

1-1-2015

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BELKHECHINE, HOUMEM; BOUDABBOUS, IMED; and HZAMI, KAOUTHAR (2015) "The prime tournaments T with $W_5(T) = T - 2$," *Turkish Journal of Mathematics*: Vol. 39: No. 4, Article 13. <https://doi.org/10.3906/mat-1412-72>

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The prime tournaments T with $|W_5(T)| = |T| - 2$

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Received: 30.12.2014

Accepted/Published Online: 20.05.2015

Printed: 30.07.2015

Abstract: We consider a tournament $T = (V, A)$. For $X \subseteq V$, the subtournament of T induced by X is $T[X] = (X, A \cap (X \times X))$. A module of T is a subset X of V such that for $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$. The trivial modules of T are \emptyset , $\{x\} (x \in V)$, and V . A tournament is prime if all its modules are trivial. For $n \geq 2$, W_{2n+1} denotes the unique prime tournament defined on $\{0, \dots, 2n\}$ such that $W_{2n+1}[\{0, \dots, 2n-1\}]$ is the usual total order. Given a prime tournament T , $W_5(T)$ denotes the set of $v \in V$ such that there is $W \subseteq V$ satisfying $v \in W$ and $T[W]$ is isomorphic to W_5 . B.J. Latka characterized the prime tournaments T such that $W_5(T) = \emptyset$. The authors proved that if $W_5(T) \neq \emptyset$, then $|W_5(T)| \geq |V| - 2$. In this article, we characterize the prime tournaments T such that $|W_5(T)| = |V| - 2$.

Key words: Tournament, prime, embedding, critical, partially critical

1. Introduction

1.1. Preliminaries

A tournament $T = (V(T), A(T))$ (or (V, A)) consists of a finite set V of vertices together with a set A of ordered pairs of distinct vertices, called arcs, such that for all $x \neq y \in V$, $(x, y) \in A$ if and only if $(y, x) \notin A$. The cardinality of T , denoted by $|T|$, is that of $V(T)$. Given a tournament $T = (V, A)$, with each subset X of V is associated the subtournament $T[X] = (X, A \cap (X \times X))$ of T induced by X . For $X \subseteq V$ (resp. $x \in V$), the subtournament $T[V \setminus X]$ (resp. $T[V \setminus \{x\}]$) is denoted by $T - X$ (resp. $T - x$). Two tournaments $T = (V, A)$ and $T' = (V', A')$ are isomorphic, which is denoted by $T \simeq T'$, if there exists an isomorphism from T onto T' , i.e. a bijection f from V onto V' such that for all $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. We say that a tournament T' embeds into T if T' is isomorphic to a subtournament of T . Otherwise, we say that T omits T' . The tournament T is said to be transitive if it omits the tournament $C_3 = (\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$. For a finite subset V of \mathbb{N} , we denote by \vec{V} the usual total order defined on V , i.e., the transitive tournament $(V, \{(i, j) : i < j\})$.

Some notations are needed. Let $T = (V, A)$ be a tournament. For two vertices $x \neq y \in V$, the notation $x \rightarrow y$ signifies that $(x, y) \in A$. Similarly, given $x \in V$ and $Y \subseteq V$, the notation $x \rightarrow Y$ (resp. $Y \rightarrow x$) means that $x \rightarrow y$ (resp. $y \rightarrow x$) for all $y \in Y$. Given $x \in V$, we set $N_T^+(x) = \{y \in V : x \rightarrow y\}$. For all

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2010 AMS Mathematics Subject Classification: 05C20, 05C60, 05C75.

$n \in \mathbb{N} \setminus \{0\}$, the set $\{0, \dots, n - 1\}$ is denoted by \mathbb{N}_n .

Let $T = (V, A)$ be a tournament. A subset I of V is a *module* [11] (or a *clan* [7]) of T provided that for all $x \in V \setminus I$, $x \rightarrow I$ or $I \rightarrow x$. For example, \emptyset , $\{x\}$, where $x \in V$, and V are modules of T , called *trivial* modules. A tournament is *prime* [4] (or *primitive* [7]) if all its modules are trivial. Notice that a tournament $T = (V, A)$ and its *dual* $T^* = (V, \{(x, y) : (y, x) \in A\})$ share the same modules. Hence, T is prime if and only if T^* is.

For $n \geq 2$, we introduce the tournament W_{2n+1} defined on \mathbb{N}_{2n+1} as follows: $W_{2n+1}[\mathbb{N}_{2n}] = \overrightarrow{\mathbb{N}_{2n}}$ and $N_{W_{2n+1}}^+(2n) = \{2i : i \in \mathbb{N}_n\}$ (see Figure 1). In 2003, B.J. Latka [8] characterized the prime tournaments omitting the tournament W_5 . In 2012, the authors were interested in the set $W_5(T)$ of the vertices x of a prime tournament $T = (V, A)$ for which there exists a subset X of V such that $x \in X$ and $T[X] \simeq W_5$. They obtained the following.

Theorem 1 ([1]) *Let T be a prime tournament into which W_5 embeds. Then $|W_5(T)| \geq |T| - 2$. If, in addition, $|T|$ is even, then $|W_5(T)| \geq |T| - 1$.*

Our main result in this paper, presented in [3] without detailed proof, gives a characterization of the class \mathcal{T} of the prime tournaments T on at least 3 vertices such that $|W_5(T)| = |T| - 2$. This answers [1, Problem 4.4].

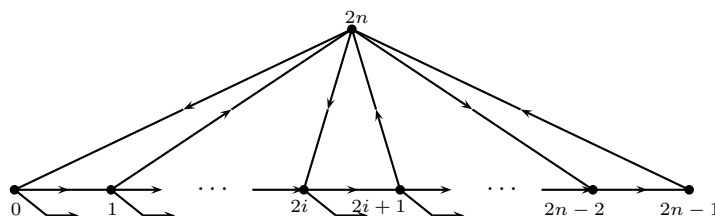


Figure 1. W_{2n+1}

1.2. Partially critical tournaments and the class \mathcal{T}

Our characterization of the tournaments of the class \mathcal{T} requires the study of their partial criticality structure, a notion introduced as a weakening of the notion of criticality defined in Section 2. These notions are defined in terms of critical vertices. A vertex x of a prime tournament T is *critical* [10] if $T - x$ is not prime. The set of noncritical vertices of a prime tournament T was introduced in [9]. It is called the *support* of T and is denoted by $\sigma(T)$. Let T be a prime tournament and let X be a subset of $V(T)$ such that $|X| \geq 3$; we say that T is *partially critical according to $T[X]$* (or $T[X]$ -critical) [6] if $T[X]$ is prime and if $\sigma(T) \subseteq X$. We will see that: for $T \in \mathcal{T}$, $V(T) \setminus W_5(T) = \sigma(T)$. Partially critical tournaments are characterized by M.Y. Sayar in [9]. In order to recall this characterization, we first introduce the tools used to this end. Given a tournament $T = (V, A)$, with each subset X of V , such that $|X| \geq 3$ and $T[X]$ is prime, are associated the following subsets of $V \setminus X$:

- $\langle X \rangle = \{x \in V \setminus X : x \rightarrow X \text{ or } X \rightarrow x\}$.
- For all $u \in X$, $X(u) = \{x \in V \setminus X : \{u, x\} \text{ is a module of } T[X \cup \{x\}]\}$.
- $\text{Ext}(X) = \{x \in V \setminus X : T[X \cup \{x\}] \text{ is prime}\}$.

The family $\{X(u) : u \in X\} \cup \{\text{Ext}(X), \langle X \rangle\}$ is denoted by p_X^T .

