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Approximate duals and nearly Parseval frames

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Abstract: In this paper we introduce approximate duality of g-frames in Hilbert C^* -modules and we show that approximate duals of g-frames in Hilbert C^* -modules share many useful properties with those in Hilbert spaces. Moreover, we obtain some new results for approximate duality of frames and g-frames in Hilbert spaces; in particular, we consider approximate duals of ε -nearly Parseval and ε -close frames.

Key words: Hilbert C^* -module, g-frame, frame, approximate duality, ε -nearly Parseval frame

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [10] in 1952 to study some problems in nonharmonic Fourier series, and they were reintroduced in 1986 by Daubechies et al. [9]. Frames have important applications in signal and image processing, wireless communications, and many other fields. There exist various generalizations of frames. A recent and general one is called g-frame [26].

As we know, duals play an important role in frame theory, especially they are used in the reconstruction of signals. It is well known that every frame in a Hilbert space has at least one dual (see [7]), and if a dual of a frame is found, then each signal can be reconstructed easily. However, it is usually difficult to calculate a dual. Here, approximate duals can be useful. Approximate duals in frame theory have important applications (see [4, 12, 27]). Approximate duality of frames in Hilbert spaces was recently investigated in [8]. Khosravi and Mirzaei Azandaryani (the present author) also introduced approximate duality of g-frames in Hilbert spaces and obtained some properties and applications of approximate duals (see [20]). In particular, it was shown that approximate duals are stable under small perturbations and they are useful for erasures (see [20, Section 3]).

Hilbert C^* -modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^* -algebra rather than in the field of complex numbers.

Frank and Larson presented a general approach to the frame theory in Hilbert C^* -modules (see [11]). They showed that every countably generated Hilbert C^* -module over a unital C^* -algebra admits a frame. It was also shown in [25] that every Hilbert C^* -module that is countably generated in the set of adjointable operators admits a frame of multipliers. Furthermore, g-frames in Hilbert C^* -modules were introduced in [16].

Frames in Hilbert C^* -modules are not trivial generalizations of Hilbert space frames due to the complex structure of C^* -algebras. Since there are important differences between the theory of Hilbert C^* -modules

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and Hilbert spaces (see Chapter 1 in [21]), it is expected that problems about frames in Hilbert C^* -modules are more complicated than those in Hilbert spaces.

In this paper we generalize the concept of approximate duality of g-frames to Hilbert C^* -modules and we get some results for approximate duals of frames and g-frames in Hilbert spaces. In particular, approximate duals of ε -nearly Parseval and ε -close frames are studied.

First, in the following section, we have a brief review of the definitions and basic properties of frames and g-frames in Hilbert C^* -modules.

In this note, all index sets are finite or countable subsets of \mathbb{Z} .

2. Frames and g-frames in Hilbert C^* -modules

Suppose that \mathfrak{A} is a unital C^* -algebra and E is a left \mathfrak{A} -module such that the linear structures of \mathfrak{A} and E are compatible. E is a pre-Hilbert \mathfrak{A} -module if E is equipped with an \mathfrak{A} -valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathfrak{A}$, such that

- (i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for each $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$;
- (ii) $\langle ax, y \rangle = a \langle x, y \rangle$, for each $a \in \mathfrak{A}$ and $x, y \in E$;
- (iii) $\langle x, y \rangle = \langle y, x \rangle^*$, for each $x, y \in E$;
- (iv) $\langle x, x \rangle \geq 0$, for each $x \in E$ and if $\langle x, x \rangle = 0$, then $x = 0$.

For each $x \in E$, we define $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ and $|x| = \langle x, x \rangle^{\frac{1}{2}}$. If E is complete with $\|\cdot\|$, it is called a *Hilbert \mathfrak{A} -module* or a *Hilbert C^* -module* over \mathfrak{A} . Let E and F be Hilbert \mathfrak{A} -modules. An operator $T : E \rightarrow F$ is called *adjointable* if there exists an operator $T^* : F \rightarrow E$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$, for each $x \in E$ and $y \in F$. Every adjointable operator T is bounded and \mathfrak{A} -linear (that is, $T(ax) = aT(x)$ for each $x \in E$ and $a \in \mathfrak{A}$). We denote the set of all adjointable operators from E into F by $\mathfrak{L}(E, F)$. $\mathfrak{L}(E, E)$ is a C^* -algebra and we denote it by $\mathfrak{L}(E)$. Note that if $\{E_i : i \in I\}$ is a sequence of Hilbert \mathfrak{A} -modules, then $\bigoplus_{i \in I} E_i$, which is the set

$$\bigoplus_{i \in I} E_i = \left\{ \{x_i\}_{i \in I} : x_i \in E_i \text{ and } \sum_{i \in I} \langle x_i, x_i \rangle \text{ is norm convergent in } \mathfrak{A} \right\},$$

is a Hilbert \mathfrak{A} -module with pointwise operations and \mathfrak{A} -valued inner product $\langle x, y \rangle = \sum_{i \in I} \langle x_i, y_i \rangle$, where $x = \{x_i\}_{i \in I}$ and $y = \{y_i\}_{i \in I}$. For each $x = \{x_i\}_{i \in I} \in \bigoplus_{i \in I} E_i$, we define $\|\cdot\|_2$ by $\|x\|_2 = \|\sum_{i \in I} \langle x_i, x_i \rangle\|^{\frac{1}{2}}$. For more details about Hilbert C^* -modules, see [21].

In this paper we focus on finitely and countably generated Hilbert C^* -modules over unital C^* -algebras. A Hilbert \mathfrak{A} -module E is *finitely generated* if there exists a finite set $\{x_1, \dots, x_n\} \subseteq E$ such that every element $x \in E$ can be expressed as an \mathfrak{A} -linear combination $x = \sum_{i=1}^n a_i x_i$, $a_i \in \mathfrak{A}$. A Hilbert \mathfrak{A} -module E is *countably generated* if there exists a countable set $\{x_i\}_{i \in I} \subseteq E$ such that E equals the norm-closure of the \mathfrak{A} -linear hull of $\{x_i\}_{i \in I}$.

Let E be a Hilbert \mathfrak{A} -module. A family $\{f_i\}_{i \in I} \subseteq E$ is a *frame* for E , if there exist real constants $0 < A \leq B < \infty$, such that for each $x \in E$,

$$A \langle x, x \rangle \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \leq B \langle x, x \rangle. \tag{1}$$

The numbers A and B are called the *lower* and *upper bound* of the frame, respectively. In this case we call it an (A, B) *frame*. The *optimal lower frame bound* is the supremum over all lower frame bounds and the *optimal upper frame bound* is the infimum over all upper frame bounds. If $A = B$, the frame is called *tight* (A -*tight*) and if $A = B = 1$, the frame is *Parseval*. If only the second inequality is required, we call it a *Bessel sequence*. If the sum in (1) converges in norm, the frame is called *standard*.

