(X, Y) = \varepsilon d\omega(F^2X, FY) - \varepsilon d\bar{\omega}(X, Y) = \varepsilon^2 d\omega(X, FY) - \varepsilon d\bar{\omega}(X, Y)$$

and since $\varepsilon^2 = 1$ we get the conclusion. □

We finish this subsection with a relationship of this formalism with the Frölicher–Nijenhuis theory. Recall that given a tensor field F of $(1, 1)$ -type it defines the following:

i) an interior product i_F ; for an r -form ω we have that $i_F\omega$ is again an r -form given by:

$$i_F\omega(X_1, \dots, X_r) := \sum_{i=1}^r \omega(X_1, \dots, FX_i, \dots, X_r), \quad r \geq 1 \text{ and } i_F f = 0, \quad \forall f \in C^\infty(M); \tag{2.17}$$

ii) an exterior F -derivative d_F with:

$$d_F := i_F \circ d - d \circ i_F. \tag{2.18}$$

Proposition 2.4 *If $\varepsilon = \pm 1$ and ω is almost F -analytic then the exterior F -derivatives of ω and $\bar{\omega}$ are:*

$$d_F\omega = \frac{1}{2}i_F \circ d\omega = \varepsilon d\bar{\omega}, \quad d_F\bar{\omega} = d\omega. \tag{2.19}$$

Proof For $r = 1$ we have:

$$i_F\omega = \varepsilon\bar{\omega} \tag{2.20}$$

and then:

$$\begin{aligned} (d_F\omega)(X, Y) &= i_F(d\omega)(X, Y) - d(\varepsilon\bar{\omega})(X, Y) \\ &= d\omega(FX, Y) + d\omega(X, FY) - \varepsilon d\bar{\omega}(X, Y) = \Omega_{F,\omega}(X, Y) + d\omega(X, FY), \end{aligned}$$

which means that $d_F\omega(\cdot, \cdot) = d\omega(\cdot, F\cdot)$. We apply the previous Corollary 2.1 to get the first part of (2.19). The second part of the required formula follows by duality. \square

Similarly to [6, 10, 16], a smooth function f on M is called *almost F -analytic* if there exists a smooth function g on M such that:

$$df \circ F = dg, \tag{2.21}$$

and in this case g is called *the corresponding function* of f . In this case g is also almost F -analytic with corresponding function εf . Let us denote by $C^\infty(M, F)$ the set of all almost F -analytic functions on M . If $f \in C^\infty(M, F)$, then by (2.20) we have:

$$d_Ff = i_F \circ df = \varepsilon d\bar{f} = df \circ F = dg. \tag{2.22}$$

Proposition 2.5 *If $f \in C^\infty(M, F)$ then df and d_Ff are both almost F -analytic.*

Proof Let $f \in C^\infty(M, F)$. Then:

$$\Omega_{F,df}(X, Y) = (d(df))(FX, Y) - \varepsilon(d(\bar{df}))(X, Y) = -(d(dg))(X, Y) = 0,$$

which says that df is almost F -analytic. The second assertion follows by setting $X \mapsto FX$ in the above relation. \square

2.2. Almost F -analytic r -forms and d_F -cohomology

In this subsection we give a generalization of previous results to r -forms for $r \geq 2$ with ε restricted to $\{-1, +1\}$ and we study the d_F -cohomology of almost analytic r -forms.

Firstly, inspired by Proposition 2.3, we introduce a class of r -forms adapted to F :

Definition 2.4 *The r -form ω is called F -symmetric if for all vector fields X_1, \dots, X_r :*

$$\omega(FX_1, \dots, X_r) = \omega(X_1, \dots, FX_i, \dots, X_r), \quad 2 \leq i \leq r. \tag{2.23}$$

Example 2.1 *i) If $\theta \in \Omega^1(M, F)$ then the 2-forms $\omega = d\theta$ and $\bar{\omega} = d\bar{\theta}$ are F -symmetric. Indeed, equation (2.16) means:*

$$d\theta(X, Y) = \varepsilon d\theta(FX, FY), \quad d\bar{\theta}(X, Y) = \varepsilon d\bar{\theta}(FX, FY)$$

and with $X \rightarrow FX$ we get the conclusion.

