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Random process generated by the incomplete Gauss sums

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Abstract: In this paper we explore a random process generated by the incomplete Gauss sums and establish an analogue of weak invariance principle for these sums. We focus our attention exclusively on a generalization of the limit distribution of the long incomplete Gauss sums given by the family of periodic functions analyzed by the author and Marklof.

Key words: Gauss sums, random process

1. Introduction

In the present paper we deal with the curves

$$\begin{aligned} [0, 1] &\rightarrow \mathbb{C} \\ t &\mapsto \mathcal{X}_q(t) = \sum_{h=1}^{[qt]} e_q(ph^2) + (qt - [qt])e_q(ph^2)|_{h=[qt]+1} \end{aligned} \quad (1.1)$$

where $q \in \mathbb{N}$, $p \in \mathbb{Z}_q^\times = \{p \leq q \mid \gcd(p, q) = 1\}$, and $e_q(x) = e^{2\pi ix/q}$. We consider p random uniformly distributed in $\mathbb{Z}_q^\times \cap q\mathcal{D}$ for some fixed $\mathcal{D} \subset \mathbb{T}$ with boundary of measure zero. It is more convenient to normalize the above curves by considering instead the map $\{t \mapsto \frac{\mathcal{X}_q(t)}{\mathcal{X}_q(1)}\}$. Our main aim is in this article to study the ensemble of these curves obtained by the incomplete Gauss sums as $q \rightarrow \infty$. The last term is added to make $\mathcal{X}_q(t)$ a continuous curve. When $t = 1$, this sum corresponds to the classical Gauss sum $\mathcal{X}_q(1)$.

This study extends the author and Marklof's [2] work on the value distribution of long incomplete Gauss sums. The above-mentioned work is later extended to the short interval case of incomplete Gauss sums by the author [3]. The classical examples of incomplete Gauss sums were also studied in the literature for many others [5, 9, 12, 13, 14]. For the higher power case, see [4, 11].

Cellarosi [1] has studied the analogous setting for theta sums $S_N(x) = \sum_{h=1}^{[tN]} e(xh^2)$ with x uniformly distributed with respect to Lebesgue measure, generalizing the limit theorems for theta sums investigated by Marklof [10] and earlier by Jurkat and van Horne [6, 7, 8]. Cellarosi's proof relies on a renormalization procedure established by means of continued fraction expansion of x and renewal-type limit theorem for the denominators of continued fraction expansion of x .

We investigate a random process generated by the values of the normalized Gauss sums $\mathcal{X}_q(t)$. We will prove a limit law for finite-dimensional distributions of such sums as $q \rightarrow \infty$. To describe the limit process let

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us define

$$\mathcal{J}^*(t) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e(n^2x + nt)}{2\pi in}, \tag{1.2}$$

and

$$\mathcal{J}(t) = t + \mathcal{J}^*(t), \tag{1.3}$$

$$\mathcal{J}^+(t) = t + \frac{1}{2}\mathcal{J}^*(t), \tag{1.4}$$

$$\mathcal{J}^-(t) = \frac{1}{2}\mathcal{J}^*(t). \tag{1.5}$$

Our main result in the paper is the following theorem. We define the following random variables. The random variable X takes the values $\pm 1 \pm i$ with equal probability and the random variable Y takes the values ± 1 with equal probability. Z takes the values $1 \pm i$ with equal probability.

We define $\epsilon_a = 1$ if $a \equiv 1 \pmod 4$, and $\epsilon_a = i$ if $a \equiv 3 \pmod 4$.

The symbol \xrightarrow{D} here denotes convergence with respect to finite-dimensional distributions. See Remark 1.1 for explanation.

Theorem 1 For each $q \in \mathbb{N}$ with a bounded number of divisors and $t \in [0, 1]$ as $q \rightarrow \infty$ we have

	q is not a square	q is a square
$q \equiv 0 \pmod 4$	$\left(\frac{\mathcal{X}_q(1)}{\sqrt{q}}, \frac{\mathcal{X}_q(t)}{\mathcal{X}_q(1)}\right) \xrightarrow{D} (X, \mathcal{J}^+(t))$	$\left(\frac{\mathcal{X}_q(1)}{\sqrt{q}}, \frac{\mathcal{X}_q(t)}{\mathcal{X}_q(1)}\right) \xrightarrow{D} (Z, \mathcal{J}^+(t))$
$q \equiv 1 \pmod 2$	$\left(\frac{\mathcal{X}_q(1)}{\epsilon_q \sqrt{q}}, \frac{\mathcal{X}_q(t)}{\mathcal{X}_q(1)}\right) \xrightarrow{D} (Y, \mathcal{J}(t))$	$\frac{\mathcal{X}_q(t)}{\epsilon_q \sqrt{q}} \xrightarrow{D} \mathcal{J}(t)$
	$q/2$ is not a square	$q/2$ is a square
$q \equiv 2 \pmod 4$	$\left(\frac{\mathcal{X}_q(1)}{\epsilon_{q/2} \sqrt{q/2}}, \frac{\mathcal{X}_q(t)}{2\mathcal{X}_q(1)}\right) \xrightarrow{D} (Y, \mathcal{J}^-(t))$	$\frac{\mathcal{X}_q(t)}{\epsilon_{q/2} \sqrt{2q}} \xrightarrow{D} \mathcal{J}^-(t)$