Let $\{E_i\}_{i \in I}$ be a sequence of Hilbert \mathfrak{A} -modules. A sequence $\Lambda = \{\Lambda_i \in \mathfrak{L}(E, E_i) : i \in I\}$ is called a *g-frame* for E with respect to $\{E_i : i \in I\}$ if there exist real constants $A, B > 0$ such that for each $x \in E$,

$$A\langle x, x \rangle \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \leq B\langle x, x \rangle.$$

A and B are *g-frame bounds* of Λ . In this case we call it an (A, B) *g-frame*. The *optimal bounds* and *tight* and *Parseval* *g-frames* are defined similarly to frames. The *g-frame* is *standard* if for each $x \in E$, the sum converges in norm. If only the second-hand inequality is required, then Λ is called a *g-Bessel sequence*.

For a standard *g-Bessel sequence* Λ , the operator $T_\Lambda : \oplus_{i \in I} E_i \rightarrow E$ defined by $T_\Lambda(\{x_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^*(x_i)$ is called the *synthesis operator* of Λ . T_Λ is adjointable and $T_\Lambda^*(x) = \{\Lambda_i x\}_{i \in I}$. Now we define the operator S_Λ on E by $S_\Lambda x = T_\Lambda T_\Lambda^*(x) = \sum_{i \in I} \Lambda_i^* \Lambda_i(x)$. If Λ is a standard (A, B) *g-frame*, then $A.Id_E \leq S_\Lambda \leq B.Id_E$.

Note that $\mathcal{F} = \{f_i\}_{i \in I}$ is a standard Bessel sequence (resp. frame) if and only if $\Lambda_{\mathcal{F}} = \{\Lambda_{f_i}\}_{i \in I}$ is a standard *g-Bessel sequence* (resp. *g-frame*), where $\Lambda_{f_i}(x) = \langle x, f_i \rangle$, for each $x \in E$ (see [16, Example 3.2]). This shows that each Bessel sequence (resp. frame) generates a *g-Bessel sequence* (resp. *g-frame*). For a standard Bessel sequence $\mathcal{F} = \{f_i\}_{i \in I}$, we denote $S_{\Lambda_{\mathcal{F}}}$ by $S_{\mathcal{F}}$.

Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be an (A, B) standard *g-frame*. We call $\tilde{\Lambda} = \{\Lambda_i S_\Lambda^{-1}\}_{i \in I}$ the *canonical g-dual* of Λ , which is a $(\frac{1}{B}, \frac{1}{A})$ standard *g-frame*. We denote the *canonical dual* of a standard frame $\mathcal{F} = \{f_i\}_{i \in I}$ by $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i \in I}$, where $\tilde{f}_i = S_{\mathcal{F}}^{-1} f_i$. Recall that if $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Gamma = \{\Gamma_i\}_{i \in I}$ are standard *g-Bessel sequences* such that $\sum_{i \in I} \Gamma_i^* \Lambda_i x = x$ or equivalently $\sum_{i \in I} \Lambda_i^* \Gamma_i x = x$, for each $x \in E$, then Γ (resp. Λ) is called a *g-dual* of Λ (resp. Γ). Also, *duals* for two standard Bessel sequences $\mathcal{F} = \{f_i\}_{i \in I}$ and $\mathcal{G} = \{g_i\}_{i \in I}$ can be defined by using the generated *g-Bessel sequences*, so \mathcal{G} (resp. \mathcal{F}) is a dual of \mathcal{F} (resp. \mathcal{G}) if $x = \sum_{i \in I} \langle x, f_i \rangle g_i$ or equivalently $x = \sum_{i \in I} \langle x, g_i \rangle f_i$, for each $x \in E$ (see [11, 13]). For more details about frames and *g-frames* in Hilbert C^* -modules, see [11, 2, 16, 28].

3. Approximate duals of g-frames in Hilbert C^* -modules

In this section all C^* -algebras are unital and all Hilbert C^* -modules are finitely or countably generated. All frames, *g-frames*, Bessel sequences, and *g-Bessel sequences* are standard. Λ and Γ denote $\{\Lambda_i \in \mathfrak{L}(E, E_i) : i \in I\}$ and $\{\Gamma_i \in \mathfrak{L}(E, E_i) : i \in I\}$, respectively. Also, $\mathcal{F} = \{f_i\}_{i \in I}$ and $\mathcal{G} = \{g_i\}_{i \in I}$ are subsets of a Hilbert C^* -module E .

For two standard *g-Bessel sequences* Λ and Γ , the operator $S_{\Gamma\Lambda}$ is defined on E by $S_{\Gamma\Lambda} = T_\Gamma T_\Lambda^*$. Since $S_{\Gamma\Lambda}^* = S_{\Lambda\Gamma}$, we have $\|Id_E - S_{\Gamma\Lambda}\| = \|(Id_E - S_{\Gamma\Lambda})^*\| = \|Id_E - S_{\Lambda\Gamma}\|$.

Now we introduce approximate duals for *g-Bessel sequences* (and also for Bessel sequences by using the generated *g-Bessel sequences*) in Hilbert C^* -modules:

Definition 3.1 (i) Two standard g -Bessel sequences Λ and Γ are approximately dual g -frames if $\|Id_E - S_{\Gamma\Lambda}\| < 1$ or equivalently $\|Id_E - S_{\Lambda\Gamma}\| < 1$. In this case, we say that Γ (resp. Λ) is an approximate g -dual of Λ (resp. Γ).

(ii) Two standard Bessel sequences \mathcal{F} and \mathcal{G} are approximately dual frames if $\Lambda_{\mathcal{F}}$ and $\Lambda_{\mathcal{G}}$ are approximately dual g -frames, i.e. $\|Id_E - S_{\Lambda_{\mathcal{G}}\Lambda_{\mathcal{F}}}\| < 1$ or equivalently $\|Id_E - S_{\Lambda_{\mathcal{F}}\Lambda_{\mathcal{G}}}\| < 1$. In this case, we say that \mathcal{G} (resp. \mathcal{F}) is an approximate dual of \mathcal{F} (resp. \mathcal{G}). We denote $S_{\Lambda_{\mathcal{G}}\Lambda_{\mathcal{F}}}$ and $S_{\Lambda_{\mathcal{F}}\Lambda_{\mathcal{G}}}$ by $S_{\mathcal{G}\mathcal{F}}$ and $S_{\mathcal{F}\mathcal{G}}$, respectively.

