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EMEK DEMİRCİ AKARSU

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Emek DEMİRCİ AKARSU*
Department of Mathematics, Faculty of Arts and Sciences, Recep Tayyip Erdoğan University, Rize, Turkey

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Abstract: In this paper we explore a random process generated by the incomplete Gauss sums and establish an analogue of weak invariance principle for these sums. We focus our attention exclusively on a generalization of the limit distribution of the long incomplete Gauss sums given by the family of periodic functions analyzed by the author and Marklof.

Key words: Gauss sums, random process

1. Introduction

In the present paper we deal with the curves

$$\begin{array}{rcl} [0,1] & \to & \mathbb{C} \\ t & \mapsto & \mathcal{X}_q(t) = \sum_{h=1}^{[qt]} e_q(ph^2) + (qt-[qt]) e_q(ph^2)\big|_{h=[qt]+1} \end{array} \tag{1.1}$$

where $q \in \mathbb{N}$, $p \in \mathbb{Z}_q^\times = \{p \le q \,|\, \gcd(p,q) = 1\}$, and $e_q(x) = \mathrm{e}^{2\pi \mathrm{i} x/q}$. We consider p random uniformly distributed in $\mathbb{Z}_q^\times \cap q\mathcal{D}$ for some fixed $\mathcal{D} \subset \mathbb{T}$ with boundary of measure zero. It is more convenient to normalize the above curves by considering instead the map $\{t \mapsto \frac{\mathcal{X}_q(t)}{\mathcal{X}_q(1)}\}$. Our main aim is in this article to study the ensemble of these curves obtained by the incomplete Gauss sums as $q \to \infty$. The last term is added to make $\mathcal{X}_q(t)$ a continuous curve. When $t = 1$, this sum corresponds to the classical Gauss sum $\mathcal{X}_q(1)$.

This study extends the author and Marklof's [2] work on the value distribution of long incomplete Gauss sums. The above-mentioned work is later extended to the short interval case of incomplete Gauss sums by the author [3]. The classical examples of incomplete Gauss sums were also studied in the literature for many others [5, 9, 12, 13, 14]. For the higher power case, see [4, 11].

Cellarosi [1] has studied the analogous setting for theta sums $S_N(x) = \sum_{h=1}^{[tN]} e(xh^2)$ with x uniformly distributed with respect to Lebesgue measure, generalizing the limit theorems for theta sums investigated by Marklof [10] and earlier by Jurkat and van Horne [6, 7, 8]. Cellarosi's proof relies on a renormalization procedure established by means of continued fraction expansion of x and renewal-type limit theorem for the denominators of continued fraction expansion of x.

We investigate a random process generated by the values of the normalized Gauss sums $\mathcal{X}_q(t)$. We will prove a limit law for finite-dimensional distributions of such sums as $q \to \infty$. To describe the limit process let

*Correspondence: emek.akarsu@erdogan.edu.tr
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us define

$$\mathcal{J}^*(t) = \sum_{n\in\mathbb{Z}_{\neq 0}} \frac{e(n^2x+nt)}{2\pi i n}, \tag{1.2}$$

and

$$\mathcal{J}(t) = t + \mathcal{J}^*(t), \tag{1.3}$$

$$\mathcal{J}^+(t) = t + \frac{1}{2}\mathcal{J}^*(t), \tag{1.4}$$

$$\mathcal{J}^-(t) = \frac{1}{2}\mathcal{J}^*(t). \tag{1.5}$$

Our main result in the paper is the following theorem. We define the following random variables. The random variable X takes the values $\pm 1 \pm \mathrm{i}$ with equal probability and the random variable Y takes the values ± 1 with equal probability. Z takes the values $1 \pm \mathrm{i}$ with equal probability.

We define $\epsilon_a = 1$ if $a \equiv 1 \bmod 4$, and $\epsilon_a = \mathrm{i}$ if $a \equiv 3 \bmod 4$.

The symbol $\xrightarrow{D}$ here denotes convergence with respect to finite-dimensional distributions. See Remark 1.1 for explanation.

Theorem 1 *For each $q \in \mathbb{N}$ with a bounded number of divisors and $t \in [0,1]$ as $q \to \infty$ we have*

	q is not a square	*q is a square*
$q \equiv 0 \bmod 4$	$\left(\frac{\mathcal{X}_q(1)}{\sqrt{q}}, \frac{\mathcal{X}_q(t)}{\mathcal{X}_q(1)}\right) \xrightarrow{D} (X, \mathcal{J}^+(t))$	$\left(\frac{\mathcal{X}_q(1)}{\sqrt{q}}, \frac{\mathcal{X}_q(t)}{\mathcal{X}_q(1)}\right) \xrightarrow{D} (Z, \mathcal{J}^+(t))$
$q \equiv 1 \bmod 2$	$\left(\frac{\mathcal{X}_q(1)}{\epsilon_q\sqrt{q}}, \frac{\mathcal{X}_q(t)}{\mathcal{X}_q(1)}\right) \xrightarrow{D} (Y, \mathcal{J}(t))$	$\frac{\mathcal{X}_q(t)}{\epsilon_q\sqrt{q}} \xrightarrow{D} \mathcal{J}(t)$
	$q/2$ is not a square	*$q/2$ is a square*
$q \equiv 2 \bmod 4$	$\left(\frac{\mathcal{X}_q(1)}{\epsilon_{q/2}\sqrt{q/2}}, \frac{\mathcal{X}_q(t)}{2\mathcal{X}_q(1)}\right) \xrightarrow{D} (Y, \mathcal{J}^-(t))$	$\frac{\mathcal{X}_q(t)}{\epsilon_{q/2}\sqrt{2q}} \xrightarrow{D} \mathcal{J}^-(t)$