Lemma 1 ([7]) Let $T = (V, A)$ be a tournament and let X be a subset of V such that $|X| \geq 3$ and $T[X]$ is prime. The nonempty elements of p_X^T constitute a partition of $V \setminus X$ and satisfy the following assertions:

- For $u \in X$, $x \in X(u)$, and $y \in V \setminus (X \cup X(u))$, if $T[X \cup \{x, y\}]$ is not prime, then $\{u, x\}$ is a module of $T[X \cup \{x, y\}]$.
- For $x \in \langle X \rangle$ and $y \in V \setminus (X \cup \langle X \rangle)$, if $T[X \cup \{x, y\}]$ is not prime, then $X \cup \{y\}$ is a module of $T[X \cup \{x, y\}]$.
- For $x \neq y \in \text{Ext}(X)$, if $T[X \cup \{x, y\}]$ is not prime, then $\{x, y\}$ is a module of $T[X \cup \{x, y\}]$.

Furthermore, $\langle X \rangle$ is divided into $X^- = \{x \in \langle X \rangle : x \rightarrow X\}$ and $X^+ = \{x \in \langle X \rangle : X \rightarrow x\}$. Similarly, for all $u \in X$, $X(u)$ is divided into $X^-(u) = \{x \in X(u) : x \rightarrow u\}$ and $X^+(u) = \{x \in X(u) : u \rightarrow x\}$. We then introduce the family $q_X^T = \{\text{Ext}(X), X^-, X^+\} \cup \{X^-(u) : u \in X\} \cup \{X^+(u) : u \in X\}$.

A graph $G = (V(G), E(G))$ (or (V, E)) consists of a finite set V of vertices together with a set E of unordered pairs of distinct vertices, called *edges*. Given a vertex x of a graph $G = (V, E)$, the set $\{y \in V, \{x, y\} \in E\}$ is denoted by $N_G(x)$. With each subset X of V is associated the *subgraph* $G[X] = (X, E \cap \binom{X}{2})$ of G induced by X . An isomorphism from a graph $G = (V, E)$ onto a graph $G' = (V', E')$ is a bijection f from V onto V' such that for all $x, y \in V$, $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$. We now introduce the graph G_{2n} defined on \mathbb{N}_{2n} , where $n \geq 1$, as follows. For all $x, y \in \mathbb{N}_{2n}$, $\{x, y\} \in E(G_{2n})$ if and only if $|y - x| \geq n$ (see Figure 2).

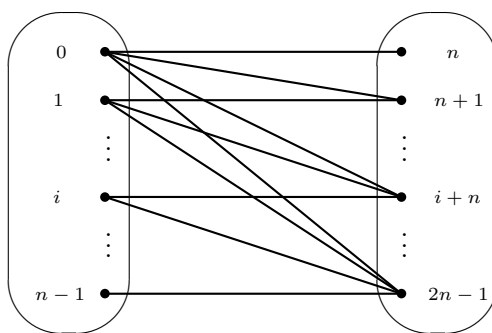


Figure 2. G_{2n}

A graph G is *connected* if for all $x \neq y \in V(G)$, there is a sequence $x_0 = x, \dots, x_m = y$ of vertices of G such that for all $i \in \mathbb{N}_m$, $\{x_i, x_{i+1}\} \in E(G)$. For example, the graph G_{2n} is connected. A *connected component* of a graph G is a maximal subset X of $V(G)$ (with respect to inclusion) such that $G[X]$ is connected. The set of the connected components of G is a partition of $V(G)$, denoted by $\mathcal{C}(G)$. Let $T = (V, A)$ be a prime tournament. With each subset X of V such that $|X| \geq 3$ and $T[X]$ is prime, is associated its *outside graph* G_X^T defined by $V(G_X^T) = V \setminus X$ and $E(G_X^T) = \{\{x, y\} \in \binom{V \setminus X}{2} : T[X \cup \{x, y\}] \text{ is prime}\}$. We now present the characterization of partially critical tournaments.

Theorem 2 ([9]) Consider a tournament $T = (V, A)$ with a subset X of V such that $|X| \geq 3$ and $T[X]$ is prime. The tournament T is $T[X]$ -critical if and only if the assertions below hold.

1. $Ext(X) = \emptyset$.
2. For all $u \in X$, the tournaments $T[X(u) \cup \{u\}]$ and $T[\langle X \rangle \cup \{u\}]$ are transitive.
3. For each $Q \in \mathcal{C}(G_X^T)$, there is an isomorphism f from G_{2n} onto $G_X^T[Q]$ such that $Q_1, Q_2 \in q_X^T$, where $Q_1 = f(\mathbb{N}_n)$ and $Q_2 = f(\mathbb{N}_{2n} \setminus \mathbb{N}_n)$. Moreover, for all $x \in Q_i$ ($i = 1$ or 2), $|N_{G_X^T}(x)| = |N_{T[Q_i]}^+(x)| + 1$ (resp. $n - |N_{T[Q_i]}^+(x)|$) if $Q_i = X^+$ or $X^-(u)$ (resp. $Q_i = X^-$ or $X^+(u)$), where $u \in X$.

The next corollary follows from Theorem 2 and Lemma 1.

Corollary 1 *Let T be a $T[X]$ -critical tournament, T is entirely determined up to isomorphism by giving $T[X]$, q_X^T and $\mathcal{C}(G_X^T)$. Moreover, the tournament T is exactly determined by giving, in addition, either the graphs $G_X^T[Q]$ for any $Q \in \mathcal{C}(G_X^T)$, or the transitive tournaments $T[Y]$ for any $Y \in q_X^T$.*

We underline the importance of Theorem 2 and Corollary 1 in our description of the tournaments of the class \mathcal{T} . Indeed, these tournaments are introduced up to isomorphism as C_3 -critical tournaments T defined by giving $\mathcal{C}(G_{\mathbb{N}_3}^T)$ in terms of the nonempty elements of $q_{\mathbb{N}_3}^T$. Figure 3 illustrates a tournament obtained from such information. We refer to [10, Discussion] for more details about this purpose.

We now introduce the class \mathcal{H} (resp. $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}$) of the C_3 -critical tournaments H (resp. I, J, K, L) such that:

- $\mathcal{C}(G_{\mathbb{N}_3}^H) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-, \mathbb{N}_3^+ \cup \mathbb{N}_3^-(1)\}$ (see Figure 3);
- $\mathcal{C}(G_{\mathbb{N}_3}^I) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^+(2), \mathbb{N}_3^+(1) \cup \mathbb{N}_3^-(0)\}$;
- $\mathcal{C}(G_{\mathbb{N}_3}^J) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(0)\}$;
- $\mathcal{C}(G_{\mathbb{N}_3}^K) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(2)\}$;
- $\mathcal{C}(G_{\mathbb{N}_3}^L) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(2), \mathbb{N}_3^+ \cup \mathbb{N}_3^-(0)\}$.

Notice that for $\mathcal{X} = \mathcal{H}, \mathcal{I}, \mathcal{J}$ or \mathcal{K} , $\{|V(T)| : T \in \mathcal{X}\} = \{2n + 1 : n \geq 3\}$ and $\{|V(T)| : T \in \mathcal{L}\} = \{2n + 1 : n \geq 4\}$. We denote by \mathcal{H}^* (resp. $\mathcal{I}^*, \mathcal{J}^*, \mathcal{K}^*, \mathcal{L}^*$) the class of the tournaments T^* , where $T \in \mathcal{H}$ (resp. $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}$).