It is clear that $S_{\Gamma\Lambda}(x) = \sum_{i \in I} \Gamma_i^* \Lambda_i(x)$ and $S_{\mathcal{G}\mathcal{F}}(x) = \sum_{i \in I} \langle x, f_i \rangle g_i$, for each $x \in E$. If Λ and Γ are g -duals, then they are approximately dual g -frames because $S_{\Lambda\Gamma} = Id_E$. Using the Neumann algorithm, we can see that $S_{\Lambda\Gamma}$ is invertible with $S_{\Lambda\Gamma}^{-1} = \sum_{n=0}^{\infty} (Id_E - S_{\Lambda\Gamma})^n$, so each $x \in E$ can be reconstructed as

$$x = \sum_{n=0}^{\infty} S_{\Lambda\Gamma} (Id_E - S_{\Lambda\Gamma})^n x, \quad x = \sum_{n=0}^{\infty} (Id_E - S_{\Lambda\Gamma})^n S_{\Lambda\Gamma} x.$$

Recall from [17] that a standard g -frame Λ is a modular g -Riesz basis if it has the following property:

if $\sum_{i \in \Omega} \Lambda_i^* g_i = 0$, where $g_i \in E_i$ and $\Omega \subseteq I$, then $g_i = 0$, for each $i \in \Omega$.

A standard frame $\{f_i\}_{i \in I}$ for E is a modular Riesz basis if it has the following property: if an \mathfrak{A} -linear combination $\sum_{i \in \Omega} a_i f_i$ with coefficients $\{a_i : i \in \Omega\} \subseteq \mathfrak{A}$ and $\Omega \subseteq I$ is equal to zero, then $a_i = 0$, for each $i \in \Omega$.

The following result is a generalization of Proposition 2.3 in [20] to Hilbert C^* -modules.

Theorem 3.2 Let Λ and Γ be approximately dual g -frames with upper bounds B and D , respectively. Then:

- (i) Λ and Γ are $(\frac{\|S_{\Gamma\Lambda}^{-1}\|^{-2}}{D}, B)$ and $(\frac{\|S_{\Lambda\Gamma}^{-1}\|^{-2}}{B}, D)$ g -frames, respectively.
- (ii) $\{\Gamma_i + \sum_{n=1}^{\infty} \Gamma_i (Id_E - S_{\Lambda\Gamma})^n\}_{i \in I}$ is a g -dual of Λ .
- (iii) For each $N \in \mathbb{N}$, define $\psi_i^N = \Gamma_i + \sum_{n=1}^N \Gamma_i (Id_E - S_{\Lambda\Gamma})^n$. Then $\Psi_N = \{\psi_i^N\}_{i \in I}$ is an approximate g -dual of Λ with $\|Id_E - S_{\Lambda\Psi_N}\| \leq \|Id_E - S_{\Lambda\Gamma}\|^{N+1} < 1$.
- (iv) If Λ is a modular g -Riesz basis, then $\widetilde{\Lambda}_i = \Gamma_i + \sum_{n=1}^{\infty} \Gamma_i (Id_E - S_{\Lambda\Gamma})^n = \lim_{N \rightarrow \infty} \psi_i^N$, for each $i \in I$.

Proof (i) Since Λ and Γ are approximately dual g -frames, $S_{\Gamma\Lambda}$ is invertible, so $\|S_{\Gamma\Lambda}^{-1}\|^{-1} \|x\| \leq \|S_{\Gamma\Lambda} x\|$, for each $x \in E$. Now by using the Cauchy-Schwarz inequality in Hilbert C^* -modules, we have

$$\begin{aligned} \|S_{\Gamma\Lambda}^{-1}\|^{-1} \|x\| \leq \|S_{\Gamma\Lambda} x\| &= \sup_{\|y\|=1} \|\langle S_{\Gamma\Lambda} x, y \rangle\| = \sup_{\|y\|=1} \left\| \sum_{i \in I} \langle \Lambda_i x, \Gamma_i y \rangle \right\| \\ &\leq \sup_{\|y\|=1} \left\| \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \right\|^{\frac{1}{2}} \left\| \sum_{i \in I} \langle \Gamma_i y, \Gamma_i y \rangle \right\|^{\frac{1}{2}} \\ &\leq \sqrt{D} \left\| \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \right\|^{\frac{1}{2}}. \end{aligned}$$

Hence:

$$\frac{\|S_{\Gamma\Lambda}^{-1}\|^{-2}\|x\|^2}{D} \leq \left\| \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \right\|,$$

and so by Theorem 3.1 in [28], Λ is a standard g -frame with the lower bound $\frac{\|S_{\Gamma\Lambda}^{-1}\|^{-2}}{D}$. Similarly, by considering $S_{\Lambda\Gamma}$ instead of $S_{\Gamma\Lambda}$ in the above conclusions, we obtain that Γ is a $(\frac{\|S_{\Lambda\Gamma}^{-1}\|^{-2}}{D}, D)$ standard g -frame.

(ii) Since $S_{\Lambda\Gamma}^{-1} = \sum_{n=0}^{\infty} (Id_E - S_{\Lambda\Gamma})^n$, we have $\Gamma_i S_{\Lambda\Gamma}^{-1} = \Gamma_i + \sum_{n=1}^{\infty} \Gamma_i (Id_E - S_{\Lambda\Gamma})^n$ and it is easy to see that $\{\Gamma_i S_{\Lambda\Gamma}^{-1}\}_{i \in I}$ is a g -dual of Λ .

(iii) For each $n = 0, \dots, N$, we have

$$\left\| \sum_{i \in I} \langle \Gamma_i (Id_E - S_{\Lambda\Gamma})^n x, \Gamma_i (Id_E - S_{\Lambda\Gamma})^n x \rangle \right\| \leq D \|(Id_E - S_{\Lambda\Gamma})^n\|^2 \|x\|^2,$$

so $\{\Gamma_i (Id_E - S_{\Lambda\Gamma})^n\}_{i \in I}$ is a standard g -Bessel sequence by Theorem 3.1 in [28] and consequently Ψ_N is a standard g -Bessel sequence. Now the result can be obtained similar to the proof of Proposition 2.3 in [20].