Remark 1.1 *The random process $\frac{\mathcal{X}_q(t)}{\mathcal{X}_q(1)}$ converges in finite dimensional distribution to the process $\mathcal{J}^*(t)$ if*

$$\frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{p\in\mathbb{Z}_q^\times\cap q\mathcal{D}} F\left(\frac{\mathcal{X}_q(t_1)}{\mathcal{X}_q(1)}, \ldots, \frac{\mathcal{X}_q(t_k)}{\mathcal{X}_q(1)}\right) \to \int_{\mathbb{T}} F(\mathcal{J}^*(t_1), \ldots, \mathcal{J}^*(t_k))\, dx \tag{1.6}$$

for every bounded continuous function $F : \mathbb{C}^k \to \mathbb{R}$.

We plot the function $\mathcal{J}^*(t) = \sum_{n\in\mathbb{Z}_{\neq 0}} \frac{e(n^2x+nt)}{2\pi i n}$ for different values of x, see Figures 1 and 2, to show how the random process generated by $\mathcal{X}_q(t)$ looks.

We now examine the vector-valued incomplete Gauss sum

$$g_{\boldsymbol{\varphi}}(p,q) = \sum_{h=1}^{q-1} \boldsymbol{\varphi}\left(\frac{h}{q}\right) e_q(ph^2), \tag{1.7}$$

where $\boldsymbol{\varphi}(x) = (\varphi_1(x), \ldots, \varphi_k(x))$ with $k \in \mathbb{Z}$ is a periodic function with period one.

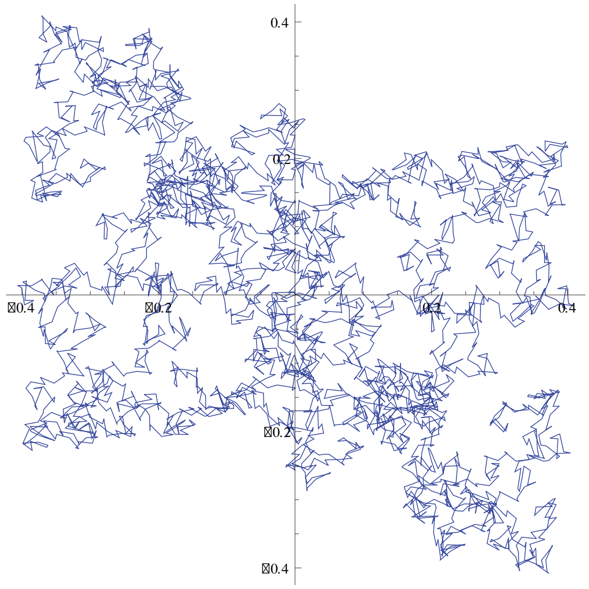

Figure 1. The plot shows the process given by the function $\mathcal{J}^*(t)$ for $x = \sqrt{2}$, t uniformly over the period $[0,1]$, and truncated at $n = 20000$.

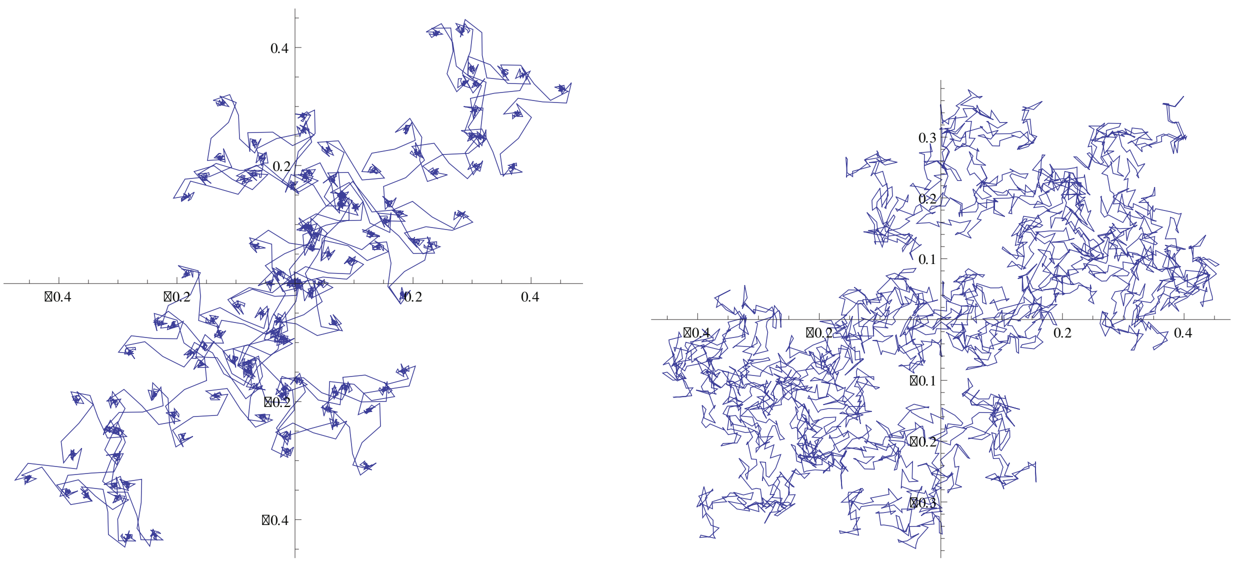

Figure 2. The plots illustrate the same as Figure 1; however, this time for $x = \pi$ on the left and for $x = \frac{\sqrt{5}+1}{2}$ (golden ratio) on the right.