Remark 1 *We have $\mathcal{H}^* = \mathcal{H}$ and $\mathcal{I}^* = \mathcal{I}$.*

Proof Let $T \in \mathcal{H}$. The permutation f of $V(T)$ defined by $f(1) = 0, f(0) = 1$, and $f(v) = v$ for all $v \in V(T) \setminus \{0, 1\}$ is an isomorphism from T^* onto a tournament T' of the class \mathcal{H} . Let now $T \in \mathcal{I}$ and let x be the unique vertex of $\mathbb{N}_3^+(2)$ such that $|N_{T[\mathbb{N}_3^+(2)]}^+(x)| = 0$. The permutation g of $V(T)$ defined by $g(1) = 0, g(0) = 1, g(x) = 2, g(2) = x$, and $g(v) = v$ for $v \in V(T) \setminus \{0, 1, 2, x\}$ is an isomorphism from T^* onto a tournament T' of the class \mathcal{I} . □

By setting $\mathcal{M} = \mathcal{H} \cup \mathcal{I} \cup \mathcal{J} \cup \mathcal{J}^* \cup \mathcal{K} \cup \mathcal{K}^* \cup \mathcal{L} \cup \mathcal{L}^*$, we state our main result as follows.

Theorem 3 *Up to isomorphism, the tournaments of the class \mathcal{T} are those of the class \mathcal{M} . Moreover, for all $T \in \mathcal{M}$, we have $V(T) \setminus W_5(T) = \sigma(T) = \{0, 1\}$.*

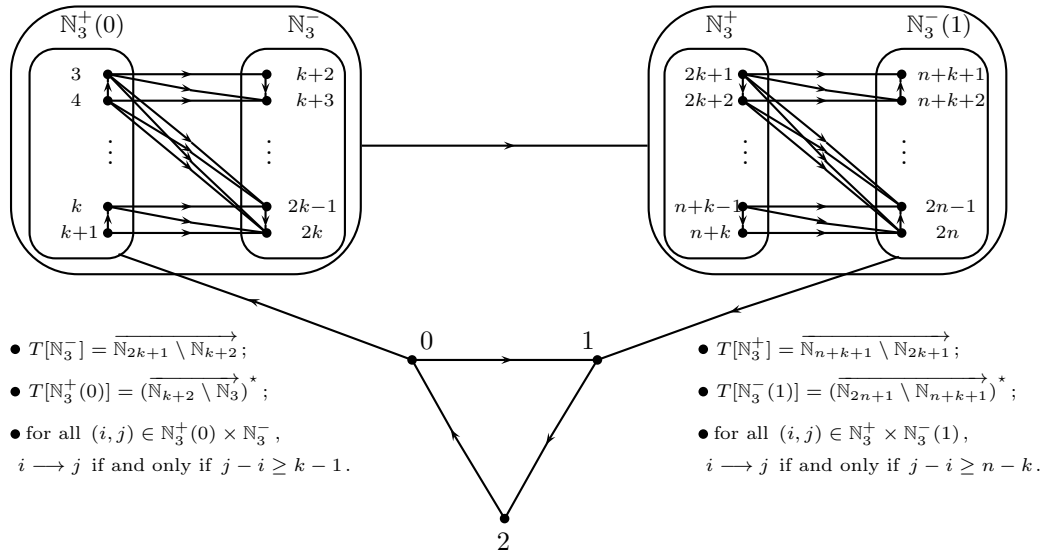


Figure 3. A tournament T of the class \mathcal{H}

2. Critical tournaments and tournaments omitting W_5

We begin by recalling the characterization of the critical tournaments and some of their properties. A prime tournament $T = (V, A)$, with $|T| \geq 3$, is *critical* if $\sigma(T) = \emptyset$, i.e. if all its vertices are critical. In order to present the critical tournaments, characterized by J.H. Schmerl and W.T. Trotter in [10], we introduce the tournaments T_{2n+1} and U_{2n+1} defined on \mathbb{N}_{2n+1} , where $n \geq 2$, as follows:

- $A(T_{2n+1}) = \{(i, j) : j - i \in \{1, \dots, n\} \pmod{2n+1}\}$ (see Figure 4).
- $A(T_{2n+1}) \setminus A(U_{2n+1}) = A(T_{2n+1}[\{n+1, \dots, 2n\}])$ (see Figure 5).

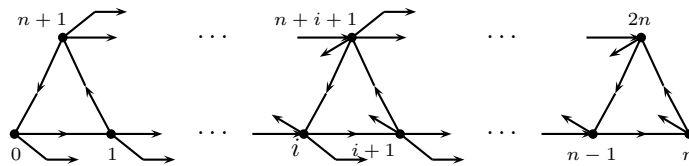


Figure 4. T_{2n+1}

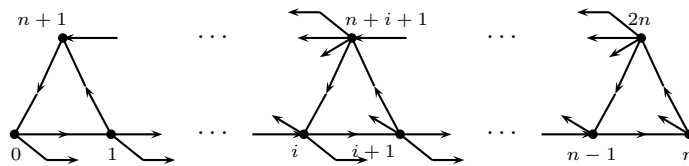


Figure 5. U_{2n+1}

Theorem 4 ([10]) *Up to isomorphism, T_{2n+1} , U_{2n+1} , and W_{2n+1} , where $n \geq 2$, are the only critical tournaments.*

Notice that a critical tournament is isomorphic to its dual. Moreover, as a tournament on 4 vertices is not prime, we have:

Fact 1 Up to isomorphism, T_5 , U_5 , and W_5 are the only prime tournaments on 5 vertices.

As mentioned in [2], the next remark follows from the definition of the critical tournaments.

Remark 2 Up to isomorphism, the prime subtournaments on at least 5 vertices of T_{2n+1} (resp. U_{2n+1} , W_{2n+1}), where $n \geq 2$, are the tournaments T_{2m+1} (resp. U_{2m+1} , W_{2m+1}), where $2 \leq m \leq n$.

To recall the characterization of the prime tournaments omitting W_5 , we introduce the Paley tournament P_7 defined on \mathbb{N}_7 by $A(P_7) = \{(i, j) : j - i \in \{1, 2, 4\} \pmod{7}\}$. Notice that for all $x \neq y \in \mathbb{N}_7$, $P_7 - x \simeq P_7 - y$, and let $B_6 = P_7 - 6$.

Theorem 5 ([8]) Up to isomorphism, the prime tournaments on at least 5 vertices and omitting W_5 are the tournaments B_6 , P_7 , T_{2n+1} , and U_{2n+1} , where $n \geq 2$.

3. Some useful configurations

In this section, we introduce a number of configurations that occur in the proof of Theorem 3. These configurations involve mainly partially critical tournaments. We begin with the two following lemmas obtained in [2].

Lemma 2 ([2]) If B_6 embeds into a prime tournament T on 7 vertices and if $T \not\cong P_7$, then $|W_5(T)| = 7$.

Lemma 3 ([2]) Let T be a U_5 -critical tournament on 7 vertices. If $T \not\cong U_7$, then $W_5(T) \cap \{3, 4\} \neq \emptyset$.

Lemma 4 specifies the C_3 -critical tournaments with a connected outside graph. It follows from the examination of the different possible configurations obtained by using Theorem 2.

Lemma 4 Given a C_3 -critical tournament T on at least 5 vertices, if $G_{\mathbb{N}_3}^T$ is connected, then T is critical. More precisely, the different configurations are as follows where $i \in \mathbb{N}_3$ and $i + 1$ is considered modulo 3.

