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HAKAN METE TAŞTAN

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Warped product skew semi-invariant submanifolds of order 1 of a locally product Riemannian manifold

Hakan Mete TAŞTAN*
Department of Mathematics, İstanbul University, İstanbul, Turkey

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Abstract: We introduce warped product skew semi-invariant submanifolds of order 1 of a locally product Riemannian manifold. We give a necessary and sufficient condition for a skew semi-invariant submanifold of order 1 to be a locally warped product. We also establish an inequality between the warping function and the squared norm of the second fundamental form for such submanifolds. The equality case is also discussed.

Key words: Locally product manifold, warped product submanifold, skew semi-invariant submanifold, invariant distribution, slant distribution

1. Introduction

The theory of submanifolds is a popular research area in differential geometry. In an almost Hermitian manifold, its almost complex structure determines several types of submanifolds. For example, holomorphic (invariant) submanifolds and totally real (anti-invariant) submanifolds are determined by the behavior of the almost complex structure. In the first case, the tangent space of the submanifolds is invariant under the action of the almost complex structure. In the second case, the tangent space of the submanifolds is anti-invariant, that is, it is mapped into the normal space. Bejancu [5] introduced the notion of CR-submanifolds of a Kählerian manifold as a natural generalization of invariant and anti-invariant submanifolds. A CR-submanifold is said to be proper if it is neither invariant nor anti-invariant. The theory of CR-submanifolds has been an interesting topic since then. Slant submanifolds are another generalization of invariant and anti-invariant submanifolds. These types of submanifolds were defined by Chen [10]. Subsequently, such submanifolds have been studied by many geometers (see [3, 8, 9, 18] and references therein). If a slant submanifold is neither invariant nor anti-invariant, then it is said to be proper. We observe that a proper CR-submanifold is never a slant submanifold. In [19], Papaghiuc introduced the notion of semi-slant submanifolds obtaining CR-submanifolds and slant submanifolds as special cases. Carriazo [9] introduced bi-slant submanifolds, which are a generalization of semi-slant submanifolds. One of the classes of such submanifolds is that of anti-slant submanifolds. These types of submanifolds are also generalizations of slant and CR-submanifolds. However, Şahin [24] called these submanifolds hemi-slant submanifolds because the name anti-slant implies that it has no slant factor. He also observed that there is no inclusion between proper hemi-slant submanifolds and proper semi-slant submanifolds. We note that hemi-slant submanifolds are also studied under the name of pseudo-slant submanifolds (see [15, 28]).

*Correspondence: hakmete@istanbul.edu.tr
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Skew CR-submanifolds of a Kählerian manifold were first defined by Ronsse in [20]. Such submanifolds are generalizations of bi-slant submanifolds. Consequently, invariant, anti-invariant, CR, slant, semi-slant, and hemi-slant submanifolds are particular cases of skew CR-submanifolds. We notice that CR-submanifolds in Kählerian manifolds correspond to semi-invariant submanifolds [6] in locally product Riemannian manifolds. Therefore, skew CR-submanifolds in Kählerian manifolds correspond to skew semi-invariant submanifolds in locally product Riemannian manifolds. The fundamental properties and further studies of skew CR-submanifolds are discussed in [20, 27]. Skew semi-invariant submanifolds of a locally product Riemannian manifold were firstly studied by Liu and Shao in [17].

The notion of warped product was initiated by Bishop and O'Neill [7]. Let M_1 and M_2 be two Riemanian manifolds with Riemannian metrics g_1 and g_2, respectively. Let f be a positive differentiable function on M_1. The warped product $M = M_1 \times_f M_2$ of M_1 and M_2 is the Riemannian manifold $(M_1 \times M_2, g)$, where