ii) More generally than i) if $\varepsilon = +1$ then a F -hybrid 2-form is F -symmetric and for $\varepsilon = -1$ an F -pure 2-form is F -symmetric. \square

Secondly, we associate a conjugate form and an $(r + 1)$ -form:

Definition 2.5 If $\omega \in \Omega^r(M)$ is F -symmetric then its F -conjugate is $\bar{\omega} = \omega_F \in \Omega^r(M)$ given by:

$$\bar{\omega}(X_1, \dots, X_r) := \frac{1}{\varepsilon} \omega(FX_1, \dots, X_r). \tag{2.24}$$

We associate $\Omega_{F,\omega} \in \Omega^{r+1}(M)$ given by:

$$\Omega_{F,\omega}(X_1, \dots, X_{r+1}) := d\omega(FX_1, \dots, X_{r+1}) - \varepsilon d\bar{\omega}(X_1, \dots, X_{r+1}). \tag{2.25}$$

Thirdly, we define the natural generalization of the previous subsection:

Definition 2.6 The F -symmetric form $\omega \in \Omega^r(M)$ is called almost F -analytic if:

$$\Omega_{F,\omega} = 0. \tag{2.26}$$

In order to unify the property that says when an F -symmetric r -form is almost F -analytic for both almost complex and paracomplex cases, we present:

Proposition 2.6 An F -symmetric r -form ω ($r \geq 1$) is almost F -analytic iff

$$\begin{aligned} &FX_1(\omega(X_2, \dots, X_{r+1})) - X_1(\omega(FX_2, \dots, X_{r+1})) = \\ &= \sum_{j=2}^{r+1} (-1)^{1+j} \omega(F[X_1, X_j] - [FX_1, X_j], X_2, \dots, \widehat{X_j}, \dots, X_{r+1}). \end{aligned} \tag{2.27}$$

Proof It follows by a direct calculation involving the definition of the exterior derivative. \square

Remark 2.1 In a more general case of $(0, r)$ -tensor fields we can consider the operator $\Phi_F : \mathcal{T}_r^0(M) \rightarrow \mathcal{T}_{r+1}^0(M)$; see [18]:

$$\begin{aligned} \Phi_F \omega(X, Y_1, \dots, Y_r) &= FX(\omega(Y_1, \dots, Y_r)) - X(\omega(FY_1, Y_2, \dots, Y_r)) \\ &+ \omega((L_{Y_1} F)X, Y_2, \dots, Y_r) + \dots + \omega(Y_1, Y_2, \dots, (L_{Y_r} F)X), \end{aligned} \tag{2.28}$$

for every vector field X, Y_1, \dots, Y_r , where L_X denotes the Lie derivative with respect to X . Then, similarly to [5, 6, 7, 10, 12, 15, 18], the tensor field ω is called almost F -analytic if $\Phi_F \omega = 0$ and for r -forms this condition is equivalent to (2.26).

Let $\Omega^r(M, F)$ be the set of almost F -analytic r -forms.

The following result is a motivation for this notion and also a generalization of the first remark above:

Proposition 2.7 If $\omega \in \Omega^r(M, F)$ then its differential is F -symmetric and its exterior F -differential of ω is:

$$d_F \omega = \frac{1}{r+1} i_F \circ d\omega = \varepsilon d\bar{\omega}. \tag{2.29}$$

Proof The first part follows directly from the skew-symmetry of $d\bar{\omega}$ and the relation:

$$d\omega(FX_1, \dots, X_{r+1}) = \varepsilon d\bar{\omega}(X_1, \dots, X_{r+1}), \tag{2.30}$$

provided by the definition. For the second part we get that $i_F\omega = \varepsilon r\bar{\omega}$ and with a similar calculus as in Proposition 2.4 we derive:

$$d_F\omega(X_1, \dots, X_{r+1}) = d\omega(FX_1, \dots, X_{r+1}),$$

by using the first part. Equation (2.29) follows then directly. □

Proposition 2.8 *The F -symmetric r -form ω is almost F -analytic if and only if $\bar{\omega}$ is almost F -analytic. If ω is almost F -analytic then ω is closed if and only if $\bar{\omega}$ is closed, and equivalently ω and $\bar{\omega}$ are d_F -closed.*