(iv) Since Λ is a modular g -Riesz basis, Corollary 4.1 in [17] yields that $\tilde{\Lambda}$ is the unique g -dual of Λ . According to part (ii), $\{\Gamma_i + \sum_{n=1}^{\infty} \Gamma_i (Id_E - S_{\Lambda\Gamma})^n\}_{i \in I}$ is also a g -dual of Λ , so $\tilde{\Lambda}_i = \Gamma_i + \sum_{n=1}^{\infty} \Gamma_i (Id_E - S_{\Lambda\Gamma})^n = \lim_{N \rightarrow \infty} \psi_i^N$. \square

As a consequence of the above theorem and Example 3.2 in [16], we obtain the following result. Parts (ii) and (iii) of the following corollary are generalizations of Proposition 3.2 in [8] to Hilbert C^* -modules.

Corollary 3.3 *Let \mathcal{F} and \mathcal{G} be approximately dual frames with upper bounds B and D , respectively. Then:*

- (i) \mathcal{F} and \mathcal{G} are $(\frac{\|S_{\mathcal{G}\mathcal{F}}^{-1}\|^{-2}}{D}, B)$ and $(\frac{\|S_{\mathcal{F}\mathcal{G}}^{-1}\|^{-2}}{B}, D)$ frames, respectively.
- (ii) $\{g_i + \sum_{n=1}^{\infty} (Id_E - S_{\mathcal{G}\mathcal{F}})^n g_i\}_{i \in I}$ is a dual of \mathcal{F} .
- (iii) For each $N \in \mathbb{N}$, define $h_i^N = g_i + \sum_{n=1}^N (Id_E - S_{\mathcal{G}\mathcal{F}})^n g_i$. Then $h_N = \{h_i^N\}_{i \in I}$ is an approximate dual of \mathcal{F} with $\|Id_E - S_{h_N \mathcal{F}}\| \leq \|Id_E - S_{\mathcal{G}\mathcal{F}}\|^{N+1} < 1$.
- (iv) If \mathcal{F} is a modular Riesz basis, then $\tilde{f}_i = g_i + \sum_{n=1}^{\infty} (Id_E - S_{\mathcal{G}\mathcal{F}})^n g_i = \lim_{N \rightarrow \infty} h_i^N$, for each $i \in I$.

We can get from the above theorem and corollary that a standard g -Bessel sequence (resp. Bessel sequence) is a standard g -frame (resp. frame) if and only if it has an approximate g -dual (resp. approximate dual).

Note that Theorem 2.5 in [20] shows that if Λ and Γ are two g -Bessel sequences in a Hilbert space H , then a necessary and sufficient condition for Λ and Γ to be approximately dual g -frames is that there exist two Bessel sequences \mathcal{F} and \mathcal{G} in H that are approximately dual frames with $S_{\Lambda\Gamma} = S_{\mathcal{F}\mathcal{G}}$. Now we have a similar result for approximate duals in Hilbert C^* -modules.

Proposition 3.4 *Let Λ and Γ be two g -Bessel sequences. Then Λ and Γ are approximately dual g -frames if and only if there exist two Bessel sequences \mathcal{F} and \mathcal{G} in E such that \mathcal{F} and \mathcal{G} are approximately dual frames with $S_{\Lambda\Gamma} = S_{\mathcal{F}\mathcal{G}}$.*

Proof Let Λ and Γ be approximately dual g-frames. As a result of Kasparov’s stabilization theorem, every finitely or countably generated Hilbert C^* -module has a standard Parseval frame (see [11, 22]). Let $\{f_{ij}\}_{j \in J_i}$ be a standard Parseval frame for E_i . It follows from Corollary 3.4 in [16] that $\mathcal{F} = \{\Lambda_i^*(f_{ij})\}_{i \in I, j \in J_i}$ and $\mathcal{G} = \{\Gamma_i^*(f_{ij})\}_{i \in I, j \in J_i}$ are standard Bessel sequences. Then for each $x \in E$, we have

$$S_{\mathcal{F}\mathcal{G}}(x) = \sum_{i \in I} \sum_{j \in J_i} \langle x, \Gamma_i^*(f_{ij}) \rangle \Lambda_i^*(f_{ij}) = \sum_{i \in I} \Lambda_i^* \Gamma_i x = S_{\Lambda\Gamma} x,$$

so $\|S_{\mathcal{F}\mathcal{G}} - Id_E\| = \|S_{\Lambda\Gamma} - Id_E\| < 1$, and the result follows. The converse is clear. □

Let \mathfrak{A} and \mathfrak{A}' be two C^* -algebras. Then $\mathfrak{A} \otimes \mathfrak{A}'$ is a C^* -algebra with the spatial norm and for each $a \in \mathfrak{A}$ and $a' \in \mathfrak{A}'$, we have $\|a \otimes a'\| = \|a\| \|a'\|$. The multiplication and involution on simple tensors are defined by $(a \otimes a')(b \otimes b') = ab \otimes a'b'$ and $(a \otimes a')^* = a^* \otimes a'^*$, respectively. As we know, if $a, a' \geq 0$, then $a \otimes a' \geq 0$.

Now let E be a Hilbert \mathfrak{A} -module and E' be a Hilbert \mathfrak{A}' -module. Then the (Hilbert C^* -module) tensor product $E \otimes E'$ is a Hilbert $\mathfrak{A} \otimes \mathfrak{A}'$ -module. The module action and inner product for simple tensors are defined by $(a \otimes a')(x \otimes x') = (ax) \otimes (a'x')$ and $\langle x \otimes x', y \otimes y' \rangle = \langle x, y \rangle \otimes \langle x', y' \rangle$, respectively. Let U and U' be adjointable operators on E and E' , respectively. Then the tensor product $U \otimes U'$ is an adjointable operator on $E \otimes E'$. Also, $(U \otimes U')^* = U^* \otimes U'^*$ and $\|U \otimes U'\| = \|U\| \|U'\|$. For more results about tensor products of C^* -algebras and Hilbert C^* -modules, see [23, 21].

Tensor products of frames and g-frames have been studied by some authors recently; see [15, 6, 16, 18].

It was proved in Proposition 3.2 in [19] that the direct sum of a countable number of g-duals (in Hilbert spaces) is a g-dual in the direct sum space but Example 2.9 in [20] shows that this is not necessarily true for approximate g-duals.