We define the Fourier series of $\boldsymbol{\varphi}$ with the sum $\sum_{n\in\mathbb{Z}} \hat{\boldsymbol{\varphi}}_n\, e(nx)$ with Fourier coefficient $\hat{\boldsymbol{\varphi}}_n$. Random variables are given by the limiting distribution of the incomplete Gauss sum

$$G_{\boldsymbol{\varphi}}(x) = \sum_{n\in\mathbb{Z}} \hat{\boldsymbol{\varphi}}_n\, e(xn^2), \tag{1.8}$$

$$G^{+}_{\boldsymbol{\varphi}}(x) = \sum_{n\in\mathbb{Z}} \hat{\boldsymbol{\varphi}}_{2n}\, e(xn^2), \tag{1.9}$$

$$G^{-}_{\boldsymbol{\varphi}}(x) = \sum_{n\in 2\mathbb{Z}+1} \hat{\boldsymbol{\varphi}}_{n}\, e(xn^2), \tag{1.10}$$

with x uniformly distributed on $\mathbb{T}$. For our application to the joint distribution of incomplete Gauss sums in (1.1) at different $t_1, \ldots, t_k$, when $\boldsymbol{\varphi}$ is a characteristic function we then have

$$\varphi_i(x) = \sum_{n\in\mathbb{Z}} \chi_{(0,t_i]}(x+n). \tag{1.11}$$

The Fourier coefficient $\hat{\boldsymbol{\varphi}}_n$ is therefore calculated as

$$\begin{aligned}\hat{\varphi}_i(n) &= \int \varphi(x)\, e(-nx)\, dx \\ &= \int \sum_{n\in\mathbb{Z}} \chi_{(0,t_i]}(x+n)\, e(-nx) dx \\ &= \int_0^{t_i} \mathrm{e}^{-2\pi \mathrm{i} n x}\, dx \\ &= \frac{[1-\mathrm{e}^{-2\pi \mathrm{i} n t_i}]}{2\pi \mathrm{i} n}.\end{aligned} \tag{1.12}$$

The theorem below is a generalization of Theorem 1 in [2]. We will take the differentiable weight function $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k)$ in the space of

$$\mathcal{B}(\mathbb{T}) = \{\boldsymbol{\varphi} \,:\, \sum_{k\in\mathbb{Z}} k^2 |\hat{\boldsymbol{\varphi}}_k| < \infty\}, \tag{1.13}$$

so that $G_{\boldsymbol{\varphi}}$ are differentiable and continuous.

The Jacobi symbol is defined for odd integers b by

$$\left(\frac{a}{b}\right) = \begin{cases} +1 & \text{if } b \nmid a \text{ and } a \text{ is a quadratic residue} \\ 0 & \text{if } b \mid a \\ -1 & \text{if } b \nmid a \text{ and } a \text{ is a quadratic nonresidue.} \end{cases} \tag{1.14}$$

This is an extension of Legendre's symbol to arbitrary odd integers b multiplicatively.

Remark that the classical Gauss sum $g_1(p,q) = \sum_{h \bmod q} e_q(ph^2)$ can be evaluated in terms of Jacobi symbol

$$g_1(p,q) = \begin{cases} (1+\mathrm{i})\, \epsilon_p^{-1} (\frac{q}{p})\, \sqrt{q} & \text{if } q \equiv 0 \bmod 4 \\ \epsilon_q (\frac{p}{q})\, \sqrt{q} & \text{if } q \equiv 1 \bmod 2 \\ 0 & \text{if } q \equiv 2 \bmod 4, \end{cases} \tag{1.15}$$

and corresponds to $\chi_q(1)$ in our case.

Theorem 2 *Fix a* $k \in \mathbb{Z}$ *and* $0 < t_1 < \ldots < t_k \leq 1$. *Fix a subset* $\mathcal{D} \subset \mathbb{T}$ *with boundary of measure zero and let each* $\varphi_i \in \mathcal{B}(\mathbb{T})$. *For each* $q \in \mathbb{N}$ *choose* $p \in \mathbb{Z}_q^\times \cap q\mathcal{D}$ *at random with uniform probability. Then as* $q \to \infty$ *along an appropriate subsequence as specified below, for any bounded continuous function* $F : \mathbb{C}^k \to \mathbb{R}$ *we have* *(i) If* $q \equiv 0 \bmod 4$ *is not a square, for every* $\sigma \in \{\pm 1 \pm \mathrm{i}\}$ *then*

$$\frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{\substack{p\in\mathbb{Z}_q^\times\cap q\mathcal{D} \\ g_1(p,q)=\sqrt{q}\,\sigma}} F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)},\dots,\frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) \to \frac{1}{4}\int_{\mathbb{T}} F(G^+_{\varphi_1}(x),\dots,G^+_{\varphi_k}(x))dx. \tag{1.16}$$