Remark 1.1 The random process $\frac{\mathcal{X}_q(t)}{\mathcal{X}_q(1)}$ converges in finite dimensional distribution to the process $\mathcal{J}^*(t)$ if

$$\frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{p \in \mathbb{Z}_q^\times \cap q\mathcal{D}} F\left(\frac{\mathcal{X}_q(t_1)}{\mathcal{X}_q(1)}, \dots, \frac{\mathcal{X}_q(t_k)}{\mathcal{X}_q(1)}\right) \rightarrow \int_{\mathbb{T}} F(\mathcal{J}^*(t_1), \dots, \mathcal{J}^*(t_k)) dx \tag{1.6}$$

for every bounded continuous function $F : \mathbb{C}^k \rightarrow \mathbb{R}$.

We plot the function $\mathcal{J}^*(t) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e(n^2x + nt)}{2\pi in}$ for different values of x , see Figures 1 and 2, to show how the random process generated by $\mathcal{X}_q(t)$ looks.

We now examine the vector-valued incomplete Gauss sum

$$g_\varphi(p, q) = \sum_{h=1}^{q-1} \varphi\left(\frac{h}{q}\right) e_q(ph^2), \tag{1.7}$$

where $\varphi(x) = (\varphi_1(x), \dots, \varphi_k(x))$ with $k \in \mathbb{Z}$ is a periodic function with period one.

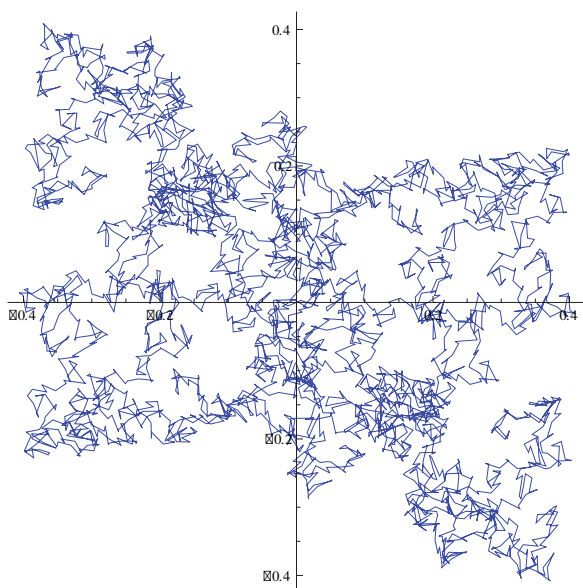


Figure 1. The plot shows the process given by the function $\mathcal{J}^*(t)$ for $x = \sqrt{2}$, t uniformly over the period $[0, 1]$, and truncated at $n = 20000$.

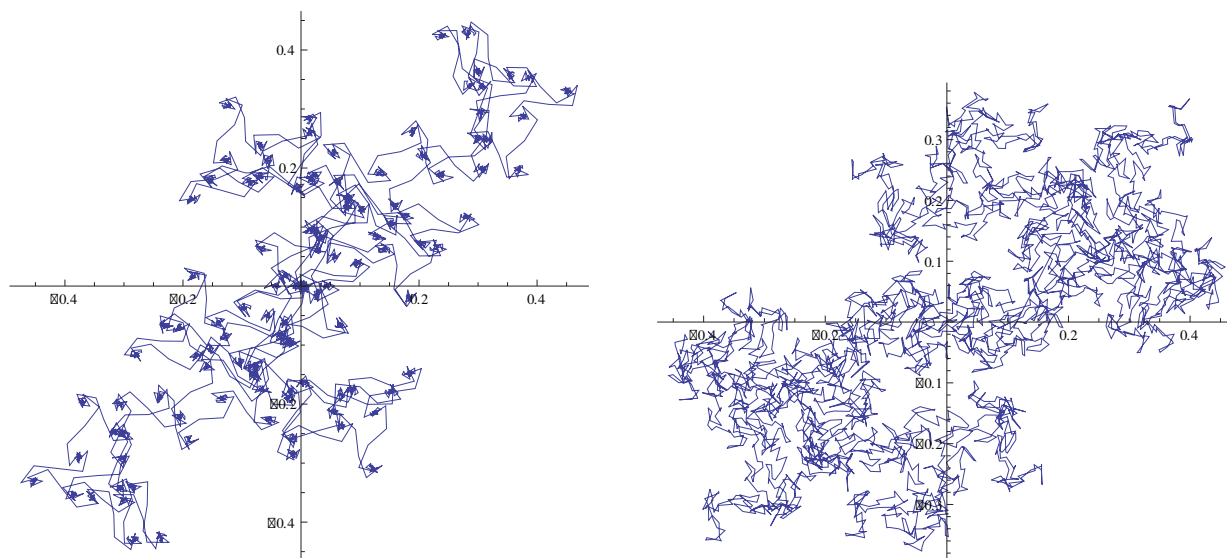


Figure 2. The plots illustrate the same as Figure 1; however, this time for $x = \pi$ on the left and for $x = \frac{\sqrt{5}+1}{2}$ (golden ratio) on the right.