$$g = g_1 + f^2 g_2 \ .$$

More explicitly, if $U \in T_pM$, then

$$\|U\|^2 = \|d\pi_1(U)\|^2 + (f^2 \circ \pi_1)\|d\pi_2(U)\|^2 \ ,$$

where $\pi_i, i = 1, 2,$ are the canonical projections $M_1 \times M_2$ onto M_1 and M_2, respectively. The function f is called the *warping function* of the warped product. If the warping function is constant, then the manifold M is said to be *trivial*. It is well known that M_1 is totally geodesic and M_2 is totally umbilical from [7]. For a warped product $M_1 \times_f M_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors that are tangent to leaves and fibers, respectively. Thus, $\mathcal{D}_1$ is obtained from tangent vectors to M_1 via horizontal lift and $\mathcal{D}_2$ is obtained by tangent vectors of M_2 via vertical lift. Let U be a vector field on M_1 and V be vector field on M_2; then from Lemma 7.3 of [7], we have

$$\nabla_U V = \nabla_V U = U(\ln f)V \ , \tag{1.1}$$

where ∇ is the Levi-Civita connection on $M_1 \times_f M_2$.

Warped product submanifolds have been studied very actively, since Chen [11] introduced the notion of CR-warped product in Kählerian manifolds. In fact, different types of warped product submanifolds of several kinds of structures have been studied in the last fourteen years. (see [2, 16, 22, 23, 24, 25, 28]). Most of the studies related to this topic can be found in the survey book [12]. Recently, Şahin [25] introduced the notion of skew CR-warped product submanifolds of Kählerian manifolds, which are generalizations of different kinds of warped product submanifolds studied by many authors. We note that the warped product skew CR-submanifolds of a cosymplectic manifold were studied in [16].

In this paper, we define and study warped product skew semi-invariant submanifolds of order 1 of a locally product Riemannian manifold. We give an example and prove a characterization theorem for the mixed totally geodesic proper skew semi-invariant submanifold using some lemmas. We also obtain an inequality between the warping function and the squared norm of the second fundamental form for such submanifolds. The equality case is also considered.

2. Preliminaries

Let $(\bar{M}, g, F)$ be a locally product Riemannian manifold (briefly, l.p.R. manifold). It means that [29] $\bar{M}$ has a tensor field F of type $(1,1)$ on $\bar{M}$ such that $\forall \bar{U}, \bar{V} \in T\bar{M}$; we have

$$F^2 = I, (F \neq \pm I), \quad g(F\bar{U}, F\bar{V}) = g(\bar{U}, \bar{V}) \quad , and \quad (\bar{\nabla}_{\bar{U}} F)\bar{V} = 0 \ , \tag{2.1}$$

where g is the Riemannian metric, $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$, and I is the identifying operator on the tangent bundle $T\bar{M}$ of $\bar{M}$.

Let M be an isometrically immersed submanifold of a l.p.R. manifold $(\bar{M}, g, F)$. Let ∇ and $\nabla^{\perp}$ be the induced and induced normal connection in M and the normal bundle $T^{\perp}M$ of M, respectively. Then for all $U, V \in TM$ and $\xi \in T^{\perp}M$ the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \tag{2.2}$$

and

$$\bar{\nabla}_U \xi = -A_{\xi} U + \nabla^{\perp}_U \xi \tag{2.3}$$

where h is the *second fundamental form* of M and A_{ξ} is the Weingarten endomorphism associated with ξ. The second fundamental form h and the *shape operator* A are related by

$$g(h(U, V), \xi) = g(A_{\xi} U, V) \ . \tag{2.4}$$

The *mean curvature vector field* H is given by $H = \frac{1}{m}(trace\, h)$, where $dim(M) = m$. The submanifold M is called *totally geodesic* in $\bar{M}$ if $h = 0$, and *minimal* if $H = 0$. If $h(U, V) = g(U, V)H$ for all $U, V \in TM$, then M is *totally umbilical*.