Proof It is sufficient to prove the implication that ω is almost F -analytic $\Rightarrow \bar{\omega}$ is almost F -analytic since:

$$\bar{\bar{\omega}}(X_1, \dots, X_r) = \frac{1}{\varepsilon}\bar{\omega}(FX_1, \dots, X_r) = \omega(F^2X_1, \dots, X_r) = \varepsilon\omega(X_1, \dots, X_r) = \frac{1}{\varepsilon}\omega(X_1, \dots, X_r) \tag{2.31}$$

and remark that almost F -analyticity is invariant with respect to scalings $\omega \rightarrow \lambda\omega$.

Firstly we must prove that $\bar{\omega}$ is F -symmetric. We have:

$$\bar{\omega}(FX_1, \dots, X_r) = \frac{1}{\varepsilon}\omega(F^2X_1, \dots, X_r) = \omega(X_1, \dots, X_r). \tag{2.32}$$

Also:

$$\begin{aligned} \bar{\omega}(X_1, \dots, FX_i, \dots, X_r) &= \frac{1}{\varepsilon}\omega(FX_1, \dots, FX_i, \dots, X_r) \\ &= \frac{1}{\varepsilon}\omega(X_1, \dots, F^2X_i, \dots, X_r) = \omega(X_1, \dots, X_r), \end{aligned}$$

which is what we claim.

Secondly, we must verify Definition 2.6. A straightforward calculation gives the generalization of (2.6):

$$\Omega_{F,\bar{\omega}}(X_1, \dots, X_{r+1}) = -\frac{1}{\varepsilon}\Omega_{F,\omega}(FX_1, \dots, X_{r+1}) \tag{2.33}$$

and the conclusion follows. □

Proposition 2.9 *If $\omega \in \Omega^r(M, F)$ then:*

$$\omega(N_F(X_1, X_2), \dots, X_{r+1}) = \bar{\omega}(N_F(X_1, X_2), \dots, X_{r+1}) = 0. \tag{2.34}$$

Proof Using the characterization of almost F -analyticity of ω from (2.27) but with $X_2 \mapsto FX_2$, we have

$$\begin{aligned} &FX_1(\omega(FX_2, \dots, X_{r+1})) - \varepsilon X_1(\omega(X_2, \dots, X_{r+1})) = \\ &= -\omega(F[X_1, FX_2] - [FX_1, FX_2], X_3, \dots, X_{r+1}) + \\ &+ \sum_{j=3}^{r+1} (-1)^{1+j}\omega(F[X_1, X_j] - [FX_1, X_j], FX_2, X_3, \dots, \widehat{X_j}, \dots, X_{r+1}). \end{aligned} \tag{2.35}$$

On the other hand, $\bar{\omega} \in \Omega^r(M, F)$, too, and using again (2.27) for $\bar{\omega}$, we have

$$\begin{aligned} & FX_1(\omega(FX_2, \dots, X_{r+1})) - \varepsilon X_1(\omega(X_2, \dots, X_{r+1})) = \\ & = -\omega(\varepsilon[X_1, X_2] - F[FX_1, X_2], X_3, \dots, X_{r+1}) + \\ & + \sum_{j=3}^{r+1} (-1)^{1+j} \omega(F[X_1, X_j] - [FX_1, X_j], FX_2, X_3, \dots, \widehat{X_j}, \dots, X_{r+1}). \end{aligned} \tag{2.36}$$

Now, by (2.35) and (2.36), the first equality follows easily. The second equality follows in a similar manner. \square

Inspired by the $\varepsilon = -1$ case, we suppose now that $m = 2n$ and for the $\varepsilon = +1$ we suppose that F is an almost paracomplex structure, i.e. the dimensions of $(+1)$ -eigenspace and (-1) -eigenspaces are both equal to n . It follows for both cases of ε the existence of local basis of vector fields of type $B = \{e_1, \dots, e_n, Fe_1, \dots, Fe_n\}$, where for the case $\varepsilon = 1$ we must have $F \neq \text{Id}$, and then there exist nontrivial F -symmetric r forms only for $r \leq n$. An important result for this choice of dimension is:

Proposition 2.10 *An F -symmetric n -form ω is almost F -analytic if and only if ω and $\bar{\omega}$ are both closed.*

Proof Suppose firstly that ω is almost F -analytic. When its differential is applied on data $\{FX_1, X_1, \dots, X_n\}$ of elements of B we have $d\omega(FX_1, X_1, \dots, X_n) = \varepsilon d\bar{\omega}(X_1, X_1, \dots, X_n) = 0$ and deduce that ω (and consequently $\bar{\omega}$) is closed. The proof of the converse part is directly from Definition 2.25 \square

We introduce now an exterior product adapted to our setting:

Definition 2.7 *The exterior F -product is the map $\wedge_F : \Omega^r(M) \times \Omega^s(M) \rightarrow \Omega^{r+s}(M)$ given by:*

$$\theta \wedge_F \omega := \theta \wedge \omega + \varepsilon \bar{\theta} \wedge \bar{\omega} \tag{2.37}$$

where \wedge is the usual exterior product of M .

A long but straightforward computation in the basis B gives:

Proposition 2.11 *Let θ and ω be F -symmetric forms.*

- i) *The $(r + s)$ -form $\theta \wedge_F \omega$ is also F -symmetric.*
- ii) *The F -conjugate of the $(r + s)$ -form above is:*

$$(\theta \wedge_F \omega)_F = \theta \wedge \bar{\omega} + \bar{\theta} \wedge \omega. \tag{2.38}$$

As a consequence, if θ and ω are almost F -analytic forms then $\theta \wedge_F \omega$ is also an almost F -analytic form.

Proposition 2.12 *Let $\omega \in \Omega^r(M, F)$ and $\theta \in \Omega^s(M, F)$, $r, s \geq 0$, where $\Omega^0(M, F) = C^\infty(M, F)$. Then:*

- i) $d_F \omega \in \Omega^{r+1}(M, F)$;
- ii) $d_F^2 \omega = 0$;
- iii) $d_F(\omega \wedge_F \theta) = d_F \omega \wedge_F \theta + (-1)^r \omega \wedge_F d_F \theta$.

Proof i) If $\omega \in \Omega^r(M, F)$ then by (2.24) and (2.30) we have:

$$d\bar{\omega} = \overline{d\omega}. \tag{2.39}$$

Now, using (2.29) and (2.39), we have:

$$\Omega_{F, d_F\omega}(X_1, \dots, X_{r+2}) = (d(\varepsilon d\bar{\omega}))(FX_1, \dots, X_{r+2}) - \varepsilon(d(d\omega))(X_1, \dots, X_{r+2}) = 0,$$

which says that $d_F\omega \in \Omega^{r+1}(M, F)$.

ii) Using (2.29), (2.31), and (2.39) we have:

$$d_F(d_F\omega) = d_F(\varepsilon d\bar{\omega}) = \varepsilon^2 d(\overline{d\omega}) = \varepsilon^2 d(d\omega) = \varepsilon d(d\omega) = 0.$$

iii) Follows using (2.29), (2.37), and (2.38). □

We notice that $(\Omega^r(M, F), \wedge_F)$ is a graded $C^\infty(M, F)$ -algebra. Also, by ii) Proposition 2.12 we have the differential complex $(\Omega^\bullet(M, F), d_F)$ and its cohomology $H^\bullet(M, F)$ is called the d_F -cohomology of almost F -analytic forms on M .

Another important property of the operator d_F is the following Poincaré type lemma:

Theorem 2.1 *Let $\omega \in \Omega^r(U, F)$, $r \geq 1$, where $U \subset M$ such that $d_F\omega = 0$ on U . Then there exists $\theta \in \Omega^{r-1}(U', F)$ where $U' \subset U$ such that $\omega = d_F\theta$ on U' .*

Proof Let ω as in the hypothesis. Taking into account that $d_F\omega = 0$ is equivalent with $d\bar{\omega} = 0$ and by applying the classical Poincaré lemma for the operator d it follows that there exists $\theta \in \Omega^r(U')$ where $U' \subset U$ and such that $\bar{\omega} = d\theta$ on U' . From $0 = \overline{d\bar{\omega}} = \overline{d\bar{\omega}} = \frac{1}{\varepsilon}d\omega$ it follows also by Poincaré lemma for the operator d that there exists $\theta_1 \in \Omega^r(U')$ where $U' \subset U$ and such that $\omega = d\theta_1$ on U' .