1. If $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^-(i) \cup \mathbb{N}_3^+(i + 1)\}$, then $T \simeq T_{2n+1}$ for some $n \geq 2$.
2. If $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^-(i) \cup \mathbb{N}_3^+(i)\}$, $\{\mathbb{N}_3^+(i) \cup \mathbb{N}_3^-(i)\}$, $\{\mathbb{N}_3^+(i) \cup \mathbb{N}_3^+(i + 1)\}$, or $\{\mathbb{N}_3^-(i) \cup \mathbb{N}_3^-(i + 1)\}$, then $T \simeq U_{2n+1}$ for some $n \geq 2$.
3. If $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^-(i) \cup \mathbb{N}_3^-(i)\}$, $\{\mathbb{N}_3^+(i) \cup \mathbb{N}_3^+(i)\}$, or $\{\mathbb{N}_3^+(i) \cup \mathbb{N}_3^-(i + 1)\}$, then $T \simeq W_{2n+1}$ for some $n \geq 2$.

For a transitive tournament T , recall that $\min T$ denotes its smallest element and $\max T$ its largest.

Lemma 5 Given a C_3 -critical tournament T on at least 5 vertices, if $T[\mathbb{N}_3 \cup e] \simeq T_5$ for all $e \in E(G_{\mathbb{N}_3}^T)$, then $T \simeq T_{2n+1}$ for some $n \geq 2$.

Proof Let T be a C_3 -critical tournament on at least 5 vertices such that for all $e \in E(G_{\mathbb{N}_3}^T)$, $T[\mathbb{N}_3 \cup e] \simeq T_5$. Given $e \in E(G_{\mathbb{N}_3}^T)$, by using Lemma 4 and Remark 2, $e = \{v, v'\}$, where $v \in \mathbb{N}_3^-(i)$, $v' \in \mathbb{N}_3^+(i + 1)$, $i \in \mathbb{N}_3$ and $i + 1$ is considered modulo 3. Then, by Theorem 2, the connected components of T are the nonempty elements of the family $\{\mathbb{N}_3^-(j) \cup \mathbb{N}_3^+(j + 1)\}_{j \in \mathbb{N}_3}$, where $j + 1$ is considered modulo 3. The tournament T is critical. Indeed, by using Theorem 2, for each $k \in \mathbb{N}_3$, $\{\max T[\mathbb{N}_3^+(k + 1) \cup \{k + 1\}], \min T[\mathbb{N}_3^-(k + 2) \cup \{k + 2\}]\}$, where

$k + 1$ and $k + 2$ are considered modulo 3, is a nontrivial module of $T - k$. It follows that $T \simeq T_{2n+1}$ for some $n \geq 2$ by Remark 2. \square

Lemma 6 *Given a U_5 -critical tournament, if $T[\mathbb{N}_5 \cup e] \simeq U_7$ for all $e \in E(G_{\mathbb{N}_5}^T)$, then $T \simeq U_{2n+1}$ for some $n \geq 2$.*

Proof The subsets X of \mathbb{N}_7 such that $U_7[X] \simeq U_5$ are the sets $\mathbb{N}_7 \setminus \{i, j\}$, where $\{i, j\} = \{0, 4\}, \{4, 1\}, \{1, 5\}, \{5, 2\}, \{2, 6\}$, or $\{6, 3\}$. By observing $q_X^{U_7}$ for such subsets X and by Theorem 2, we deduce that the elements of $\mathcal{C}(G_{\mathbb{N}_5}^T)$ are the nonempty elements among the following six sets: $\mathbb{N}_5^+ \cup \mathbb{N}_5^-(0)$, $\mathbb{N}_5^+(0) \cup \mathbb{N}_5^+(3)$, $\mathbb{N}_5^-(1) \cup \mathbb{N}_5^-(3)$, $\mathbb{N}_5^+(1) \cup \mathbb{N}_5^+(4)$, $\mathbb{N}_5^-(2) \cup \mathbb{N}_5^-(4)$, and $\mathbb{N}_5^- \cup \mathbb{N}_5^+(2)$. Suppose first that $|\mathcal{C}(G_{\mathbb{N}_5}^T)| = 6$. The tournament T is critical. Indeed, by using Theorem 2, $\{\min T[\mathbb{N}_5^+], \max T[\mathbb{N}_5^+(3)]\}$ (resp. $\{\min T[\mathbb{N}_5^-(3)], \max T[\mathbb{N}_5^+(4)]\}$), $\{\min T[\mathbb{N}_5^-(4)], \max T[\mathbb{N}_5^-]\}$, $\{\min T[\mathbb{N}_5^-(1)], \max T[\mathbb{N}_5^+(0)]\}$, $\{\min T[\mathbb{N}_5^-(2)], \max T[\mathbb{N}_5^+(1)]\}$) is a nontrivial module of $T - 0$ (resp. $T - 1, T - 2, T - 3, T - 4$). By Remark 2, $T \simeq U_{2n+1}$ for some $n \geq 8$. Suppose now that $|\mathcal{C}(G_{\mathbb{N}_5}^T)| \leq 5$. Then T embeds into a U_5 -critical tournament T' with $|\mathcal{C}(G_{\mathbb{N}_5}^{T'})| = 6$. By the first case, $T' \simeq U_{2n+1}$ for some $n \geq 8$ and thus $T \simeq U_{2n+1}$ for some $n \geq 2$ by Remark 2. \square

Lemma 7 *Let $T = (V, A)$ be a $T[X]$ -critical tournament with $|V \setminus X| \geq 2$, let $Q = \mathbb{N}_{2n}$ be a connected component of G_X^T such that $G_X^T[Q] = G_{2n}$, and let $e = \{i, i+n\}$, where $i \in \mathbb{N}_n$. Then the tournament $T - e$ is $T[X]$ -critical. Moreover, Q is included in any subset Z of V such that $T[Z] \simeq W_5$ and $Z \cap (V \setminus (Q \cup W_5(T - e))) \neq \emptyset$.*

Proof For $n \geq 2$, the function

$$f_i : Q \setminus e \longrightarrow \mathbb{N}_{2n-2}$$

$$k \longmapsto \begin{cases} k & \text{if } 0 \leq k \leq i - 1 \\ k - 1 & \text{if } i + 1 \leq k \leq n + i - 1 \\ k - 2 & \text{if } n + i + 1 \leq k \leq 2n - 1, \end{cases}$$

is an isomorphism from $G_{2n} - e$ onto G_{2n-2} . It follows from Theorem 2 that $T - e$ is $T[X]$ -critical. Now suppose that there is $Z \subseteq V$ such that $T[Z] \simeq W_5$ and $Z \cap (V \setminus (Q \cup W_5(T - e))) \neq \emptyset$. Therefore, we have $|Z \cap e| = 1$ or $e \subset Z$. Suppose for a contradiction that $|Z \cap e| = 1$, and set $\{z\} = Z \cap e$. As $\text{Ext}(V \setminus e) = \emptyset$, then by Lemma 1, either $z \in \langle V' \rangle$ or $z \in V'(u)$, where $V' = V \setminus e$ and $u \in V'$. If $z \in \langle V' \rangle$, then $Z \setminus \{z\}$ is a nontrivial module of $T[Z]$, a contradiction. If $z \in V'(u)$, then $u \notin Z$, otherwise $\{u, z\}$ is a nontrivial module of $T[Z]$. Thus, $T[Z'] \simeq W_5$, where $Z' = (Z \setminus \{z\}) \cup \{u\} \subset V \setminus e$. A contradiction because $Z' \cap (V \setminus W_5(T - e)) \neq \emptyset$. Finally, for all $e' \in \{\{j, j+n\} : j \in \mathbb{N}_n\}$, the bijection f from $V \setminus e$ onto $V \setminus e'$, defined by $f|_{V \setminus Q} = \text{Id}_{V \setminus Q}$ and $f|_{Q \setminus e} = f_j^{-1} \circ f_i$, is an isomorphism from $T - e$ onto $T - e'$. It follows that $V \setminus (Q \cup W_5(T - e')) = V \setminus (Q \cup W_5(T - e))$. Thus, as proved above, $e' \subset Z$, so that $Q \subset Z$. \square

4. Proof of Theorem 3

We begin by establishing the partial criticality structure of the tournaments of the class \mathcal{T} . For this purpose, we use the notion of minimal tournaments for two vertices. Given a prime tournament $T = (V, A)$ of cardinality ≥ 3 and two distinct vertices $x \neq y \in V$, T is said to be *minimal* for $\{x, y\}$ (or $\{x, y\}$ -*minimal*) when for all proper subset X of V , if $\{x, y\} \subset X$ ($|X| \geq 3$), then $T[X]$ is not prime. These tournaments were introduced and characterized by A. Cournier and P. Ille in [5]. From this characterization, the following fact, observed in [1], is obtained by a simple and quick verification.