It was also shown in [20, Proposition 2.10] and [18, Corollary 3.8] (by using resolutions of the identity) that the tensor product of two g-duals (in Hilbert spaces) gives a g-dual in the tensor product space. In the following example, we show that the result does not necessarily hold for approximate g-duals:

Example 3.5 Let H be a separable Hilbert space (as a special case of a Hilbert C^* -module) and $\Lambda = \{\Lambda_i\}_{i \in I}$ be an A -tight g-frame with $\sqrt{2} < A < 2$. It is easy to see that Λ is an approximate g-dual of itself. Now the proof of Corollary 2.2 in [18] yields that $\Lambda \otimes \Lambda = \{\Lambda_i \otimes \Lambda_j\}_{i, j \in I}$ is an A^2 -tight g-frame, so $S_{(\Lambda \otimes \Lambda)(\Lambda \otimes \Lambda)} = S_{(\Lambda \otimes \Lambda)} = A^2 \cdot Id_{(H \otimes H)}$. Thus, $\|S_{(\Lambda \otimes \Lambda)(\Lambda \otimes \Lambda)} - Id_{(H \otimes H)}\| = A^2 - 1 > 1$. This means that $\Lambda \otimes \Lambda$ is not an approximate g-dual of itself.

Now we consider tensor products of g-duals and approximate g-duals in Hilbert C^* -modules. In the following proposition $\Lambda' = \{\Lambda'_j \in \mathfrak{L}(E', E'_j) : j \in J\}$, $\Gamma' = \{\Gamma'_j \in \mathfrak{L}(E', E'_j) : j \in J\}$, $\mathcal{F}' = \{f'_j\}_{j \in J}$ and $\mathcal{G}' = \{g'_j\}_{j \in J} \subseteq E'$, where E' and E'_j ’s are Hilbert \mathfrak{A}' -modules.

Proposition 3.6 (i) Let Γ be an approximate g-dual (resp. a g-dual) of Λ . If Γ' is a g-dual of Λ' , then $\Gamma \otimes \Gamma' = \{\Gamma_i \otimes \Gamma'_j\}_{i \in I, j \in J}$ is an approximate g-dual (resp. a g-dual) of $\Lambda \otimes \Lambda' = \{\Lambda_i \otimes \Lambda'_j\}_{i \in I, j \in J}$.

(ii) Let \mathcal{G} be an approximate dual (resp. a dual) of \mathcal{F} . If \mathcal{G}' is a dual of \mathcal{F}' , then $\mathcal{G} \otimes \mathcal{G}' = \{g_i \otimes g'_j\}_{i \in I, j \in J}$ is an approximate dual (resp. a dual) of $\mathcal{F} \otimes \mathcal{F}' = \{f_i \otimes f'_j\}_{i \in I, j \in J}$.

Proof (i) First suppose that Γ and Γ' are approximate g-dual and g-dual of Λ and Λ' , respectively. It follows from Theorem 2.2.5 in [23] that $0 \leq S_\Lambda \otimes S_{\Lambda'} \leq \|S_\Lambda \otimes S_{\Lambda'}\| \cdot Id_{E \otimes E'} \leq BB' \cdot Id_{E \otimes E'}$, where B and B' are upper bounds of Λ and Λ' , respectively. Hence, Lemma 4.1 in [21] implies that $0 \leq \langle (S_\Lambda \otimes S_{\Lambda'})z, z \rangle \leq BB' \langle z, z \rangle$, for each $z \in E \otimes E'$. Now it is easy to obtain that $\sum_{(i,j) \in I \times J} \langle (\Lambda_i \otimes \Lambda'_j)z, (\Lambda_i \otimes \Lambda'_j)z \rangle$ converges in norm and

$$\left\| \sum_{(i,j) \in I \times J} |(\Lambda_i \otimes \Lambda'_j)z|^2 \right\| = \left\| \langle (S_\Lambda \otimes S_{\Lambda'})z, z \rangle \right\| \leq BB' \|z\|^2,$$

so $\Lambda \otimes \Lambda'$ is a standard g-Bessel sequence by Theorem 3.1 in [28] (also, see [16, Section 5]). Similarly, we can get that $\Gamma \otimes \Gamma'$ is a standard g-Bessel sequence. It is also easy to see that

$$S_{(\Gamma \otimes \Gamma')(\Lambda \otimes \Lambda')}(x \otimes x') = (S_{\Gamma_\Lambda} \otimes S_{\Gamma'_{\Lambda'}})(x \otimes x') = (S_{\Gamma_\Lambda} \otimes Id_{E'})(x \otimes x'),$$

for each $x \otimes x' \in E \otimes E'$, and since the operators are bounded, we have $S_{(\Gamma \otimes \Gamma')(\Lambda \otimes \Lambda')} = S_{\Gamma_\Lambda} \otimes Id_{E'}$. Therefore

$$\|S_{(\Gamma \otimes \Gamma')(\Lambda \otimes \Lambda')} - Id_{(E \otimes E')}\| = \|(S_{\Gamma_\Lambda} - Id_E) \otimes Id_{E'}\| = \|S_{\Gamma_\Lambda} - Id_E\| < 1.$$

This means that $\Gamma \otimes \Gamma'$ is an approximate g-dual of $\Lambda \otimes \Lambda'$. It is clear that if Γ and Γ' are g-duals of Λ and Λ' , respectively, then $S_{(\Gamma \otimes \Gamma')(\Lambda \otimes \Lambda')} = Id_{(E \otimes E')}$, so $\Gamma \otimes \Gamma'$ is a g-dual of $\Lambda \otimes \Lambda'$.

(ii) We can get the result by using Example 3.2 in [16] and part (i). □

Note that Proposition 2.10, Corollary 2.11 in [20], and part (ii) of Corollary 3.8 in [18] are special cases of the above proposition.

Now we show that approximate duals in Hilbert C^* -modules are stable under small perturbations. The following result is analogous to part (i) of Theorem 3.1 in [20] that we need in the next section.