Similar arguments as in the proof of Theorem 1 from [9] show that both θ and θ_1 are almost F -analytic and $\theta = \overline{\theta_1}$. Now, the proof follows easily since $\omega = d\theta_1 = \varepsilon d\bar{\theta} = d_F\theta$. □

We notice that $\ker\{d_F : \Omega^0(U, F) \rightarrow \Omega^1(U, F)\} \cong \tilde{\mathbb{R}}$ where $\tilde{\mathbb{R}}$ is the sheaf of germs associated to the constant pre-sheaf \mathbb{R} . Also, consider $\Phi^r(M, F)$ the sheaf of germs of almost F -analytic r -forms on M and $i : \tilde{\mathbb{R}} \rightarrow \Phi^0(M, F)$ the natural inclusion. The sheaves $\Phi^r(M, F)$ are fine and taking into account Theorem 2.1 it follows that the following sequence of sheaves:

$$0 \longrightarrow \tilde{\mathbb{R}} \xrightarrow{i} \Phi^0(M, F) \xrightarrow{d_F} \Phi^1(M, F) \xrightarrow{d_F} \dots \xrightarrow{d_F} \Phi^n(M, F) \xrightarrow{d_F} 0$$

is a fine resolution of $\tilde{\mathbb{R}}$ and we denote by $H^r(M, F; \tilde{\mathbb{R}})$ the cohomology groups of M with coefficients in the sheaf $\tilde{\mathbb{R}}$. Thus, we obtain a de Rham theorem for the d_F -cohomology of almost F -analytic forms, namely:

Theorem 2.2 *Then d_F -cohomology groups of almost F -analytic forms on M are given by:*

- i) $H^0(M, F; \tilde{\mathbb{R}}) \cong \mathbb{R}$,
- ii) $H^r(M, F; \tilde{\mathbb{R}}) \cong H^r(M, F)$, $1 \leq r \leq n - 1$,
- iii) $H^n(M, F; \tilde{\mathbb{R}}) \cong \Omega^n(M, F)/d_F(\Omega^{n-1}(M, F))$,

iv) $H^r(M, F; \tilde{\mathbb{R}}) = 0, n + 1 \leq r \leq 2n.$

We consider now a deformation of almost F -analytic forms with pairs of almost F -analytic functions:

Definition 2.8 Fix $\omega \in \Omega^r(M)$ an almost F -analytic form and $\alpha, \beta \in C^\infty(M).$ The (α, β) -deformation of ω is the r -form:

$$\omega_{\alpha, \beta} := \alpha\omega + \beta\bar{\omega}. \tag{2.40}$$

Since $\omega_{\alpha, \beta}$ is an F -symmetric form it is natural to ask in what conditions regarding these functions the new r -form is also an almost F -analytic one:

Proposition 2.13 The F -symmetric r -form $\omega_{\alpha, \beta}$ is almost F -analytic if and only if α is almost F -analytic with corresponding function $\beta.$

Proof We have:

$$(\omega_{\alpha, \beta})_F = \alpha\bar{\omega} + \frac{\beta}{\varepsilon}\omega = (\omega_F)_{\alpha, \beta}. \tag{2.41}$$

The proof is easy to see in the case $r = 1$ where the almost F -analyticity of $\omega_{\alpha, \beta}$ means:

$$d\alpha(FX)\omega(Y) + \frac{1}{\varepsilon}d\beta(FX)\omega(FY) = d\alpha(X)\omega(FY) + d\beta(X)\omega(Y) \tag{2.42}$$

for all vector fields $X, Y.$ A detailed proof for $r \geq 2$ can be found in [3] for the case of almost (para) complex Lie algebroids. □