Fact 2 ([1, 5]) *Up to isomorphism, the tournaments C_3 and U_5 are the unique minimal tournaments for two vertices T such that $|W_5(T)| \leq |T| - 2$. Moreover, $\{3, 4\}$ is the unique unordered pair of vertices for which U_5 is minimal.*

Proposition 1 *Let $T = (V, A)$ be a tournament of the class \mathcal{T} . Then the vertices of $W_5(T)$ are critical and there exists $z \in W_5(T)$ such that $T[(V \setminus W_5(T)) \cup \{z\}] \simeq C_3$. In particular, T is $T[(V \setminus W_5(T)) \cup \{z\}]$ -critical.*

Proof By Theorem 1, $|T|$ is odd and ≥ 7 . First, suppose by contradiction that there is $\alpha \in W_5(T)$ such that $T - \alpha$ is prime. Since $|T - \alpha|$ is even and ≥ 6 with $|V(T - \alpha) \setminus W_5(T - \alpha)| \geq 2$, then by Theorems 1 and 5, $T - \alpha \simeq B_6$ and $T \not\cong P_7$. A contradiction by Lemma 2. Second, let X be a minimal subset of V such that $V \setminus W_5(T) \subset X$ ($|X| \geq 3$) and $T[X]$ is prime, so that $T[X]$ is $(V \setminus W_5(T))$ -minimal. By Fact 2, $T[X] \simeq C_3$ or U_5 . Suppose, toward a contradiction that $T[X] \simeq U_5$ and take $T[X] = U_5$. By Fact 2, $V \setminus W_5(T) = \{3, 4\}$. As T is U_5 -critical, then by Lemma 6 and Theorem 5, there exists $e \in E(G_X^T)$ such that $T[X \cup e]$ is prime and not isomorphic to U_7 . It follows from Lemma 3, that there exists a subset Z of $X \cup e$ such that $T[Z] \simeq W_5$ and $Z \cap (V \setminus W_5(T)) \neq \emptyset$, a contradiction. \square

Now, we prove Theorem 3 for tournaments on 7 vertices.

Proposition 2 *Up to isomorphism, the class \mathcal{M} and the class \mathcal{T} have the same tournaments on 7 vertices. Moreover, for each tournament T on 7 vertices of the class \mathcal{M} , we have $V(T) \setminus W_5(T) = \sigma(T) = \{0, 1\}$.*

Proof Let $T = (V, A)$ be a tournament on 7 vertices of the class \mathcal{M} . $T \in \mathcal{M} \setminus (\mathcal{L} \cup \mathcal{L}^*)$ because the tournaments of the class \mathcal{L} have at least 9 vertices. Let $e \in E(G_{\mathbb{N}_3}^T)$. By Lemma 4, $T - e \simeq U_5$ or T_5 . By Lemma 7, if there exists a subset $Z \subset V$ such that $T[Z] \simeq W_5$, then $e \subset Z$. It follows that $V \setminus \mathbb{N}_3 \subset Z$. Thus $V \setminus W_5(T) = \{0, 1\}$ by verifying that $T - \{1, 2\} \not\cong W_5$, $T - \{0, 2\} \not\cong W_5$ and $T - \{0, 1\} \simeq W_5$. As T is C_3 -critical, $\sigma(T) = \{0, 1\}$ from the following. First, $T - 2$ is not prime because $\{0\} \cup \mathbb{N}_3^- \cup \mathbb{N}_3^+(0)$ (resp. $\{1\} \cup \mathbb{N}_3^+(0)$, $\{0, 1\} \cup \mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(1)$, $\{1\} \cup \mathbb{N}_3^+(0)$) is a nontrivial module of $T - 2$ if $T \in \mathcal{H}$ (resp. \mathcal{I} , \mathcal{J} , \mathcal{K}). Second, by Lemma 1, we have $\text{Ext}(X) = \{0, 1\}$, where $X = V \setminus \{0, 1\}$, because $\{0, 1\} \cap \langle X \rangle = \emptyset$, and for all $u \in X$, $\{0, 1\} \cap X(u) = \emptyset$ because $V \setminus W_5(T) = \{0, 1\}$.

Conversely, let T be a tournament on 7 vertices of the class \mathcal{T} . By Proposition 1, we can assume that T is C_3 -critical with $V(T) \setminus W_5(T) \subset \mathbb{N}_3$. By Lemma 4 and Theorem 5, $|\mathcal{C}(G_{\mathbb{N}_3}^T)| = 2$. We distinguish the following cases.

- $\mathbb{N}_3^+ \neq \emptyset$ and $\mathbb{N}_3^- \neq \emptyset$. By Theorem 2, $|\mathbb{N}_3^-| = |\mathbb{N}_3^+| = 1$. Therefore, we can assume that $\mathbb{N}_3(0) \neq \emptyset$ and $\mathbb{N}_3(2) = \emptyset$. It suffices to verify that $|\mathbb{N}_3(0)| = |\mathbb{N}_3^+(0)| = 1$ because, in this case, by using Theorem 2 and Lemma 4, $T \in \mathcal{H}$. By using again Theorem 2 and Lemma 4, we verify the following. First, if $|\mathbb{N}_3(0)| = 2$,

then $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^+ \cup \mathbb{N}_3^-(0), \mathbb{N}_3^- \cup \mathbb{N}_3^+(0)\}$. Therefore, $T - \{0, 1\} \simeq T - \{0, 2\} \simeq W_5$, a contradiction. Second, if $|\mathbb{N}_3^-(0)| = 1$, then $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^+ \cup \mathbb{N}_3^-(0), \mathbb{N}_3^- \cup \mathbb{N}_3^+(1)\}$. Therefore, $T \simeq U_7$, a contradiction by Theorem 5.