(ii) If $q \equiv 1 \bmod 2$ *is not a square, for every* $\sigma \in \{\pm 1\}$ *then*

$$\frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{\substack{p\in\mathbb{Z}_q^\times\cap q\mathcal{D} \\ g_1(p,q))=\epsilon_q\sqrt{q}\,\sigma}} F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)},\dots,\frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) \to \frac{1}{2}\int_{\mathbb{T}} F(G_{\varphi_1}(x),\dots,G_{\varphi_k}(x))dx. \tag{1.17}$$

(iii) If $q \equiv 2 \bmod 4$ *and* $q/2$ *is not a square, for every* $\sigma \in \{\pm 1\}$ *then*

$$\frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{\substack{p\in\mathbb{Z}_q^\times\cap q\mathcal{D} \\ g_1(p,q)=\epsilon_{q/2}\sqrt{q/2}\,\sigma}} F\left(\frac{g_{\varphi_1}(p,q)}{2g_1(p,q)},\dots,\frac{g_{\varphi_k}(p,q)}{2g_1(p,q)}\right) \to \frac{1}{2}\int_{\mathbb{T}} F(G^-_{\varphi_1}(x),\dots,G^-_{\varphi_k}(x))dx. \tag{1.18}$$

(iv) If $q \equiv 0 \bmod 4$ *is a square, for every* $\sigma \in \{1 \pm \mathrm{i}\}$ *then*

$$\frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{\substack{p\in\mathbb{Z}_q^\times\cap q\mathcal{D} \\ g_1(p,q)=\sqrt{q}\,\sigma}} F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)},\dots,\frac{g_{\varphi_k}(p,q)}{g_1(p,q))}\right) \to \frac{1}{4}\int_{\mathbb{T}} F(G^+_{\varphi_1}(x),\dots,G^+_{\varphi_k}(x))dx. \tag{1.19}$$

(v) If $q \equiv 1 \bmod 2$ *is a square, then*

$$\frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{p\in\mathbb{Z}_q^\times\cap q\mathcal{D}} F\left(\frac{g_{\varphi_1}(p,q)}{\epsilon_q\sqrt{q}},\dots,\frac{g_{\varphi_k}(p,q)}{\epsilon_q\sqrt{q}}\right) \to \frac{1}{2}\int_{\mathbb{T}} F(G_{\varphi_1}(x),\dots,G_{\varphi_k}(x))dx. \tag{1.20}$$

(vi) If $q \equiv 2 \bmod 4$ *and* $q/2$ *is a square, then*

$$\frac{1}{\#(\mathbb{Z}_q^\times \cap q\mathcal{D})} \sum_{p\in\mathbb{Z}_q^\times\cap q\mathcal{D}} F\left(\frac{g_{\varphi_1}(p,q)}{\epsilon_{q/2}\sqrt{2q}},\dots,\frac{g_{\varphi_k}(p,q)}{\epsilon_{q/2}\sqrt{2q}}\right) \to \frac{1}{2}\int_{\mathbb{T}} F(G^-_{\varphi_1}(x),\dots,G^-_{\varphi_k}(x))dx. \tag{1.21}$$

We are able to extend the statements of Theorem 2 to the Riemann integrable case, with the condition that q has a bounded number of divisors. In order to do this we need to estimate mean square

$$M_{2,\boldsymbol{\varphi}}(q) = \frac{1}{\phi(q)\,|\mathcal{D}|} \sum_{p\in\mathbb{Z}_q^\times \cap q\mathcal{D}} \|g_{\boldsymbol{\varphi}}(p,q)\|^2 \tag{1.22}$$

where $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k)$.

Theorem 3 *Fix a $k \in \mathbb{Z}$ and $0 < t_1 < \ldots < t_k \leq 1$. Fix a subset $\mathcal{D} \subset \mathbb{T}$ with boundary of measure zero and let each φ_i be Riemann integrable. Theorem 2 holds for any sequence of $q \to \infty$ as long as q has a bounded number of divisors.*

Note that this is an extension of Theorem 2 in the paper [2].

2. Proof of Theorem 2

Before going through the proof of the theorem we need to state two theorems from [2], which are used in the proof.

Theorem 4 (Demirci Akarsu-Marklof [2]) *For each $\varphi_i \in \mathcal{B}(\mathbb{T})$,*

$$g_{\varphi_i}(p,q) = \begin{cases} g_1(p,q)\, G^+_{\varphi_i}\big(-\frac{\overline{p}}{q}\big) & \text{if } q \equiv 0 \bmod 4 \\ g_1(p,q)\, G_{\varphi_i}\big(-\frac{\overline{4p}}{q}\big) & \text{if } q \equiv 1 \bmod 2 \\ 2g_1(2p,q/2)\, G^-_{\varphi_i}\big(-\frac{\overline{8p}}{q/2}\big) & \text{if } q \equiv 2 \bmod 4. \end{cases} \tag{2.1}$$

In the first and second case, $\overline{x}$ denotes the inverse of $x \bmod q$, in the third the inverse $\bmod\ q/2$.