Proposition 3.7 *Let Λ be a g-Bessel sequence and $\Psi = \{\psi_i\}_{i \in I}$ be an approximate g-dual (resp. a g-dual) of Λ with upper bound C . If Γ is a sequence such that $\Gamma - \Lambda$ is a g-Bessel sequence with upper bound K and $CK < (1 - \|Id_E - S_{\Psi_\Lambda}\|)^2$ (resp. $CK < 1$), then Γ and Ψ are approximately dual g-frames.*

Proof Let Ω be a finite subset of I and B be an upper bound for Λ . Then

$$\left\| \sum_{i \in \Omega} \langle \Gamma_i x, \Gamma_i x \rangle \right\|^{\frac{1}{2}} \leq \|\{\Lambda_i x\}_{i \in \Omega}\|_2 + \|\{\Gamma_i x - \Lambda_i x\}_{i \in \Omega}\|_2 \leq (\sqrt{B} + \sqrt{K})\|x\|,$$

for each $x \in E$. Thus, by Theorem 3.1 in [28], Γ is a standard g-Bessel sequence. Now by using the Cauchy-Schwarz inequality in Hilbert C^* -modules, for each $x \in E$, we have

$$\begin{aligned} \|(Id_E - S_{\Psi_\Gamma})x\| &\leq \|(Id_E - S_{\Psi_\Lambda})x\| + \|(S_{\Psi_\Lambda} - S_{\Psi_\Gamma})x\| \\ &\leq \|(Id_E - S_{\Psi_\Lambda})x\| + \sup_{\|y\|=1} \left\{ \left\| \sum_{i \in I} |(\Lambda_i - \Gamma_i)x|^2 \right\|^{\frac{1}{2}} \left\| \sum_{i \in I} |\psi_i y|^2 \right\|^{\frac{1}{2}} \right\} \\ &\leq \|(Id_E - S_{\Psi_\Lambda})x\| + \sqrt{CK}\|x\| \leq (\|Id_E - S_{\Psi_\Lambda}\| + \sqrt{CK})\|x\|. \end{aligned}$$

Hence, $\|Id_E - S_{\Psi_\Gamma}\| \leq \|Id_E - S_{\Psi_\Lambda}\| + \sqrt{CK} < 1$. Also, if Λ and Ψ are g-duals, then $S_{\Psi_\Lambda} = Id_E$ and we have $\|Id_E - S_{\Psi_\Gamma}\| \leq \sqrt{CK} < 1$. □

The following result is a generalization of Proposition 3.10 in [20] to Hilbert C^* -modules.

Proposition 3.8 *Let $0 \leq \lambda_1, \lambda_2 < 1$, $A, B, \varepsilon > 0$, and $K = \lambda_1 + \frac{\varepsilon}{\sqrt{A}} + \frac{\lambda_2[(1+\lambda_1)\sqrt{A}+\varepsilon]}{\sqrt{A}(1-\lambda_2)}$.*

(i) *If Λ is an (A, B) g -frame and Γ is a sequence satisfying*

$$\left\| \sum_{i \in \Omega} (\Lambda_i^* - \Gamma_i^*) f_i \right\| \leq \lambda_1 \left\| \sum_{i \in \Omega} \Lambda_i^* f_i \right\| + \lambda_2 \left\| \sum_{i \in \Omega} \Gamma_i^* f_i \right\| + \varepsilon \left\| \sum_{i \in \Omega} |f_i|^2 \right\|^{\frac{1}{2}}, \tag{2}$$

for each finite subset $\Omega \subseteq I$, $f_i \in E_i$ with $K < 1$, then $\tilde{\Lambda}$ is an approximate g -dual of Γ and Γ is a g -frame.

(ii) *If $\mathcal{F} = \{f_i\}_{i \in I}$ is an (A, B) frame and $\mathcal{G} = \{g_i\}_{i \in I}$ is a sequence satisfying*

$$\left\| \sum_{i \in \Omega} a_i f_i - \sum_{i \in \Omega} a_i g_i \right\| \leq \lambda_1 \left\| \sum_{i \in \Omega} a_i f_i \right\| + \lambda_2 \left\| \sum_{i \in \Omega} a_i g_i \right\| + \varepsilon \left\| \sum_{i \in \Omega} |a_i|^2 \right\|^{\frac{1}{2}},$$

for each finite subset $\Omega \subseteq I$, $\{a_i\}_{i \in \Omega} \subseteq \mathfrak{A}$ with $K < 1$, then $\tilde{\mathcal{F}}$ is an approximate dual of \mathcal{G} and \mathcal{G} is a frame.

Proof (i) Suppose that $\{e_{i,k}\}_{k \in J_i}$ is a Parseval frame for E_i and $\{c_{i,k}\}_{i \in \Omega, k \in \Omega_i}$ is a finite subset of \mathfrak{A} , where Ω and Ω_i s are finite index sets. Since Λ is an (A, B) standard g -frame, Corollary 3.4 in [16] yields that $\{u_{i,k} = \Lambda_i^*(e_{i,k}) : i \in I, k \in J_i\}$ is an (A, B) standard frame. Now for $v_{i,k} = \Gamma_i^*(e_{i,k})$, we have

$$\begin{aligned} \left\| \sum_{i \in \Omega} \sum_{k \in \Omega_i} c_{i,k} (u_{i,k} - v_{i,k}) \right\| &= \left\| \sum_{i \in \Omega} (\Lambda_i^* - \Gamma_i^*) \left(\sum_{k \in \Omega_i} c_{i,k} e_{i,k} \right) \right\| \\ &\leq \lambda_1 \left\| \sum_{i \in \Omega} \sum_{k \in \Omega_i} c_{i,k} u_{i,k} \right\| + \lambda_2 \left\| \sum_{i \in \Omega} \sum_{k \in \Omega_i} c_{i,k} v_{i,k} \right\| \\ &\quad + \varepsilon \left\| \sum_{i \in \Omega} \sum_{k \in \Omega_i} |c_{i,k}|^2 \right\|^{\frac{1}{2}}. \end{aligned}$$

Hence, Theorem 3.2 in [14] implies that $\{v_{i,k} = \Gamma_i^*(e_{i,k}) : i \in I, k \in J_i\}$ is a standard Bessel sequence with upper bound $\frac{[(1+\lambda_1)\sqrt{B}+\varepsilon]^2}{(1-\lambda_2)^2}$, so (by [16, Theorem 3.3]) Γ is a standard g -Bessel sequence with upper bound $\frac{[(1+\lambda_1)\sqrt{B}+\varepsilon]^2}{(1-\lambda_2)^2}$. Thus, for each $\{f_i\}_{i \in I} \in \oplus_{i \in I} E_i$, the series $\sum_{i \in I} \Gamma_i^* f_i$ converges in E and from (2), we can get

$$\left\| \sum_{i \in I} (\Lambda_i^* - \Gamma_i^*) f_i \right\| \leq \lambda_1 \left\| \sum_{i \in I} \Lambda_i^* f_i \right\| + \lambda_2 \left\| \sum_{i \in I} \Gamma_i^* f_i \right\| + \varepsilon \left\| \sum_{i \in I} |f_i|^2 \right\|^{\frac{1}{2}}. \tag{3}$$

Since for each $x \in E$, $\{\tilde{\Lambda}_i x\}_{i \in I} \in \oplus_{i \in I} E_i$ and $\frac{1}{A}$ is an upper bound for $\tilde{\Lambda}$, by using (3), we have

$$\|S_{\Gamma \tilde{\Lambda}} x\| - \|x\| \leq \|S_{\Gamma \tilde{\Lambda}} x - x\| \leq \left(\lambda_1 + \frac{\varepsilon}{\sqrt{A}}\right) \|x\| + \lambda_2 \|S_{\Gamma \tilde{\Lambda}} x\|.$$

Now we can obtain a result similar to the proof of Proposition 3.10 in [20].