- $\langle \mathbb{N}_3 \rangle = \emptyset$. By Theorem 2, we can assume that $|\mathbb{N}_3^-(0)| = |\mathbb{N}_3^+(0)| = 1$. We have $|\mathbb{N}_3(1)| = 1$. Otherwise, by Theorem 2 and Lemma 4, we can suppose that $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^+(1), \mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(1)\}$. Therefore, $T - \{1, 2\} \simeq T - \{0, 2\} \simeq W_5$, a contradiction. We have also $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3(2), \mathbb{N}_3^-(0) \cup \mathbb{N}_3(1)\}$. Otherwise, again by Theorem 2 and Lemma 4, $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^+(1), \mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(2)\}$, so that $T \simeq U_7$, a contradiction by Theorem 5. Thus, we distinguish four cases. If $|\mathbb{N}_3^-(2)| = |\mathbb{N}_3^+(1)| = 1$, then $T \simeq T_7$, which contradicts Theorem 5. If $|\mathbb{N}_3^+(2)| = |\mathbb{N}_3^-(1)| = 1$, then $T - \{0, 2\} \simeq T - \{0, 1\} \simeq W_5$, a contradiction. If $|\mathbb{N}_3^+(2)| = |\mathbb{N}_3^+(1)| = 1$, then $T \in \mathcal{I}$. If $|\mathbb{N}_3^-(2)| = |\mathbb{N}_3^-(1)| = 1$, then T is isomorphic to a tournament of the class \mathcal{I} with $V(T) \setminus W_5(T) = \{0, 2\}$.
- $\emptyset \neq \langle \mathbb{N}_3 \rangle \in q_{\mathbb{N}_3}^T$. By interchanging T and T^* , we can suppose that $\langle \mathbb{N}_3 \rangle = \mathbb{N}_3^-$. In this case, $|\mathbb{N}_3^-| = 1$ by Theorem 2. First, suppose that $|\mathbb{N}_3(0)| = 2$ and $|\mathbb{N}_3(1)| = 1$. By Theorem 2 and Lemma 4, $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(0) \cup \mathbb{N}_3(1)\}$. We have $|\mathbb{N}_3^+(1)| = 1$, otherwise $T \simeq U_7$, a contradiction by Theorem 5. Thus, T is isomorphic to a tournament of the class \mathcal{K} with $V(T) \setminus W_5(T) = \{0, 2\}$. Second, suppose that $|\mathbb{N}_3(0)| = 1$ and $|\mathbb{N}_3(1)| = 2$. Again by Theorem 2 and Lemma 4, $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(0)\}$, so that $T \in \mathcal{J}$. Lastly, suppose that $|\mathbb{N}_3(0)| = |\mathbb{N}_3(1)| = 1$. By Theorem 2 and Lemma 4, we can suppose that $\mathcal{C}(G_{\mathbb{N}_3}^T) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3(0) \cup \mathbb{N}_3(2)\}$. By Lemma 4, we distinguish only three cases. If $|\mathbb{N}_3^-(2)| = |\mathbb{N}_3^-(0)| = 1$, then $T - \{0, 1\} \simeq T - \{1, 2\} \simeq W_5$, a contradiction. If $|\mathbb{N}_3^+(0)| = |\mathbb{N}_3^+(2)| = 1$, then $T \simeq U_7$, which contradicts Theorem 5. If $|\mathbb{N}_3^-(2)| = |\mathbb{N}_3^+(0)| = 1$, then $T \in \mathcal{K}$.

□

We complete our structural study of the tournaments of the class \mathcal{T} by the following two corollaries.

Corollary 2 *Let T be a C_3 -critical tournament such that $V(T) \setminus W_5(T) = \{0, 1\}$. Then there exist $Q \neq Q' \in \mathcal{C}(G_{\mathbb{N}_3}^T)$ and a tournament R on 7 vertices of the class \mathcal{M} such that for all $e \in E(G_{\mathbb{N}_3}^T[Q])$ and for all $e' \in E(G_{\mathbb{N}_3}^T[Q'])$, there exists an isomorphism f from R onto $T[\mathbb{N}_3 \cup e \cup e']$. Moreover, $f(0) = 0$, $f(1) = 1$ and we have:*

1. *If $R \in \mathcal{H} \cup \mathcal{J} \cup \mathcal{J}^*$, then $f(2) = 2$;*
2. *If $R \in \mathcal{I} \cup \mathcal{K} \cup \mathcal{K}^*$, then $f(2) = 2$ or $\mathbb{N}_3(2) = \{f(2)\}$.*

Proof To begin, notice the following remark: given a $D[X]$ -critical tournament D , for any edges a and b belonging to a same connected component of G_X^D , we have $D[X \cup a] \simeq D[X \cup b]$. Therefore, by Fact 1, Lemma 5, and Theorem 5, there exists $Q \in \mathcal{C}(G_{\mathbb{N}_3}^T)$ such that for all $a \in E(G_{\mathbb{N}_3}^T[Q])$, $T[\mathbb{N}_3 \cup a] \simeq U_5$. By Lemma 4 and Remark 2, the tournament $T[\mathbb{N}_3 \cup Q]$ is isomorphic to U_{2n+1} , for some $n \geq 2$, and does not admit a prime subtournament on 7 vertices other than U_7 . Therefore, by Lemma 6, Theorem 5, and the remark above, there exists $Q' \in \mathcal{C}(G_{\mathbb{N}_3}^T) \setminus \{Q\}$ such that for all $e \in E(G_{\mathbb{N}_3}^T[Q])$ and for all $e' \in E(G_{\mathbb{N}_3}^T[Q'])$, $T[\mathbb{N}_3 \cup e \cup e']$ is prime and not isomorphic to U_7 . Moreover, $T[\mathbb{N}_3 \cup e \cup e'] \not\simeq P_7$ because the vertices of P_7 are all noncritical.

Likewise, $T[\mathbb{N}_3 \cup e \cup e'] \not\cong T_7$ by Remark 2. It follows from Theorem 5 and Proposition 2 that there exists an isomorphism f from a tournament R on 7 vertices of the class \mathcal{M} onto $T[\mathbb{N}_3 \cup e \cup e']$. As $(0, 1) \in A(R) \cap A(T)$ and $V(R) \setminus W_5(R) = V(T) \setminus W_5(T) = \{0, 1\}$ by Proposition 2, then f fixes 0 and 1. If $R \in \mathcal{H} \cup \mathcal{J} \cup \mathcal{J}^*$, then f fixes 2 because 2 is the unique vertex x of R such that $R[\{0, 1, x\}] \simeq C_3$. If $R \in \mathcal{I} \cup \mathcal{K} \cup \mathcal{K}^*$, then $|\{x \in V(R) : R[\{0, 1, x\}] \simeq C_3\}| = 2$. Therefore, $f(2) = 2$ or α , where α is the unique vertex of $\mathbb{N}_3(2)$ in the tournament $T[\mathbb{N}_3 \cup e \cup e']$. \square

Corollary 3 For all $T \in \mathcal{T}$, we have $V(T) \setminus W_5(T) = \sigma(T)$.

Proof Let T be a tournament of the class \mathcal{T} such that $V(T) \setminus W_5(T) = \{0, 1\}$. By Proposition 1, we can assume that T is C_3 -critical. By the same proposition, it suffices to prove that $\{0, 1\} \subseteq \sigma(T)$. By Corollary 2, there is a subset X of $V(T)$ such that $\mathbb{N}_3 \subset X$ and $T[X]$ is isomorphic to a tournament on 7 vertices of the class \mathcal{M} . Suppose for a contradiction that T admits a critical vertex $i \in \{0, 1\}$, and let $Y = X \setminus \{i\}$. By Proposition 2, $T[Y]$ is prime. As T is $T[Y]$ -critical, then $i \notin \text{Ext}(Y)$ by Theorem 2. This is a contradiction because $T[X]$ is prime. \square

Now, we prove that $\mathcal{M} \subseteq \mathcal{T}$. More precisely:

Proposition 3 For all tournament T of the class \mathcal{M} , we have $V(T) \setminus W_5(T) = \sigma(T) = \{0, 1\}$.