The order of $\mathbb{Z}_q^\times$ is denoted by Euler's totient function $\phi(q)$.

Theorem 5 (Demirci Akarsu-Marklof [2]) *Let $f \in \mathrm{C}(\mathbb{T}^2)$. Then the following convergence holds uniformly in $t \in \mathbb{Z}_q^\times$ as $q \to \infty$:*

(i) For any sequence of q,

$$\frac{1}{\phi(q)} \sum_{p\in\mathbb{Z}_q^\times} f\left(\frac{p}{q}, \frac{t\overline{p}}{q}\right) \to \int_{\mathbb{T}^2} f(x)dx. \tag{2.2}$$

(ii) If $q \equiv 0 \bmod 4$ is not a square then, for every $\sigma \in \{\pm 1, \pm \mathrm{i}\}$,

$$\frac{1}{\phi(q)} \sum_{\substack{p\in\mathbb{Z}_q^\times \\ \epsilon_p(\frac{q}{p})=\sigma}} f\left(\frac{p}{q}, \frac{t\overline{p}}{q}\right) \to \frac{1}{4}\int_{\mathbb{T}^2} f(x)dx. \tag{2.3}$$

(iii) If $q \equiv 0 \bmod 4$ then, for every $\sigma \in \{\pm 1\}$,

$$\frac{1}{\phi(q)} \sum_{\substack{p\in\mathbb{Z}_q^\times \\ p\equiv\sigma \bmod 4}} f\left(\frac{p}{q}, \frac{t\overline{p}}{q}\right) \to \frac{1}{2}\int_{\mathbb{T}^2} f(x)dx. \tag{2.4}$$

(iv) If $q \equiv 1 \bmod 2$ *is not a square then, for every* $\sigma \in \{\pm 1\}$*,*

$$\frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ (\frac{p}{q}) = \sigma}} f\left(\frac{p}{q}, \frac{t\overline{p}}{q}\right) \to \frac{1}{2} \int_{\mathbb{T}^2} f(x)dx. \tag{2.5}$$

Proof

Case (i): $q \equiv 0 \bmod 4$, not a square. We need to show that for any bounded continuous $F : \mathbb{C}^k \to \mathbb{R}$ we have

$$\begin{aligned} &\frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ \epsilon_p (\frac{q}{p}) = \sigma}} \chi_{\mathcal{D}}\left(\frac{p}{q}\right) F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) \\ &\qquad \to \frac{|\mathcal{D}|}{4} \int_{\mathbb{T}} F(G^+_{\varphi_1}(x), \dots, G^+_{\varphi_k}(x))dx. \end{aligned} \tag{2.6}$$

By Theorem 4 (i), (2.6) equals

$$\begin{aligned} &\frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ \epsilon_p (\frac{q}{p}) = \sigma}} \chi_{\mathcal{D}}\left(\frac{p}{q}\right) F\left(G^+_{\varphi_1}\left(-\frac{\overline{p}}{q}\right), \dots, G^+_{\varphi_k}\left(-\frac{\overline{p}}{q}\right)\right) \\ &\qquad \to \frac{|\mathcal{D}|}{4} \int_{\mathbb{T}} F(G^+_{\varphi_1}(x), \dots, G^+_{\varphi_k}(x))dx. \end{aligned} \tag{2.7}$$

If we choose the test function

$$f(x_1, x_2) = \chi_{\mathcal{D}}(x_1) F(G^+_{\varphi_1}(-x_2), \dots, G^+_{\varphi_k}(-x_2)), \tag{2.8}$$

the proof then uses the approximation argument in which $\chi_{\mathcal{D}}$ is approximated by a continuous function (see Remark 5 in [2] for more details). As $G^+_{\varphi_1}, \dots, G^+_{\varphi_k}$ and F are continuous, the result then follows by Case (ii) of Theorem 5.

Case (ii): $q \equiv 1 \bmod 2$ and not a square. We in this case have

$$\begin{aligned} &\frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ (\frac{p}{q}) = \sigma}} \chi_{\mathcal{D}}\left(\frac{p}{q}\right) F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) \\ &\qquad \to \frac{|\mathcal{D}|}{2} \int_{\mathbb{T}} F(G_{\varphi_1}(x), \dots, G_{\varphi_k}(x))dx. \end{aligned} \tag{2.9}$$

In view of Theorem 4 (ii), this statement reduces to

$$\begin{aligned} &\frac{1}{\phi(q)} \sum_{\substack{p \in \mathbb{Z}_q^\times \\ (\frac{p}{q}) = \sigma}} \chi_{\mathcal{D}}\left(\frac{p}{q}\right) F\left(G_{\varphi_1}\left(-\frac{\overline{4p}}{q}\right), \dots, G_{\varphi_k}\left(-\frac{\overline{4p}}{q}\right)\right) \\ &\qquad \to \frac{|\mathcal{D}|}{2} \int_{\mathbb{T}} F(G_{\varphi_1}(x), \dots, G_{\varphi_k}(x))dx. \end{aligned} \tag{2.10}$$

We conclude this by Theorem 5 (iv).