We define the Fourier series of φ with the sum $\sum_{n \in \mathbb{Z}} \hat{\varphi}_n e(nx)$ with Fourier coefficient $\hat{\varphi}_n$. Random variables are given by the limiting distribution of the incomplete Gauss sum

$$G_\varphi(x) = \sum_{n \in \mathbb{Z}} \hat{\varphi}_n e(xn^2), \tag{1.8}$$

$$G_{\varphi}^{+}(x) = \sum_{n \in \mathbb{Z}} \hat{\varphi}_{2n} e(xn^2), \tag{1.9}$$

$$G_{\varphi}^{-}(x) = \sum_{n \in 2\mathbb{Z}+1} \hat{\varphi}_n e(xn^2), \tag{1.10}$$

with x uniformly distributed on \mathbb{T} . For our application to the joint distribution of incomplete Gauss sums in (1.1) at different t_1, \dots, t_k , when φ is a characteristic function we then have

$$\varphi_i(x) = \sum_{n \in \mathbb{Z}} \chi_{(0,t_i]}(x+n). \tag{1.11}$$

The Fourier coefficient $\hat{\varphi}_n$ is therefore calculated as

$$\begin{aligned} \hat{\varphi}_i(n) &= \int \varphi(x) e(-nx) dx \\ &= \int \sum_{n \in \mathbb{Z}} \chi_{(0,t_i]}(x+n) e(-nx) dx \\ &= \int_0^{t_i} e^{-2\pi i n x} dx \\ &= \frac{[1 - e^{-2\pi i n t_i}]}{2\pi i n}. \end{aligned} \tag{1.12}$$

The theorem below is a generalization of Theorem 1 in [2]. We will take the differentiable weight function $\varphi = (\varphi_1, \dots, \varphi_k)$ in the space of

$$\mathcal{B}(\mathbb{T}) = \{ \varphi : \sum_{k \in \mathbb{Z}} k^2 |\hat{\varphi}_k| < \infty \}, \tag{1.13}$$

so that G_{φ} are differentiable and continuous.

The Jacobi symbol is defined for odd integers b by

$$\left(\frac{a}{b}\right) = \begin{cases} +1 & \text{if } b \nmid a \text{ and } a \text{ is a quadratic residue} \\ 0 & \text{if } b \mid a \\ -1 & \text{if } b \nmid a \text{ and } a \text{ is a quadratic nonresidue.} \end{cases} \tag{1.14}$$

This is an extension of Legendre’s symbol to arbitrary odd integers b multiplicatively.

Remark that the classical Gauss sum $g_1(p, q) = \sum_{h \bmod q} e_q(ph^2)$ can be evaluated in terms of Jacobi symbol

$$g_1(p, q) = \begin{cases} (1+i) \epsilon_p^{-1}\left(\frac{q}{p}\right) \sqrt{q} & \text{if } q \equiv 0 \pmod{4} \\ \epsilon_q\left(\frac{p}{q}\right) \sqrt{q} & \text{if } q \equiv 1 \pmod{2} \\ 0 & \text{if } q \equiv 2 \pmod{4}, \end{cases} \tag{1.15}$$

and corresponds to $\chi_q(1)$ in our case.

Theorem 2 Fix a $k \in \mathbb{Z}$ and $0 < t_1 < \dots < t_k \leq 1$. Fix a subset $\mathcal{D} \subset \mathbb{T}$ with boundary of measure zero and let each $\varphi_i \in \mathcal{B}(\mathbb{T})$. For each $q \in \mathbb{N}$ choose $p \in \mathbb{Z}_q^{\times} \cap q\mathcal{D}$ at random with uniform probability. Then as $q \rightarrow \infty$ along an appropriate subsequence as specified below, for any bounded continuous function $F : \mathbb{C}^k \rightarrow \mathbb{R}$ we have

(i) If $q \equiv 0 \pmod 4$ is not a square, for every $\sigma \in \{\pm 1 \pm i\}$ then

$$\begin{aligned} & \frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{\substack{p \in \mathbb{Z}_q^\times \cap q\mathcal{D} \\ g_1(p,q) = \sqrt{q}\sigma}} F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) \\ & \rightarrow \frac{1}{4} \int_{\mathbb{T}} F(G_{\varphi_1}^+(x), \dots, G_{\varphi_k}^+(x)) dx. \end{aligned} \tag{1.16}$$