(ii) Since $\Lambda_{f_i}^*(a) = af_i$ and $\Lambda_{g_i}^*(a) = ag_i$, for each $a \in \mathfrak{A}$, the result follows from part (i) for $\Lambda_i = \Lambda_{f_i}$ and $\Gamma_i = \Lambda_{g_i}$. \square

4. Approximate duals and ε -nearly Parseval frames

In this section, we consider ε -nearly Parseval frames in Hilbert spaces, which are useful in applications (see [5]). We obtain some results for approximate duals of ε -nearly Parseval and ε -close frames (since Hilbert spaces are special cases of Hilbert C^* -modules, we do not state the definitions of frames, g-frames, and approximate duals in Hilbert spaces separately).

ε -nearly Parseval frames were defined in [5] and we have the following definition:

Definition 4.1 *Suppose that H is a separable Hilbert space and $\{H_i\}_{i \in I}$ is a sequence of separable Hilbert spaces.*

(i) *Let Λ_i be a bounded operator from H into H_i and $\varepsilon < 1$. We say that $\Lambda = \{\Lambda_i\}_{i \in I}$ is an ε -nearly Parseval g-frame if for each $f \in H$*

$$(1 - \varepsilon)\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq (1 + \varepsilon)\|f\|^2.$$

(ii) *Let $\{f_i\}_{i \in I}$ be a sequence in H and $\varepsilon < 1$. We say that $\{f_i\}_{i \in I}$ is an ε -nearly Parseval frame if for each $f \in H$*

$$(1 - \varepsilon)\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq (1 + \varepsilon)\|f\|^2.$$

It is clear that if $\varepsilon = 0$, then an ε -nearly Parseval g-frame is a Parseval g-frame and so it is a g-dual of itself. Now we have the following result for approximate duals:

Theorem 4.2 (i) *If Λ is an ε -nearly Parseval g-frame, then it is an approximate g-dual of itself.*

(ii) *If $\{f_i\}_{i \in I}$ is an ε -nearly Parseval frame, then it is an approximate dual of itself.*

Proof (i) Since Λ is an ε -nearly Parseval g-frame, we have $(1 - \varepsilon).Id_H \leq S_\Lambda \leq (1 + \varepsilon).Id_H$, so $-\varepsilon.Id_H \leq (S_\Lambda - Id_H) \leq \varepsilon.Id_H$. Because $S_{\Lambda\Lambda} = S_\Lambda$, we obtain that $\|S_{\Lambda\Lambda} - Id_H\| \leq \varepsilon < 1$. This means that Λ is an approximate g-dual of itself.

(ii) Since frames are special cases of g-frames, we get the result from part (i). \square

Note that if $\{\Lambda_i\}_{i \in I}$ is a g-Bessel sequence and $J \subset I$, then we denote the optimal upper bound of $\{\Lambda_i\}_{i \in J^c}$ by $B(J^c)$.

It was shown in Theorem 3.1 in [20] that if $\{\Lambda_i\}_{i \in I}$ is a Parseval g-frame and $B(J^c) < 1$, then $\{\Lambda_i\}_{i \in J}$ is an approximate g-dual of itself. Now we have the following result for ε -nearly Parseval g-frames:

Proposition 4.3 (i) *Let Λ be an ε -nearly Parseval g-frame and $J \subset I$ such that $B(J^c) < 1 - \varepsilon$. Then $\{\Lambda_i\}_{i \in J}$ is an approximate g-dual of itself.*

(ii) Let $\{f_i\}_{i \in I}$ be an ε -nearly Parseval frame and $J \subset I$ such that $B(J^c) < 1 - \varepsilon$. Then $\{f_i\}_{i \in J}$ is an approximate dual of itself.

Proof (i) We have

$$\begin{aligned} \sum_{i \in J} \|\Lambda_i f\|^2 &= \sum_{i \in I} \|\Lambda_i f\|^2 - \sum_{i \in J^c} \|\Lambda_i f\|^2 \\ &\geq \sum_{i \in I} \|\Lambda_i f\|^2 - B(J^c) \|f\|^2 \geq \left((1 - \varepsilon) - B(J^c) \right) \|f\|^2. \end{aligned}$$

Hence, $\left(1 - (\varepsilon + B(J^c))\right)$ is a lower bound for $\{\Lambda_i\}_{i \in J}$. Therefore, $\{\Lambda_i\}_{i \in J}$ is an ε' -nearly Parseval g-frame, where $\varepsilon' = (\varepsilon + B(J^c))$. Now by Theorem 4.2, $\{\Lambda_i\}_{i \in J}$ is an approximate g-dual of itself.

(ii) The result follows from part (i). □

Recall that two sequences $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ in H are ε -close if $\sum_{i \in I} \|f_i - g_i\|^2 \leq \varepsilon^2$ (see [5, Definition 2.4]).

Example 4.4 Let $H = \mathbb{C}^2$, $\{e_1, e_2\}$ be the standard orthonormal basis for H and $\frac{2}{3} < \varepsilon < 1$. For $\mathcal{F} = \{\sqrt{\frac{\varepsilon}{2}}e_1, e_2\}$ and $\mathcal{G} = \{0, e_2\}$, it is easy to see that \mathcal{F} is an ε -nearly Parseval frame that is ε -close to \mathcal{G} , but \mathcal{F} and \mathcal{G} are not approximately dual frames because \mathcal{G} is not a frame.

Now we have the following result:

Proposition 4.5 (i) Let $\mathcal{F} = \{f_i\}_{i \in I}$ be an ε -nearly Parseval frame with upper bound A . If $\{g_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$ are ε -close with $\sqrt{A}\varepsilon < 1 - \|Id_H - S_{\mathcal{F}}\|$, then \mathcal{F} is an approximate dual of $\{g_i\}_{i \in I}$.

(ii) If \mathcal{F} in part (i) is also an A -tight frame with $\sqrt{A}\varepsilon < 1 - |1 - A|$, then \mathcal{F} is an approximate dual of $\{g_i\}_{i \in I}$.