Case (iii): $q \equiv 2 \bmod 4$, $q/2$ is not a square. Following the same strategy as above, we deduce that the claim of the theorem is equivalent to

$$\begin{aligned}&\frac{1}{\phi(q)} \sum_{\substack{p\in\mathbb{Z}_q^\times \\ (\frac{2p}{q/2})=\sigma}} \chi_{\mathcal{D}}\Big(\frac{p}{q}\Big) F\Big(G^-_{\varphi_1}\Big(-\frac{\overline{8p}}{q/2}\Big),\ldots,G^-_{\varphi_k}\Big(\frac{\overline{8p}}{q/2}\Big)\Big)\\ &\qquad\to \frac{|\mathcal{D}|}{2}\int_{\mathbb{T}} F(G^-_{\varphi_1}(x),\ldots,G^-_{\varphi_k}(x))dx.\end{aligned} \tag{2.11}$$

We substitute $q = 2q_0$ and $p = 2p_0 + q_0$, i.e., $q_0 = q/2$ and $p_0 = \frac{1}{4}(2p - q)$. Hence (2.11) is equivalent to

$$\begin{aligned}&\frac{1}{\phi(q)} \sum_{\substack{p\in\mathbb{Z}_{q_0}^\times \\ (\frac{p_0}{q_0})=\sigma}} \chi_{\mathcal{D}}\Big(\frac{p_0}{q_0}+\frac{1}{2}\Big) F\Big(G^-_{\varphi_1}\Big(-\frac{\overline{16p_0}}{q_0}\Big),\ldots,G^-_{\varphi_k}\Big(-\frac{\overline{16p_0}}{q_0}\Big)\Big)\\ &\qquad\to \frac{|\mathcal{D}|}{2}\int_{\mathbb{T}} F(G^-_{\varphi_1}(x),\ldots,G^-_{\varphi_k}(x))dx,\end{aligned} \tag{2.12}$$

which then follows by Theorem 5 (iv).

Case (iv): $q \equiv 0 \bmod 4$, is a square. We use the same process as in Case (i), and note that the condition $\epsilon_p = 1$ ($\epsilon_p = \mathrm{i}$) is equivalent to $p \equiv 1 \bmod 4$ ($p \equiv -1 \bmod 4$). The statement follows from Theorem 5 (iii).

Case (v): $q \equiv 1 \bmod 2$, a square. Analogous to Case (ii), but this time we employ Theorem 5 (i).

Case (vi): $q \equiv 2 \bmod 4$, $q/2$ is a square. This is analogous to Case (iii), except that we use Theorem 5 (i).

□

3. Proof of Theorem 3

The lemma below is the key tool to be used in the proof of Theorem 3 for Riemann integrable weight $\boldsymbol{\varphi}$. We estimate the second moment of $M_{2,\boldsymbol{\varphi}}(q)$ (recall Equation (1.22)).

Lemma 1 *Fix a positive integer N. Then there exists a constant $C_N > 0$ such that any subsequences of $q \to \infty$ as long as q has a bounded number of divisors, for Riemann integrable function $\boldsymbol{\varphi}$, we have*

$$\limsup_{\substack{q\to\infty \\ d(q)\le N}} \frac{M_{2,\boldsymbol{\varphi}}(q)}{q} \le \frac{C_N}{|\mathcal{D}|}\|\boldsymbol{\varphi}\|_2^2, \tag{3.1}$$

where $\|\boldsymbol{\varphi}\|_2^2 = \|\varphi_1\|_2^2 + \ldots + \|\varphi_k\|_2^2$.

Proof [Proof of Lemma 1] We have

$$\begin{aligned}M_{2,\boldsymbol{\varphi}}(q) &\le \frac{1}{|\mathcal{D}|\phi(q)} \sum_{p\in\mathbb{Z}_q^\times} \|g_{\boldsymbol{\varphi}}(p,q)\|^2\\ &\le \frac{q}{|\mathcal{D}|\phi(q)} \sum_{p\in\mathbb{Z}_q^\times} (|g_{\varphi_1}(p,q)|^2 + \ldots + |g_{\varphi_k}(p,q)|^2).\end{aligned} \tag{3.2}$$

By Lemma 1 in [2] we simply get

$$\limsup_{\substack{q\to\infty\\ d(q)\le N}} \frac{M_{2,\boldsymbol{\varphi}}(q)}{q} \le \frac{C_N}{|\mathcal{D}|}\|\boldsymbol{\varphi}\|_2^2. \tag{3.3}$$

□

In the below lemma, we use the tightness argument, which is as follows: the sequence probability measures defined by the value distribution of incomplete Gauss sums is tight. Following the Helly–Prokhorov theorem, this means that every sequence contains a convergent subsequence. In other words, the sequence is relatively compact.