(ii) If $q \equiv 1 \pmod 2$ is not a square, for every $\sigma \in \{\pm 1\}$ then

$$\begin{aligned} & \frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{\substack{p \in \mathbb{Z}_q^\times \cap q\mathcal{D} \\ g_1(p,q) = \epsilon_q \sqrt{q}\sigma}} F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) \\ & \rightarrow \frac{1}{2} \int_{\mathbb{T}} F(G_{\varphi_1}(x), \dots, G_{\varphi_k}(x)) dx. \end{aligned} \tag{1.17}$$

(iii) If $q \equiv 2 \pmod 4$ and $q/2$ is not a square, for every $\sigma \in \{\pm 1\}$ then

$$\begin{aligned} & \frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{\substack{p \in \mathbb{Z}_q^\times \cap q\mathcal{D} \\ g_1(p,q) = \epsilon_{q/2} \sqrt{q/2}\sigma}} F\left(\frac{g_{\varphi_1}(p,q)}{2g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q)}{2g_1(p,q)}\right) \\ & \rightarrow \frac{1}{2} \int_{\mathbb{T}} F(G_{\varphi_1}^-(x), \dots, G_{\varphi_k}^-(x)) dx. \end{aligned} \tag{1.18}$$

(iv) If $q \equiv 0 \pmod 4$ is a square, for every $\sigma \in \{1 \pm i\}$ then

$$\begin{aligned} & \frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{\substack{p \in \mathbb{Z}_q^\times \cap q\mathcal{D} \\ g_1(p,q) = \sqrt{q}\sigma}} F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) \\ & \rightarrow \frac{1}{4} \int_{\mathbb{T}} F(G_{\varphi_1}^+(x), \dots, G_{\varphi_k}^+(x)) dx. \end{aligned} \tag{1.19}$$

(v) If $q \equiv 1 \pmod 2$ is a square, then

$$\begin{aligned} & \frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{p \in \mathbb{Z}_q^\times \cap q\mathcal{D}} F\left(\frac{g_{\varphi_1}(p,q)}{\epsilon_q \sqrt{q}}, \dots, \frac{g_{\varphi_k}(p,q)}{\epsilon_q \sqrt{q}}\right) \\ & \rightarrow \frac{1}{2} \int_{\mathbb{T}} F(G_{\varphi_1}(x), \dots, G_{\varphi_k}(x)) dx. \end{aligned} \tag{1.20}$$

(vi) If $q \equiv 2 \pmod 4$ and $q/2$ is a square, then

$$\begin{aligned} & \frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{p \in \mathbb{Z}_q^\times \cap q\mathcal{D}} F\left(\frac{g_{\varphi_1}(p,q)}{\epsilon_{q/2} \sqrt{2q}}, \dots, \frac{g_{\varphi_k}(p,q)}{\epsilon_{q/2} \sqrt{2q}}\right) \\ & \rightarrow \frac{1}{2} \int_{\mathbb{T}} F(G_{\varphi_1}^-(x), \dots, G_{\varphi_k}^-(x)) dx. \end{aligned} \tag{1.21}$$

We are able to extend the statements of Theorem 2 to the Riemann integrable case, with the condition that q has a bounded number of divisors. In order to do this we need to estimate mean square

$$M_{2,\varphi}(q) = \frac{1}{\phi(q)|\mathcal{D}|} \sum_{p \in \mathbb{Z}_q^\times \cap q\mathcal{D}} \|g_\varphi(p, q)\|^2 \tag{1.22}$$

where $\varphi = (\varphi_1, \dots, \varphi_k)$.

Theorem 3 Fix a $k \in \mathbb{Z}$ and $0 < t_1 < \dots < t_k \leq 1$. Fix a subset $\mathcal{D} \subset \mathbb{T}$ with boundary of measure zero and let each φ_i be Riemann integrable. Theorem 2 holds for any sequence of $q \rightarrow \infty$ as long as q has a bounded number of divisors.

Note that this is an extension of Theorem 2 in the paper [2].

2. Proof of Theorem 2

Before going through the proof of the theorem we need to state two theorems from [2], which are used in the proof.

Theorem 4 (Demirci Akarsu-Marklof [2]) For each $\varphi_i \in \mathcal{B}(\mathbb{T})$,

$$g_{\varphi_i}(p, q) = \begin{cases} g_1(p, q) G_{\varphi_i}^+(-\frac{\bar{p}}{q}) & \text{if } q \equiv 0 \pmod{4} \\ g_1(p, q) G_{\varphi_i}(-\frac{4\bar{p}}{q}) & \text{if } q \equiv 1 \pmod{2} \\ 2g_1(2p, q/2) G_{\varphi_i}^-(-\frac{8\bar{p}}{q/2}) & \text{if } q \equiv 2 \pmod{4}. \end{cases} \tag{2.1}$$

In the first and second case, \bar{x} denotes the inverse of $x \pmod{q}$, in the third the inverse mod $q/2$.