Proof (i) Since \mathcal{F} is an ε -nearly Parseval frame, it is an approximate dual of itself by Theorem 4.2. Also, for each $f \in H$, we have

$$\sum_{i \in I} |\langle f, f_i - g_i \rangle|^2 \leq \|f\|^2 \sum_{i \in I} \|f_i - g_i\|^2 \leq \varepsilon^2 \|f\|^2.$$

By considering $C = A$ and $K = \varepsilon^2$, we obtain that $\{f_i - g_i\}_{i \in I}$ is a Bessel sequence with upper bound K and $\sqrt{CK} = \varepsilon\sqrt{A} < 1 - \|Id_H - S_{\mathcal{F}}\|$. Now the result follows from Proposition 3.7 (or [20, Theorem 3.1]).

(ii) The result follows from part (i) by considering $S_{\mathcal{F}} = A.Id_H$. □

Remark 4.6 We can obtain from part (ii) of the above proposition that if $\mathcal{F} = \{f_i\}_{i \in I}$ is an A -tight frame with $A \leq 1$, then each sequence $\{g_i\}_{i \in I}$ that is ε -close to \mathcal{F} with $\varepsilon < \sqrt{A}$ is an approximate dual of \mathcal{F} . Especially if \mathcal{F} is a Parseval frame that is ε -close to $\mathcal{G} = \{g_i\}_{i \in I}$, then \mathcal{F} and \mathcal{G} are approximately dual frames.

Let $\mathcal{F} = \{f_i\}_{i=1}^n$ be an ε -nearly Parseval frame for a d -dimensional Hilbert space H . As we know (see [7, Theorem 5.3.4] and [3]), $\{S_{\mathcal{F}}^{-\frac{1}{2}} f_i\}_{i=1}^n$ is the closest Parseval frame to \mathcal{F} . It was proved in [5, Proposition 3.1 and

Remark 3.2] that the relation $\sum_{i=1}^n \|S_{\mathcal{F}}^{-1} f_i - f_i\|^2 = d(2 - \varepsilon - 2\sqrt{1 - \varepsilon}) \leq \frac{d\varepsilon^2}{4}$ holds if $\{\frac{1}{\sqrt{1-\varepsilon}} f_i\}_{i=1}^n$ is a Parseval frame (or equivalently $S_{\mathcal{F}}^{-1} = \frac{1}{\sqrt{1-\varepsilon}} Id_H$, so we have $\sum_{i=1}^n \|\frac{1}{\sqrt{1-\varepsilon}} f_i - f_i\|^2 = d(2 - \varepsilon - 2\sqrt{1 - \varepsilon}) \leq \frac{d\varepsilon^2}{4}$). It was also shown in Example 2.4 in [24] that if $\mathcal{F} = \{f_i\}_{i=1}^n$ is an ε -nearly Parseval frame for a finite-dimensional Hilbert space H , then $\{\frac{1}{2}(f_i + S_{\mathcal{F}}^{-1} f_i)\}_{i=1}^n$ is a $(1, 1 + \frac{\varepsilon}{4})$ frame, which is much closer to \mathcal{F} than $\{S_{\mathcal{F}}^{-1} f_i\}_{i=1}^n$ with better frame bounds compared with the bounds of \mathcal{F} . Therefore, the frames of the forms $\{\frac{1}{\sqrt{1-\varepsilon}} f_i\}_{i \in I}$ and $\{\frac{1}{2}(f_i + S_{\mathcal{F}}^{-1} f_i)\}_{i \in I}$ are useful in applications (also, see [1, Section 3]). Now we have the following results:

Proposition 4.7 (i) Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be an ε -nearly Parseval g -frame with $\varepsilon < \frac{1}{3}$. Then $\{\frac{1}{\sqrt{1-\varepsilon}} \Lambda_i\}_{i \in I}$ is an approximate g -dual of itself.

(ii) Let $\{f_i\}_{i \in I}$ be an ε -nearly Parseval frame with $\varepsilon < \frac{1}{3}$. Then $\{\frac{1}{\sqrt{1-\varepsilon}} f_i\}_{i \in I}$ is an approximate dual of itself.

Proof (i) We have $(1 - \varepsilon) Id_H \leq S_{\Lambda} \leq (1 + \varepsilon) Id_H$, so $0 \leq \frac{S_{\Lambda}}{1-\varepsilon} - Id_H \leq \frac{2\varepsilon}{1-\varepsilon} Id_H$. Therefore, $\|\frac{S_{\Lambda}}{1-\varepsilon} - Id_H\| \leq \frac{2\varepsilon}{1-\varepsilon} < 1$. This means that $\{\frac{1}{\sqrt{1-\varepsilon}} \Lambda_i\}_{i \in I}$ and $\{\frac{1}{\sqrt{1-\varepsilon}} \Lambda_i\}_{i \in I}$ are approximately dual g -frames.

(ii) The result follows from part (i). □

Proposition 4.8 Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame and $\mathcal{G} = \{g_i\}_{i \in I}$, where $g_i = \frac{1}{2}(f_i + S_{\mathcal{F}}^{-1} f_i)$.

(i) If $\|S_{\mathcal{F}} - Id_H\| < 2$, then \mathcal{F} and \mathcal{G} are approximately dual frames.

(ii) If \mathcal{F} is an ε -nearly Parseval frame, then \mathcal{F} and \mathcal{G} are approximately dual frames.

(iii) If $\mathcal{F} = \{f_i\}_{i \in I}$ is an A -tight frame with $A < 3$, then \mathcal{F} and \mathcal{G} are approximately dual frames.

Proof (i) For each $f \in H$, we have

$$S_{\mathcal{F}\mathcal{G}}(f) = \frac{1}{2} \left(\sum_{i \in I} \langle f, f_i \rangle f_i + \sum_{i \in I} \langle f, S_{\mathcal{F}}^{-1} f_i \rangle f_i \right) = \frac{1}{2} (S_{\mathcal{F}} f + f).$$

Hence:

$$\|S_{\mathcal{F}\mathcal{G}} - Id_H\| = \frac{1}{2} \|S_{\mathcal{F}} - Id_H\| < 1.$$

This means that \mathcal{F} and \mathcal{G} are approximately dual frames.

(ii) Since \mathcal{F} is an ε -nearly Parseval frame, \mathcal{F} is an approximate dual of itself by Theorem 4.2, and so $\|S_{\mathcal{F}} - Id_H\| < 1$. Now we get the result from part (i).

(iii) Since \mathcal{F} is A -tight, we have $S_{\mathcal{F}} = A Id_H$, so $\|S_{\mathcal{F}} - Id_H\| = |A - 1| < 2$, and the result follows from part (i). □

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