Lemma 2 *Let* $\boldsymbol{\varphi}$ *be a Riemann integrable function. Then, for every* $\epsilon > 0$, $\delta > 0$ *there exists a smooth function* $\boldsymbol{\psi}$ *such that for the subsequence of* q *specified in Lemma 1,*

$$\limsup_{\substack{q\to\infty\\ d(q)\le N}} \frac{1}{\phi(q)}\big|\{p\in\mathbb{Z}_q^\times : q^{-1/2}\|g_{\boldsymbol{\varphi}}(p,q)-g_{\boldsymbol{\psi}}(p,q)\|>\delta\}\big| < \epsilon. \tag{3.4}$$

Proof

By Chebyshev's inequality we have

$$\limsup_{\substack{q\to\infty\\ d(q)\le N}} \frac{1}{\phi(q)}\big|\{p\in\mathbb{Z}_q^\times : q^{-1/2}\|(g_{\varphi_1}(p,q),\ldots,g_{\varphi_k}(p,q))\|>\delta\}\big| < \frac{M_{2,\boldsymbol{\varphi}}(q)}{\delta^2\, q}. \tag{3.5}$$

By Lemma 1, there exists $R_\epsilon > 0$ such that

$$\limsup_{\substack{q\to\infty\\ d(q)\le N}} \frac{1}{\phi(q)}\big|\{p\in\mathbb{Z}_q^\times : q^{-1/2}\|(g_{\varphi_1}(p,q),\ldots,g_{\varphi_k}(p,q))\|>R_\epsilon\}\big| < \epsilon\,\|\boldsymbol{\varphi}\|_2^2. \tag{3.6}$$

Since

$$\begin{aligned}(g_{\varphi_1}(p,q),\ldots,g_{\varphi_k}(p,q)) &- (g_{\psi_1}(p,q),\ldots,g_{\psi_k}(p,q))\\ &= (g_{\varphi_1-\psi_1}(p,q),\ldots,g_{\varphi_k-\psi_k}(p,q))\end{aligned} \tag{3.7}$$

and each $\varphi_1-\psi_1,\ldots,\varphi_k-\psi_k$ is Riemann integrable, we get

$$\begin{aligned}\limsup_{\substack{q\to\infty\\ d(q)\le N}} \frac{1}{\phi(q)}\big|\{p\in\mathbb{Z}_q^\times :\\ q^{-1/2}\|(g_{\varphi_1}(p,q)-g_{\psi_1}(p,q)),\ldots,(g_{\varphi_k}(p,q)-g_{\psi_k}(p,q))\|>\delta\}\big| < \frac{M_{2,\boldsymbol{\varphi}-\boldsymbol{\psi}}(q)}{\delta^2\, q}.\end{aligned} \tag{3.8}$$

We then have via (3.7)

$$\begin{aligned}\limsup_{\substack{q\to\infty\\ d(q)\le N}} \frac{1}{\phi(q)}\big|\{p\in\mathbb{Z}_q^\times : q^{-1/2}\|(g_{\varphi_1-\psi_1}(p,q),\ldots,g_{\varphi_k-\psi_k}(p,q))\|>\delta\}\big|\\ < \frac{M_{2,\boldsymbol{\varphi}-\boldsymbol{\psi}}(q)}{\delta^2\, q}.\end{aligned} \tag{3.9}$$

The proof then follows by Equations (3.5) and (3.6). □

Proof [The proof of Theorem 3]

We only go through the case $q \equiv 0 \bmod 4$; the other cases are similar.

Lemma 2 tells us that any sequence of $q \to \infty$ with $d(q) \le N$ contains a subsequence $\{q_j\}$ with the property: there is a probability measure ν (depending on the sequence chosen, $\boldsymbol{\varphi}$ and $\mathcal{D}$) on $\{\pm 1 \pm \mathrm{i}\} \times \mathbb{C}$ such that for any $\sigma \in \{\pm 1 \pm \mathrm{i}\}$ and any bounded continuous function $F : \mathbb{C}^k \to \mathbb{R}$ we have

$$\lim_{j\to\infty} \frac{1}{|\mathcal{D}|\,\phi(q_j)} \sum_{\substack{p\in\mathbb{Z}_{q_j}^{\times}\cap q_j\mathcal{D} \\ \epsilon_p(\frac{q_j}{p})=\sigma}} F\left(\frac{g_{\varphi_1}(p,q_j)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q_j)}{g_1(p,q)}\right) = \int_{\mathbb{C}} F(z)\, \nu_{\boldsymbol{\varphi}}(\sigma, dz). \tag{3.10}$$

We claim that for every $F \in \mathrm{C}_0^{\infty}(\mathbb{C}^k)$

$$\lim_{\substack{q\to\infty \\ d(q)\le N}} \frac{1}{|\mathcal{D}|\,\phi(q)} \sum_{\substack{p\in\mathbb{Z}_{q}^{\times}\cap q\mathcal{D} \\ \epsilon_p(\frac{q}{p})=\sigma}} F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) = \int_{\mathbb{C}} F(z)\, \nu_{\boldsymbol{\varphi}}(\sigma, dz) \tag{3.11}$$

holds and it thus implies that ν is unique and the full sequence of q converges.