The order of \mathbb{Z}_q^\times is denoted by Euler's totient function $\phi(q)$.

Theorem 5 (Demirci Akarsu-Marklof [2]) Let $f \in C(\mathbb{T}^2)$. Then the following convergence holds uniformly in $t \in \mathbb{Z}_q^\times$ as $q \rightarrow \infty$:

(i) For any sequence of q ,

$$\frac{1}{\phi(q)} \sum_{p \in \mathbb{Z}_q^\times} f\left(\frac{p}{q}, \frac{t\bar{p}}{q}\right) \rightarrow \int_{\mathbb{T}^2} f(x)dx. \tag{2.2}$$

(ii) If $q \equiv 0 \pmod{4}$ is not a square then, for every $\sigma \in \{\pm 1, \pm i\}$,

$$\frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ \epsilon_p(\frac{p}{q}) = \sigma}} f\left(\frac{p}{q}, \frac{t\bar{p}}{q}\right) \rightarrow \frac{1}{4} \int_{\mathbb{T}^2} f(x)dx. \tag{2.3}$$

(iii) If $q \equiv 0 \pmod{4}$ then, for every $\sigma \in \{\pm 1\}$,

$$\frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ p \equiv \sigma \pmod{4}}} f\left(\frac{p}{q}, \frac{t\bar{p}}{q}\right) \rightarrow \frac{1}{2} \int_{\mathbb{T}^2} f(x)dx. \tag{2.4}$$

(iv) If $q \equiv 1 \pmod 2$ is not a square then, for every $\sigma \in \{\pm 1\}$,

$$\frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ \left(\frac{p}{q}\right) = \sigma}} f\left(\frac{p}{q}, \frac{\overline{p}}{q}\right) \rightarrow \frac{1}{2} \int_{\mathbb{T}^2} f(x) dx. \tag{2.5}$$

Proof

Case (i): $q \equiv 0 \pmod 4$, not a square. We need to show that for any bounded continuous $F : \mathbb{C}^k \rightarrow \mathbb{R}$ we have

$$\begin{aligned} \frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ \epsilon_p\left(\frac{p}{q}\right) = \sigma}} \chi_{\mathcal{D}}\left(\frac{p}{q}\right) F\left(\frac{g_{\varphi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\varphi_k}(p, q)}{g_1(p, q)}\right) \\ \rightarrow \frac{|\mathcal{D}|}{4} \int_{\mathbb{T}} F(G_{\varphi_1}^+(x), \dots, G_{\varphi_k}^+(x)) dx. \end{aligned} \tag{2.6}$$

By Theorem 4 (i), (2.6) equals

$$\begin{aligned} \frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ \epsilon_p\left(\frac{p}{q}\right) = \sigma}} \chi_{\mathcal{D}}\left(\frac{p}{q}\right) F\left(G_{\varphi_1}^+\left(-\frac{\overline{p}}{q}\right), \dots, G_{\varphi_k}^+\left(-\frac{\overline{p}}{q}\right)\right) \\ \rightarrow \frac{|\mathcal{D}|}{4} \int_{\mathbb{T}} F(G_{\varphi_1}^+(x), \dots, G_{\varphi_k}^+(x)) dx. \end{aligned} \tag{2.7}$$

If we choose the test function

$$f(x_1, x_2) = \chi_{\mathcal{D}}(x_1) F(G_{\varphi_1}^+(-x_2), \dots, G_{\varphi_k}^+(-x_2)), \tag{2.8}$$

the proof then uses the approximation argument in which $\chi_{\mathcal{D}}$ is approximated by a continuous function (see Remark 5 in [2] for more details). As $G_{\varphi_1}^+, \dots, G_{\varphi_k}^+$ and F are continuous, the result then follows by Case (ii) of Theorem 5.

Case (ii): $q \equiv 1 \pmod 2$ and not a square. We in this case have

$$\begin{aligned} \frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ \left(\frac{p}{q}\right) = \sigma}} \chi_{\mathcal{D}}\left(\frac{p}{q}\right) F\left(\frac{g_{\varphi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\varphi_k}(p, q)}{g_1(p, q)}\right) \\ \rightarrow \frac{|\mathcal{D}|}{2} \int_{\mathbb{T}} F(G_{\varphi_1}(x), \dots, G_{\varphi_k}(x)) dx. \end{aligned} \tag{2.9}$$

In view of Theorem 4 (ii), this statement reduces to

$$\begin{aligned} \frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ \left(\frac{p}{q}\right) = \sigma}} \chi_{\mathcal{D}}\left(\frac{p}{q}\right) F\left(G_{\varphi_1}\left(-\frac{\overline{4p}}{q}\right), \dots, G_{\varphi_k}\left(-\frac{\overline{4p}}{q}\right)\right) \\ \rightarrow \frac{|\mathcal{D}|}{2} \int_{\mathbb{T}} F(G_{\varphi_1}(x), \dots, G_{\varphi_k}(x)) dx. \end{aligned} \tag{2.10}$$

We conclude this by Theorem 5 (iv).