Proof Let T be a tournament on $(2n + 1)$ vertices of the class \mathcal{M} for some $n \geq 3$. By Corollary 3, it suffices to prove that $V(T) \setminus W_5(T) = \{0, 1\}$. We proceed by induction on n . By Proposition 2, the statement is satisfied for $n = 3$. Let now $n \geq 4$. Therefore, either T is a tournament on 9 vertices of the class $\mathcal{L} \cup \mathcal{L}^*$ or there is $Q \in \mathcal{C}(G_{\mathbb{N}_3}^T)$ such that $|Q| \geq 4$. In the first case, for all $e \in E(G_{\mathbb{N}_3}^T)$, $T - e$ is isomorphic to U_7 or to a tournament on 7 vertices of the class $\mathcal{K} \cup \mathcal{K}^*$. Therefore, if there exists a subset Z of $V(T)$ such that $Z \cap \{0, 1\} \neq \emptyset$ and $T[Z] \simeq W_5$, then, for all $e \in E(G_{\mathbb{N}_3}^T)$, $e \subset Z$ by Lemma 7. Thus, $V(T) \setminus \mathbb{N}_3 \subset Z$, a contradiction. As, furthermore, W_5 embeds into T , then $V(T) \setminus W_5(T) = \{0, 1\}$ by Theorem 1. In the second case, let $Q \in \mathcal{C}(G_{\mathbb{N}_3}^T)$ such that $|Q| \geq 4$. Let $\mathcal{X} = \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K},$ or \mathcal{L} . For $T \in \mathcal{X}$, by Lemma 7, there is $e \in E(G_{\mathbb{N}_3}^T[Q])$ such that $T - e$ is C_3 -critical. Moreover, $T - e$ is isomorphic to a tournament of the class \mathcal{X} because $\mathcal{C}(G_{\mathbb{N}_3}^{T-e})$ is as described in the same class. By induction hypothesis, W_5 embeds into $T - e$, and thus into T . By Theorem 1, it suffices to verify that $\{0, 1\} \subseteq V(T) \setminus W_5(T)$. Therefore, suppose that there exists $Z \subset V(T)$ such that $Z \cap \{0, 1\} \neq \emptyset$ and $T[Z] \simeq W_5$. By induction hypothesis and by Lemma 7, $Q \subset Z$, so that $Z \subset Q \cup \mathbb{N}_3$. This is a contradiction by Theorem 5, because $T[\mathbb{N}_3 \cup Q] \simeq U_{|Q|+3}$ or $T_{|Q|+3}$ by Lemma 4. \square

We are now ready to construct the tournaments of the class \mathcal{T} . We partition these tournaments T according to the following invariant $c(T)$. For $T \in \mathcal{T}$, $c(T)$ is the minimum of $|\mathcal{C}(G_{\sigma(T) \cup \{x\}}^T)|$, the minimum being taken over all the vertices x of $W_5(T)$ such that $T[\sigma(T) \cup \{x\}] \simeq C_3$. Notice that $c(T) = c(T^*)$. As T is $T[\sigma(T) \cup \{x\}]$ -critical by Proposition 1, then $c(T) \leq 4$. Moreover, $c(T) \geq 2$ by Lemma 4. Proposition 1 leads us to classify the tournaments T of the class \mathcal{T} according to the different values of $c(T)$. We will see that $c(T) = 2$ or 3. Theorem 3 results from Propositions 3, 4, 5, and 6.

Proposition 4 Up to isomorphism, the tournaments T of the class \mathcal{T} such that $c(T) = 2$ are those of the class $\mathcal{M} \setminus (\mathcal{L} \cup \mathcal{L}^*)$.

Proof For all $T \in \mathcal{M} \setminus (\mathcal{L} \cup \mathcal{L}^*)$, we have $T \in \mathcal{T}$ by Proposition 3, and $c(T) = 2$ by Lemma 4. Now let T be a tournament on $(2n + 1)$ vertices of the class \mathcal{T} such that $c(T) = 2$. By Proposition 1, we can assume that T is C_3 -critical with $V(T) \setminus W_5(T) = \{0, 1\}$ and $|\mathcal{C}(G_{\mathbb{N}_3}^T)| = 2$. By Corollary 2 and by interchanging T and T^* , there is a tournament R on 7 vertices of the class $\mathcal{H} \cup \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$ such that for all $e \in E(G_{\mathbb{N}_3}^T[Q])$ and for all $e' \in E(G_{\mathbb{N}_3}^T[Q'])$, there exists an isomorphism f , fixing 0 and 1, from R onto $T[\mathbb{N}_3 \cup e \cup e']$, where Q and Q' are the two different connected components of $G_{\mathbb{N}_3}^T$. If $f(2) = 2$, then, by Theorem 2, T and R are in the same class \mathcal{H} , \mathcal{I} , \mathcal{J} , or \mathcal{K} . Suppose now that $f(2) \neq 2$. By Corollary 2, $R \in \mathcal{I} \cup \mathcal{K}$. If $R \in \mathcal{I}$ (resp. \mathcal{K}), then $T[\mathbb{N}_3 \cup e \cup e']$ is a tournament on 7 vertices of the class \mathcal{I}' (resp. \mathcal{K}') of the C_3 -critical tournaments Z such that $\mathcal{C}(G_{\mathbb{N}_3}^Z) = \{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+(1), \mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(2)\}$ (resp. $\mathcal{C}(G_{\mathbb{N}_3}^Z) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(1) \cup \mathbb{N}_3^+(2)\}$). By Theorem 2, $T \in \mathcal{I}'$ (resp. \mathcal{K}'). Moreover, by considering the vertex $\alpha = \min T[\mathbb{N}_3^-(2)]$ (resp. $\max T[\mathbb{N}_3^+(2)]$) and by using Corollary 3, T is also $T[\{0, 1, \alpha\}]$ -critical with $\mathcal{C}(G_{\{0,1,\alpha\}}^T) = \{\{0, 1, \alpha\}^-(0) \cup \{0, 1, \alpha\}^+(1), \{0, 1, \alpha\}^+(0) \cup \{0, 1, \alpha\}^+(\alpha)\}$ (resp. $\mathcal{C}(G_{\{0,1,\alpha\}}^T) = \{\{0, 1, \alpha\}^+(1) \cup \{0, 1, \alpha\}^-, \{0, 1, \alpha\}^+(0) \cup \{0, 1, \alpha\}^-(\alpha)\}$). It follows that T is isomorphic to a tournament of the class \mathcal{I} (resp. \mathcal{K}). \square

Proposition 5 *Up to isomorphism, the tournaments T of the class \mathcal{T} such that $c(T) = 3$ are those of the class $\mathcal{L} \cup \mathcal{L}^*$.*

Proof Let T be a tournament of the class $\mathcal{L} \cup \mathcal{L}^*$. $T \in \mathcal{T}$ by Proposition 3. Moreover, $c(T) = 3$ by Theorem 2. Indeed, it suffices to observe that for all $x \in \{i \in V(T) \setminus \mathbb{N}_3 : T[\{0, 1, i\}] \simeq C_3\} = \mathbb{N}_3^-(2)$, we have $\max T[\mathbb{N}_3^+(1)] \in X^+(1)$, $\min T[\mathbb{N}_3^-] \in X^-$, $\min T[\mathbb{N}_3^+] \in X^+$, $\max T[\mathbb{N}_3^-(0)] \in X^-(0)$ and $2 \in X^+(x)$, where $X = \{0, 1, x\}$.