To prove the existence of limit (3.11), notice that since $F \in \mathrm{C}_0^{\infty}(\mathbb{C}^k)$ we have $|F(\mathbf{w}) - F(\mathbf{z})| \le C \min\{1, \|\mathbf{w} - \mathbf{z}\|\}$ for some constant $C > 0$. Therefore, we have

$$\begin{aligned}
&\frac{1}{|\mathcal{D}|\,\phi(q)} \sum_{\substack{p\in\mathbb{Z}_{q}^{\times}\cap q\mathcal{D} \\ \epsilon_p(\frac{q}{p})=\sigma}} \left| F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) - F\left(\frac{g_{\psi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\psi_k}(p,q)}{g_1(p,q)}\right) \right| \\
&\le \frac{C}{|\mathcal{D}|\,\phi(q)} \sum_{\substack{p\in\mathbb{Z}_{q}^{\times}\cap q\mathcal{D} \\ \epsilon_p(\frac{q}{p})=\sigma}} \min\left\{1, \left\| \left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) - \left(\frac{g_{\psi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\psi_k}(p,q)}{g_1(p,q)}\right) \right\| \right\} \\
&\le \frac{C}{|\mathcal{D}|\,\phi(q)} \sum_{p\in\mathbb{Z}_{q}^{\times}} \min\left\{1, \left\| \left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) - \left(\frac{g_{\psi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\psi_k}(p,q)}{g_1(p,q)}\right) \right\| \right\} \\
&\le \frac{C}{|\mathcal{D}|\,\phi(q)} \sum_{p\in\mathbb{Z}_{q}^{\times}} \min\left\{1, \left\| \frac{g_{\varphi_1-\psi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\varphi_k-\psi_k}(p,q)}{g_1(p,q)} \right\| \right\} \\
&\le \frac{C}{|\mathcal{D}|} (2^{1/2}\delta + \epsilon).
\end{aligned} \tag{3.12}$$

The sequence

$$\lim_{q\to\infty} \frac{1}{|\mathcal{D}|\,\phi(q)} \sum_{\substack{p\in\mathbb{Z}_{q}^{\times}\cap q\mathcal{D} \\ \epsilon_p(\frac{q}{p})=\sigma}} F\left(\frac{g_{\psi_1}(p,q)}{g_1(p,q)}, \dots, \frac{g_{\psi_k}(p,q)}{g_1(p,q)}\right) \tag{3.13}$$

defines a Cauchy sequence, as (3.11) is satisfied for the smooth function $\boldsymbol{\psi}$ by Theorem 2. By the upper bound (3.12), the triangle inequality and the fact that (3.13) is a Cauchy sequence, it is now observed that the sequence

$$\lim_{q\to\infty} \frac{1}{|\mathcal{D}|\,\phi(q)} \sum_{\substack{p\in\mathbb{Z}_q^\times\cap q\mathcal{D}\\ \epsilon_p(\frac{q}{p})=\sigma}} F\left(\frac{g_{\varphi_1}(p,q)}{g_1(p,q)},\ldots,\frac{g_{\varphi_k}(p,q)}{g_1(p,q)}\right) \tag{3.14}$$

is also a Cauchy sequence; therefore the claim is proved. We have thus shown that $\nu_{\boldsymbol{\varphi}}$ is unique and the full sequence of q converges for every bounded continuous F.

Since $\boldsymbol{\psi}$ converges to $\boldsymbol{\varphi}$, (3.13)$\to$(3.14) holds by the bound (3.12). This concludes the proof of Theorem 3 for the Riemann integrable case. □

The proof of Theorem 1

In particular, if we take $\boldsymbol{\varphi} = (\chi_{(0,t_1]},\ldots,\chi_{(0,t_k]})$ above, it proves Theorem 1.

References

[1] Cellarosi F. Limiting curlicue measures for theta sums. Ann Inst Henri Poincaré Probab Stat 2011; 47: 466–497.

[2] Demirci Akarsu E, Marklof J. The value distribution of incomplete Gauss sums. Mathematika 2013; 59: 381–398.

[3] Demirci Akarsu E. Short incomplete Gauss sums and rational points on metaplectic horocycles. Int J Number Thr 2014; 10: 1553–1576.

[4] Evans R, Minei M, Yee B. Incomplete higher order Gauss sums. J Math Anal Appl 2003; 281: 454–476.

[5] Fiedler H, Jurkat WB, Körner O. Asymptotic expansions of finite theta series. Acta Arith 1977; 32: 129–146.

[6] Jurkat WB, van Horne JW. The proof of the central limit theorem for theta sums. Duke Math J 1981; 48: 873–885.

[7] Jurkat WB, van Horne JW. On the central limit theorem for theta series. Michigan Math J 1982; 29: 65–67.

[8] Jurkat WB, van Horne JW. The uniform central limit theorem for theta sums. Duke Math J 1983; 50: 649–666.

[9] Lehmer DH. Incomplete Gauss sums. Mathematika 1976; 23: 125–135.

[10] Marklof J. Limit theorems for Theta sums. Duke Math J 1999; 97: 127–153.

[11] Montgomery HL, Vaughan RC, Wooley TD. Some remarks on Gauss sums associated with kth powers. Math Proc Cambridge Philos Soc 1995; 118: 21–33.

[12] Oskolkov KI. On functional properties of incomplete Gaussian sums. Canadian J Math 1991; 43: 182–212.

[13] Paris RB. An asymptotic approximation for incomplete Gauss sums. J Comput Appl Math 2005; 180: 461–477.

[14] Paris RB. An asymptotic approximation for incomplete Gauss sums II. J Comput Appl Math 2008; 212: 16–30.