Case (iii): $q \equiv 2 \pmod 4$, $q/2$ is not a square. Following the same strategy as above, we deduce that the claim of the theorem is equivalent to

$$\begin{aligned} & \frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ (\frac{2p}{q/2}) = \sigma}} \chi_{\mathcal{D}}\left(\frac{p}{q}\right) F\left(G_{\varphi_1}^-\left(-\frac{\overline{8p}}{q/2}\right), \dots, G_{\varphi_k}^-\left(\frac{\overline{8p}}{q/2}\right)\right) \\ & \rightarrow \frac{|\mathcal{D}|}{2} \int_{\mathbb{T}} F(G_{\varphi_1}^-(x), \dots, G_{\varphi_k}^-(x)) dx. \end{aligned} \tag{2.11}$$

We substitute $q = 2q_0$ and $p = 2p_0 + q_0$, i.e., $q_0 = q/2$ and $p_0 = \frac{1}{4}(2p - q)$. Hence (2.11) is equivalent to

$$\begin{aligned} & \frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_{q_0}^\times \\ (\frac{p_0}{q_0}) = \sigma}} \chi_{\mathcal{D}}\left(\frac{p_0}{q_0} + \frac{1}{2}\right) F\left(G_{\varphi_1}^-\left(-\frac{\overline{16p_0}}{q_0}\right), \dots, G_{\varphi_k}^-\left(-\frac{\overline{16p_0}}{q_0}\right)\right) \\ & \rightarrow \frac{|\mathcal{D}|}{2} \int_{\mathbb{T}} F(G_{\varphi_1}^-(x), \dots, G_{\varphi_k}^-(x)) dx, \end{aligned} \tag{2.12}$$

which then follows by Theorem 5 (iv).

Case (iv): $q \equiv 0 \pmod 4$, is a square. We use the same process as in Case (i), and note that the condition $\epsilon_p = 1$ ($\epsilon_p = i$) is equivalent to $p \equiv 1 \pmod 4$ ($p \equiv -1 \pmod 4$). The statement follows from Theorem 5 (iii).

Case (v): $q \equiv 1 \pmod 2$, a square. Analogous to Case (ii), but this time we employ Theorem 5 (i).

Case (vi): $q \equiv 2 \pmod 4$, $q/2$ is a square. This is analogous to Case (iii), except that we use Theorem 5 (i). □

3. Proof of Theorem 3

The lemma below is the key tool to be used in the proof of Theorem 3 for Riemann integrable weight φ . We estimate the second moment of $M_{2,\varphi}(q)$ (recall Equation (1.22)).

Lemma 1 Fix a positive integer N . Then there exists a constant $C_N > 0$ such that any subsequences of $q \rightarrow \infty$ as long as q has a bounded number of divisors, for Riemann integrable function φ , we have

$$\limsup_{\substack{q \rightarrow \infty \\ d(q) \leq N}} \frac{M_{2,\varphi}(q)}{q} \leq \frac{C_N}{|\mathcal{D}|} \|\varphi\|_2^2, \tag{3.1}$$

where $\|\varphi\|_2^2 = \|\varphi_1\|_2^2 + \dots + \|\varphi_k\|_2^2$.

Proof [Proof of Lemma 1] We have

$$\begin{aligned} M_{2,\varphi}(q) & \leq \frac{1}{|\mathcal{D}|\phi(q)} \sum_{p \in \mathbb{Z}_q^\times} \|g_{\varphi}(p, q)\|^2 \\ & \leq \frac{q}{|\mathcal{D}|\phi(q)} \sum_{p \in \mathbb{Z}_q^\times} (|g_{\varphi_1}(p, q)|^2 + \dots + |g_{\varphi_k}(p, q)|^2). \end{aligned} \tag{3.2}$$

By Lemma 1 in [2] we simply get

$$\limsup_{\substack{q \rightarrow \infty \\ d(q) \leq N}} \frac{M_{2,\varphi}(q)}{q} \leq \frac{C_N}{|\mathcal{D}|} \|\varphi\|_2^2. \tag{3.3}$$

□

In the below lemma, we use the tightness argument, which is as follows: the sequence probability measures defined by the value distribution of incomplete Gauss sums is tight. Following the Helly–Prokhorov theorem, this means that every sequence contains a convergent subsequence. In other words, the sequence is relatively compact.