Now let T be a tournament on $(2n + 1)$ vertices of \mathcal{T} such that $c(T) = 3$. By Proposition 1, we can assume that T is C_3 -critical with $V(T) \setminus W_5(T) = \{0, 1\}$ and $|\mathcal{C}(G_{\mathbb{N}_3}^T)| = 3$. By Corollary 2 and by interchanging T and T^* , there is a tournament R on 7 vertices of the class $\mathcal{H} \cup \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$ such that for all $e \in E(G_{\mathbb{N}_3}^T[Q])$ and $e' \in E(G_{\mathbb{N}_3}^T[Q'])$, there exists an isomorphism f , which fixes 0 and 1, from R onto $T[\mathbb{N}_3 \cup e \cup e']$, where $Q \neq Q' \in \mathcal{C}(G_{\mathbb{N}_3}^T)$. Take $e'' \in E(G_{\mathbb{N}_3}^T[Q''])$, where $Q'' = \mathcal{C}(G_{\mathbb{N}_3}^T) \setminus \{Q, Q'\}$. Suppose, toward a contradiction, that $R \in \mathcal{H} \cup \mathcal{J}$. By Theorem 2 and by Corollary 2, if $R \in \mathcal{H}$ (resp. $R \in \mathcal{J}$), then $\{Q, Q'\} = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(1) \cup \mathbb{N}_3^+\}$ (resp. $\{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(1)\}$). Therefore, by Lemma 4, $Q'' = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^+(2)\}$, $\{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+(1)\}$ or $\{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(2)\}$ (resp. $\{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(2)\}$ or $\{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-(2)\}$). We verify that in each of these cases, either $T[\{0\} \cup e \cup e'']$, $T[\{0\} \cup e' \cup e'']$, $T[\{1\} \cup e \cup e'']$ or $T[\{1\} \cup e' \cup e'']$ is isomorphic to W_5 , a contradiction. Therefore, $R \in \mathcal{I} \cup \mathcal{K}$. By Corollary 2, $f(2) = 2$ or α , where α is the unique vertex of $\mathbb{N}_3(2)$ in $T[\mathbb{N}_3 \cup e \cup e']$.

Suppose, again by contradiction, that $R \in \mathcal{I}$. We begin by the case where $f(2) = 2$. By Theorem 2, we can suppose that $Q = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^+(2)\}$ and $Q' = \{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+(1)\}$. By Lemma 4, $Q'' = \{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+\}$, $\{\mathbb{N}_3^-(2) \cup \mathbb{N}_3^+\}$, or $\{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(2)\}$. If $Q'' = \{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+\}$ (resp. $\{\mathbb{N}_3^-(2) \cup \mathbb{N}_3^+\}$), then $T[\{0\} \cup e \cup e''] \simeq W_5$ (resp. $T[\{1\} \cup e' \cup e''] \simeq W_5$), a contradiction. If $Q'' = \{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(2)\}$, then, by taking $X = \{0, 1, x\}$, where $x = \min T[\mathbb{N}_3^-(2)]$, we obtain a contradiction because, by Corollary 3, T is $T[X]$ -critical with $|\mathcal{C}(G_X^T)| = 2$. Indeed, $\mathcal{C}(G_X^T) = \{X^-(0) \cup X^+(1), X^+(0) \cup X^+(x)\}$, with $X^-(0) = \mathbb{N}_3^-(0)$, $X^+(1) = \mathbb{N}_3^+(1)$, $X^+(0) = \mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(1)$,

and $X^+(x) = \mathbb{N}_3^+(2) \cup \{2\} \cup (\mathbb{N}_3^-(2) \setminus \{x\})$. Now if $f(2) = \alpha$, then we obtain again a contradiction. Indeed, by replacing T by T^* and by interchanging the vertices 0 and 1, $\{Q, Q'\} = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^+(2), \mathbb{N}_3^-(0) \cup \mathbb{N}_3^+(1)\}$ as in the case where $f(2) = 2$.

At present, $R \in \mathcal{K}$. We begin by the case where $f(2) = 2$. By Theorem 2, we can suppose that $Q = \{\mathbb{N}_3^- \cup \mathbb{N}_3^+(1)\}$ and $Q' = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(2)\}$. By Lemma 4, $Q'' = \{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+\}$, $\{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+\}$, $\{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(1)\}$ or $\{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+(2)\}$. If $Q'' = \{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+\}$ (resp. $\{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(0)\}$), then $T[\{0\} \cup e \cup e''] \simeq W_5$ (resp. $T[\{1\} \cup e' \cup e''] \simeq W_5$), a contradiction. If $Q'' = \{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+(2)\}$, then, by taking $X = \{0, 1, x\}$, where $x = \max T[\mathbb{N}_3^+(2)]$, we have a contradiction because, by Corollary 3, T is $T[X]$ -critical with $|\mathcal{C}(G_X^T)| = 2$. Indeed, $\mathcal{C}(G_X^T) = \{X^- \cup X^+(1), X^+(0) \cup X^-(x)\}$, with $X^- = \mathbb{N}_3^-$, $X^+(1) = \mathbb{N}_3^+(1)$, $X^+(0) = \mathbb{N}_3^-(1) \cup \mathbb{N}_3^+(0)$ and $X^-(x) = \mathbb{N}_3^-(2) \cup \{2\} \cup (\mathbb{N}_3^+(2) \setminus \{x\})$. If $Q'' = \{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+\}$, then $T \in \mathcal{L}$. Now suppose that $f(2) = \alpha$. By Theorem 2, we can suppose that $Q = \{\mathbb{N}_2^+(1) \cup \mathbb{N}_2^-\}$ and $Q' = \{\mathbb{N}_2^-(1) \cup \mathbb{N}_3^+(2)\}$. By Lemma 4, $Q'' = \{\mathbb{N}_3^-(2) \cup \mathbb{N}_3^+\}$, $\{\mathbb{N}_3^-(2) \cup \mathbb{N}_3(0)\}$, or $\{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+\}$. If $Q'' = \{\mathbb{N}_3^-(2) \cup \mathbb{N}_3^+\}$ or $\{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+(2)\}$, then $T[\{0\} \cup e \cup e''] \simeq W_5$, a contradiction. If $Q'' = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(2)\}$, then we obtain the same configuration giving $|\mathcal{C}(G_X^T)| = 2$ in the case where $f(2) = 2$. If $Q'' = \{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+\}$, then T is isomorphic to a tournament of the class \mathcal{L}^* . \square

Proposition 6 For any tournament T of the class \mathcal{T} , we have $c(T) = 2$ or 3 .

Proof Let T be a tournament on $(2n + 1)$ vertices of the class \mathcal{T} for some $n \geq 3$. We proceed by induction on n . By Propositions 4 and 5, the statement is satisfied for $n = 3$ and for $n = 4$. Let $n \geq 5$. By Proposition 1, we can assume that T is C_3 -critical with $V(T) \setminus W_5(T) = \{0, 1\}$. By Theorem 2 and Lemma 4, $2 \leq c(T) \leq 4$. Therefore, we only consider the case where $|\mathcal{C}(G_{\mathbb{N}_3}^T)| = 4$. By Corollary 2, there exist $Q \neq Q' \in \mathcal{C}(G_{\mathbb{N}_3}^T)$ and a tournament R on 7 vertices of the class \mathcal{M} , such that for all $e \in E(G_{\mathbb{N}_3}^T[Q])$ and for all $e' \in E(G_{\mathbb{N}_3}^T[Q'])$, $T[\mathbb{N}_3 \cup e \cup e'] \simeq R$. By Lemma 7, there exists $e'' \in E(G_{\mathbb{N}_3}^T[Q''])$, where $Q'' \in \mathcal{C}(G_{\mathbb{N}_3}^T) \setminus \{Q, Q'\}$, such that $T - e''$ is C_3 -critical. As W_5 embeds into $T - e''$, then $V(T - e'') \setminus W_5(T - e'') = \{0, 1\}$ by Theorem 1. Therefore, $T - e'' \in \mathcal{T}$. By induction hypothesis, $c(T - e'') = 2$ or 3 . By Theorem 2, if $c(T - e'') = 2$, then $c(T) = 2$ or 3 . Therefore, suppose that $c(T - e'') = 3$. By Proposition 5 and by interchanging T and T^* , we can assume that $T - e'' \in \mathcal{L}$. By Theorem 2 and by taking $e'' = \{x, x'\}$, we can assume that $x \in \mathbb{N}_3^-(1)$ and $x' \in \mathbb{N}_3^+(2)$. Thus, for $X = \{0, 1, x'\}$, we have $T[X] \simeq C_3$ and $X^+(x') = \emptyset$. It follows from Theorem 2 that $c(T) < 4$. \square

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