This results yields the introduction of the set:

$$\tilde{C}^\infty(M, F) = \{(\alpha, \beta) \in C^\infty(M, F) \times C^\infty(M, F); \quad d\beta = d\alpha \circ F\}. \tag{2.43}$$

A straightforward computation gives that $\tilde{C}^\infty(M, F)$ is a commutative algebra with respect to the product:

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) := (\alpha_1\alpha_2 + \varepsilon\beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1), \tag{2.44}$$

having as a unit the pair of the constant functions $(1, 0) \in \tilde{C}^\infty(M, F).$ The inverse of the element $(\alpha, \beta) \in \tilde{C}^\infty(M, F)$ different from $(0, 0)$ is the pair $\left(\frac{\alpha}{\alpha^2 - \varepsilon\beta^2}, \frac{-\beta}{\alpha^2 - \varepsilon\beta^2}\right);$ for the case $\varepsilon = +1$ we also exclude the cases $(\alpha, \pm\alpha).$

Let us introduce the set of pairs of forms:

$$\tilde{\Omega}^r(M, F) = \{(\omega, \bar{\omega}); \omega \in \Omega^r(M, F)\}. \tag{2.45}$$

Proposition 2.13 says that $\tilde{\Omega}^r(M, F)$ is a $\tilde{C}^\infty(M, F)$ -module for all $1 \leq r \leq n$ and hence the set

$$\tilde{\Omega}(M, F) = \sum_{r=1}^n \tilde{\Omega}^r(M, F)$$

is a graded $\tilde{C}^\infty(M, F)$ -algebra. We consider the wedge product

$$(\omega, \bar{\omega})\tilde{\wedge}(\theta, \bar{\theta}) = (\omega \wedge_F \theta, (\omega \wedge_F \theta)_F) \tag{2.46}$$

and the operator

$$D : \tilde{\Omega}^r(M, F) \rightarrow \tilde{\Omega}^{r+1}(M, F), \quad D(\omega, \bar{\omega}) = (d\omega, d\bar{\omega}). \tag{2.47}$$

It follows that:

- i) D is a local operator and \mathbb{R} -linear;
- ii) for every $(\omega, \bar{\omega}) \in \tilde{\Omega}^r(M, F)$ and $(\theta, \bar{\theta}) \in \tilde{\Omega}^s(M, F)$ we have

$$D [(\omega, \bar{\omega}) \tilde{\wedge} (\theta, \bar{\theta})] = D(\omega, \bar{\omega}) \tilde{\wedge} (\theta, \bar{\theta}) + (-1)^r (\omega, \bar{\omega}) \tilde{\wedge} D(\theta, \bar{\theta});$$

- iii) $D^2 = (0, 0)$;

and an associated cohomology of the differential complex $(\tilde{\Omega}(M, F), D)$ can be considered exactly as in [9].

3. Almost analytic forms on almost para-Norden manifolds and examples

We continue with the setting of Subsection 2.2, namely $\varepsilon = \pm 1$, but we add a Riemannian metric g to our framework, which satisfies

$$g(FX, Y) = \varepsilon g(X, FY). \tag{3.1}$$

Then:

- a) for $\varepsilon = -1$ the triple (M, F, g) is an usual almost Hermitian manifold,
- b) for $\varepsilon = +1$ the triple (M, F, g) is an almost para-Norden manifold; see, for instance, [11].

In order to unify these cases we get the following formula:

$$g(FX, FY) = g(X, Y), \quad \forall X, Y \in \mathcal{X}(M). \tag{3.2}$$

The fundamental 2-form of an almost Hermitian manifold is $\omega(X, Y) := g(X, FY)$, which is not F -symmetric, since $\omega(FX, Y) = -\omega(X, FY)$, while the symmetric bilinear form $\omega(X, Y) := g(X, FY)$ associated to an almost para-Norden manifold is F -symmetric.

The characterization of almost analyticity of differential forms on almost Hermitian manifolds in terms of their harmonicity was studied in [13]. In order to unify these results for both cases presented above, in this section we extend some similar results for the case of almost para-Norden manifolds.