Lemma 2 *Let φ be a Riemann integrable function. Then, for every $\epsilon > 0$, $\delta > 0$ there exists a smooth function ψ such that for the subsequence of q specified in Lemma 1,*

$$\limsup_{\substack{q \rightarrow \infty \\ d(q) \leq N}} \frac{1}{\phi(q)} |\{p \in \mathbb{Z}_q^\times : q^{-1/2} \|g_\varphi(p, q) - g_\psi(p, q)\| > \delta\}| < \epsilon. \tag{3.4}$$

Proof

By Chebyshev’s inequality we have

$$\limsup_{\substack{q \rightarrow \infty \\ d(q) \leq N}} \frac{1}{\phi(q)} |\{p \in \mathbb{Z}_q^\times : q^{-1/2} \|(g_{\varphi_1}(p, q), \dots, g_{\varphi_k}(p, q))\| > \delta\}| < \frac{M_{2,\varphi}(q)}{\delta^2 q}. \tag{3.5}$$

By Lemma 1, there exists $R_\epsilon > 0$ such that

$$\limsup_{\substack{q \rightarrow \infty \\ d(q) \leq N}} \frac{1}{\phi(q)} |\{p \in \mathbb{Z}_q^\times : q^{-1/2} \|(g_{\varphi_1}(p, q), \dots, g_{\varphi_k}(p, q))\| > R_\epsilon\}| < \epsilon \|\varphi\|_2^2. \tag{3.6}$$

Since

$$\begin{aligned} (g_{\varphi_1}(p, q), \dots, g_{\varphi_k}(p, q)) - (g_{\psi_1}(p, q), \dots, g_{\psi_k}(p, q)) \\ = (g_{\varphi_1 - \psi_1}(p, q), \dots, g_{\varphi_k - \psi_k}(p, q)) \end{aligned} \tag{3.7}$$

and each $\varphi_1 - \psi_1, \dots, \varphi_k - \psi_k$ is Riemann integrable, we get

$$\begin{aligned} \limsup_{\substack{q \rightarrow \infty \\ d(q) \leq N}} \frac{1}{\phi(q)} |\{p \in \mathbb{Z}_q^\times : \\ q^{-1/2} \|(g_{\varphi_1}(p, q) - g_{\psi_1}(p, q)), \dots, (g_{\varphi_k}(p, q) - g_{\psi_k}(p, q))\| > \delta\}| < \frac{M_{2,\varphi-\psi}(q)}{\delta^2 q}. \end{aligned} \tag{3.8}$$

We then have via (3.7)

$$\begin{aligned} \limsup_{\substack{q \rightarrow \infty \\ d(q) \leq N}} \frac{1}{\phi(q)} |\{p \in \mathbb{Z}_q^\times : q^{-1/2} \|(g_{\varphi_1 - \psi_1}(p, q), \dots, g_{\varphi_k - \psi_k}(p, q))\| > \delta\}| \\ < \frac{M_{2,\varphi-\psi}(q)}{\delta^2 q}. \end{aligned} \tag{3.9}$$

The proof then follows by Equations (3.5) and (3.6). □

Proof [The proof of Theorem 3]

We only go through the case $q \equiv 0 \pmod 4$; the other cases are similar.

Lemma 2 tells us that any sequence of $q \rightarrow \infty$ with $d(q) \leq N$ contains a subsequence $\{q_j\}$ with the property: there is a probability measure ν (depending on the sequence chosen, φ and \mathcal{D}) on $\{\pm 1 \pm i\} \times \mathbb{C}$ such that for any $\sigma \in \{\pm 1 \pm i\}$ and any bounded continuous function $F : \mathbb{C}^k \rightarrow \mathbb{R}$ we have

$$\lim_{j \rightarrow \infty} \frac{1}{|\mathcal{D}| \phi(q_j)} \sum_{\substack{p \in \mathbb{Z}_{q_j}^\times \cap q_j \mathcal{D} \\ \epsilon_p(\frac{q_j}{p}) = \sigma}} F\left(\frac{g_{\varphi_1}(p, q_j)}{g_1(p, q_j)}, \dots, \frac{g_{\varphi_k}(p, q_j)}{g_1(p, q_j)}\right) = \int_{\mathbb{C}} F(z) \nu_{\varphi}(\sigma, dz). \tag{3.10}$$

We claim that for every $F \in C_0^\infty(\mathbb{C}^k)$

$$\lim_{\substack{q \rightarrow \infty \\ d(q) \leq N}} \frac{1}{|\mathcal{D}| \phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \cap q \mathcal{D} \\ \epsilon_p(\frac{q}{p}) = \sigma}} F\left(\frac{g_{\varphi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\varphi_k}(p, q)}{g_1(p, q)}\right) = \int_{\mathbb{C}} F(z) \nu_{\varphi}(\sigma, dz) \tag{3.11}$$

holds and it thus implies that ν is unique and the full sequence of q converges.