The metric g yields the Hodge star operator \star and the orthonormal basis B of the type discussed above. Hence, similar to the almost Hermitian case, see Proposition 2.3 in [13, p. 77], a direct computation yields:

Proposition 3.1 *If the n -form ω is F -symmetric on the almost para-Norden manifold (M^{2n}, F, g) then $\star\omega$ is also F -symmetric.*

The important consequence of this result is:

Proposition 3.2 *If ω is an almost F -analytic n -form on the almost para-Norden manifold (M^{2n}, F, g) then $\star\omega$ is also almost F -analytic.*

We arrive now to the main result of this section, which provides a large class of almost F -analytic forms:

Proposition 3.3 *An F -symmetric n -form on the almost para-Norden manifold (M, g, F) is almost F -analytic if and only if ω and $\bar{\omega}$ are both harmonic.*

Proof It is a direct consequence of $d\omega = d(\star\omega) = 0$. □

Suppose $n = 2$ and $\varepsilon = -1$. By using the corollary 18 of [8, p. 208] it results that on a compact, oriented surface M^2 with positive Ricci (equivalently Gaussian, if M is embedded in \mathbb{R}^3) curvature at one point we have $\Omega^1(M, F) = 0$.

We end this section with some examples of (almost) F -analytic forms. In order to find large classes of almost F -analytic forms we suppose now that F is integrable. Then we call *F -analytic forms* the differential forms studied until now.

The integrability of F yields the local coordinates $\{x^i, y^i; 1 \leq i \leq n\}$ such that the expression of F is:

$$F\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad F\left(\frac{\partial}{\partial y^i}\right) = \varepsilon \frac{\partial}{\partial x^i}. \tag{3.3}$$

Let $\omega = a_i dx^i + b_i dy^i$ be a 1-form on M ; hence, $\bar{\omega} = \varepsilon b_i dx^i + a_i dy^i$. The F -analyticity of ω means:

$$FX(\omega(Y)) - \omega([FX, Y]) = X(\omega(FY)) - \omega(F[X, Y]), \tag{3.4}$$

and the choice of X, Y in the basis $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}; 1 \leq i \leq n\}$ gives the following characterization:

Theorem 3.1 *The 1-form ω is an F -analytic form if and only if its coefficients satisfy the ε -Cauchy–Riemann equations:*

$$\frac{\partial a_j}{\partial y^i} = \frac{\partial b_j}{\partial x^i}, \quad \frac{\partial a_j}{\partial x^i} = \varepsilon \frac{\partial b_j}{\partial y^i}. \tag{3.5}$$

Similarly, the pair of smooth functions (α, β) belongs to $\tilde{C}^\infty(M, F)$ if and only if α and β satisfies the ε -Cauchy–Riemann equations (3.5).

A natural framework where quadratic endomorphisms are involved is provided by ε -contact structures, namely triples (φ, ξ, η) consisting of an endomorphism, a vector field, and a 1-form on M^{2n+1} satisfying:

$$\varphi^2 = \varepsilon(I_M - \eta \otimes \xi), \quad \eta(\xi) = 1. \tag{3.6}$$

For $\varepsilon = -1$ we get the almost contact geometry [2], while for $\varepsilon = +1$ we have the almost paracontact geometry [19]. On the product manifold $M \times \mathbb{R}$ we consider:

$$J(X, a \frac{d}{dt}) = (\varphi X + \varepsilon a \xi, \eta(X) \frac{d}{dt}), \tag{3.7}$$

and a straightforward computation yields that $J^2 = \varepsilon I_{M \times \mathbb{R}}$. For the 1-form $\omega_b = \eta + bdt$ with $b \in \mathbb{R}$, its conjugate with respect to J is:

$$(\omega_b)_J = \varepsilon b \eta + dt, \tag{3.8}$$

and then ω_b is almost J -analytic form if and only if:

$$d\eta(\varphi X, Y) = b\varepsilon d\eta(X, Y) \tag{3.9}$$

for all vector fields X, Y . In particular, if (M, φ, ξ, η) is ε -cosymplectic, i.e. η is closed, then all ω_b are almost J -analytic.

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