To prove the existence of limit (3.11), notice that since $F \in C_0^\infty(\mathbb{C}^k)$ we have $|F(\mathbf{w}) - F(\mathbf{z})| \leq C \min\{1, \|\mathbf{w} - \mathbf{z}\|\}$ for some constant $C > 0$. Therefore, we have

$$\begin{aligned} & \frac{1}{|\mathcal{D}| \phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \cap q \mathcal{D} \\ \epsilon_p(\frac{q}{p}) = \sigma}} \left| F\left(\frac{g_{\varphi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\varphi_k}(p, q)}{g_1(p, q)}\right) - F\left(\frac{g_{\psi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\psi_k}(p, q)}{g_1(p, q)}\right) \right| \\ & \leq \frac{C}{|\mathcal{D}| \phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \cap q \mathcal{D} \\ \epsilon_p(\frac{q}{p}) = \sigma}} \min \left\{ 1, \left\| \left(\frac{g_{\varphi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\varphi_k}(p, q)}{g_1(p, q)}\right) - \left(\frac{g_{\psi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\psi_k}(p, q)}{g_1(p, q)}\right) \right\| \right\} \\ & \leq \frac{C}{|\mathcal{D}| \phi(q)} \sum_{p \in \mathbb{Z}_q^\times} \min \left\{ 1, \left\| \left(\frac{g_{\varphi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\varphi_k}(p, q)}{g_1(p, q)}\right) - \left(\frac{g_{\psi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\psi_k}(p, q)}{g_1(p, q)}\right) \right\| \right\} \\ & \leq \frac{C}{|\mathcal{D}| \phi(q)} \sum_{p \in \mathbb{Z}_q^\times} \min \left\{ 1, \left\| \frac{g_{\varphi_1 - \psi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\varphi_k - \psi_k}(p, q)}{g_1(p, q)} \right\| \right\} \\ & \leq \frac{C}{|\mathcal{D}|} (2^{1/2} \delta + \epsilon). \end{aligned} \tag{3.12}$$

The sequence

$$\lim_{q \rightarrow \infty} \frac{1}{|\mathcal{D}| \phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \cap q \mathcal{D} \\ \epsilon_p(\frac{q}{p}) = \sigma}} F\left(\frac{g_{\psi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\psi_k}(p, q)}{g_1(p, q)}\right) \tag{3.13}$$

defines a Cauchy sequence, as (3.11) is satisfied for the smooth function ψ by Theorem 2. By the upper bound (3.12), the triangle inequality and the fact that (3.13) is a Cauchy sequence, it is now observed that the sequence

$$\lim_{q \rightarrow \infty} \frac{1}{|\mathcal{D}|} \phi(q) \sum_{\substack{p \in \mathbb{Z}_q^\times \cap q\mathcal{D} \\ \epsilon_p(\frac{q}{p}) = \sigma}} F\left(\frac{g_{\varphi_1}(p, q)}{g_1(p, q)}, \dots, \frac{g_{\varphi_k}(p, q)}{g_1(p, q)}\right) \quad (3.14)$$

is also a Cauchy sequence; therefore the claim is proved. We have thus shown that ν_φ is unique and the full sequence of q converges for every bounded continuous F .

Since ψ converges to φ , (3.13) \rightarrow (3.14) holds by the bound (3.12). This concludes the proof of Theorem 3 for the Riemann integrable case. \square

The proof of Theorem 1

In particular, if we take $\varphi = (\chi_{(0, t_1]}, \dots, \chi_{(0, t_k]})$ above, it proves Theorem 1.

References

- [1] Cellarosi F. Limiting curlicue measures for theta sums. *Ann Inst Henri Poincaré Probab Stat* 2011; 47: 466–497.
- [2] Demirci Akarsu E, Marklof J. The value distribution of incomplete Gauss sums. *Mathematika* 2013; 59: 381–398.
- [3] Demirci Akarsu E. Short incomplete Gauss sums and rational points on metaplectic horocycles. *Int J Number Thr* 2014; 10: 1553–1576.
- [4] Evans R, Minei M, Yee B. Incomplete higher order Gauss sums. *J Math Anal Appl* 2003; 281: 454–476.
- [5] Fiedler H, Jurkat WB, Körner O. Asymptotic expansions of finite theta series. *Acta Arith* 1977; 32: 129–146.
- [6] Jurkat WB, van Horne JW. The proof of the central limit theorem for theta sums. *Duke Math J* 1981; 48: 873–885.
- [7] Jurkat WB, van Horne JW. On the central limit theorem for theta series. *Michigan Math J* 1982; 29: 65–67.
- [8] Jurkat WB, van Horne JW. The uniform central limit theorem for theta sums. *Duke Math J* 1983; 50: 649–666.
- [9] Lehmer DH. Incomplete Gauss sums. *Mathematika* 1976; 23: 125–135.
- [10] Marklof J. Limit theorems for Theta sums. *Duke Math J* 1999; 97: 127–153.
- [11] Montgomery HL, Vaughan RC, Wooley TD. Some remarks on Gauss sums associated with k th powers. *Math Proc Cambridge Philos Soc* 1995; 118: 21–33.
- [12] Oskolkov KI. On functional properties of incomplete Gaussian sums. *Canadian J Math* 1991; 43: 182–212.
- [13] Paris RB. An asymptotic approximation for incomplete Gauss sums. *J Comput Appl Math* 2005; 180: 461–477.
- [14] Paris RB. An asymptotic approximation for incomplete Gauss sums II. *J Comput Appl Math* 2008; 212: 16–30.