3. Skew semi-invariant submanifolds of order 1 of a locally product Riemannian manifold

Let $\bar{M}$ be a l.p.R. manifold with a Riemannian metric g and almost product structure F. Let M be a Riemannian submanifold isometrically immersed in $\bar{M}$. For any $U \in TM$, we write

$$FU = TU + NU \ . \tag{3.1}$$

Here TU is the tangential part of FU and NU is the normal part of FU. Similarly, for any $\xi \in T^{\perp}M$, we put

$$F\xi = t\xi + \omega\xi \ , \tag{3.2}$$

where $t\xi$ is the tangential part of $F\xi$ and $\omega\xi$ is the normal part of $F\xi$. Then, using (2.1), (3.1), and (3.2), we have

$$\begin{aligned} &(a) \quad T^2 + tN = I, \qquad (b) \quad \omega^2 + Nt = I, \\ &(c) \quad NT + \omega N = 0, \qquad (d) \quad Tt + t\omega = 0. \end{aligned} \tag{3.3}$$

Using (2.1) and (3.1), we have $g(T^2U, V) = g(T^2V, U)$ for all $U, V \in TM$. It means that T^2 is a symmetric operator on the tangent space $T_pM, p \in M$. Therefore, its eigenvalues are real and diagonalizable. Moreover, its eigenvalues are bounded by 0 and 1. For each $p \in M$, we set

$$\mathcal{D}^{\lambda}_p = Ker\{T^2 - \lambda^2(p)I\}_p \, ,$$

where I is the identity endomorphism and $\lambda(p)$ belongs to closed interval $[0,1]$ such that $\lambda^2(p)$ is an eigenvalue of T_p^2. Since T_p^2 is symmetric and diagonalizable, there is some integer k such that $\lambda_1^2(p),...,\lambda_k^2(p)$ are distinct eigenvalues of T_p^2, and T_pM can be decomposed as a direct sum of mutually orthogonal eigenspaces, i.e.

$$T_pM = \mathcal{D}_p^{\lambda_1} \oplus ... \oplus \mathcal{D}_p^{\lambda_k}.$$

For $i \in \{1,...,k\}$, $\mathcal{D}_p^{\lambda_i}$ is a T-invariant subspace of T_pM. We note that $\mathcal{D}_p^0 = KerT_p$ and $\mathcal{D}_p^1 = KerN_p$. $\mathcal{D}_p^0$ is the maximal anti F-invariant subspace of T_pM, whereas $\mathcal{D}_p^1$ is the maximal F-invariant subspace of T_pM. From now on, we denote the distributions $\mathcal{D}^0$ and $\mathcal{D}^1$ by $\mathcal{D}^\perp$ and $\mathcal{D}^T$, respectively.

Definition 3.1 *([17]) Let M be a submanifold of a l.p.R. manifold $\bar{M}$. Then M is said to be a* generic submanifold *if there exists an integer k and functions $\lambda_i, i \in \{1,...,k\}$ defined on M with values in $(0,1)$ such that*

(i) *Each $\lambda_i^2(p), i \in \{1,...,k\}$ is a distinct eigenvalue of T_p^2 with*

$$T_pM = \mathcal{D}_p^\perp \oplus \mathcal{D}_p^T \oplus \mathcal{D}_p^{\lambda_1} \oplus ... \oplus \mathcal{D}_p^{\lambda_k}$$

for $p \in M$.

(ii) *The dimensions of $\mathcal{D}^\perp$, $\mathcal{D}^T$, and $\mathcal{D}^{\lambda_i}, for 1 \leq i \leq k$ are independent of $p \in M$.*
Moreover, if each λ_i is constant on M, then we say that M is a skew semi-invariant submanifold *of $\bar{M}$.*

Let M be a skew semi-invariant submanifold of a l.p.R. manifold $\bar{M}$ as in definition 3.1. Then we observe the following special cases:

(a) If $k=0$ and $\mathcal{D}^\perp = \{0\}$, then M is an invariant submanifold [1].

(b) If $k=0$ and $\mathcal{D}^T = \{0\}$, then M is an anti-invariant submanifold [1].

(c) If $k=0$, then M is a semi-invariant submanifold [6].

(d) If $\mathcal{D}^\perp = \{0\} = \mathcal{D}^T$ and $k=1$, then M is a slant submanifold [21].

(e) If $\mathcal{D}^\perp = \{0\}, \mathcal{D}^T \neq \{0\}$ and $k=1$, then M is a semi-slant submanifold [21].

(f) If $\mathcal{D}^T = \{0\}, \mathcal{D}^\perp \neq \{0\}$ and $k=1$, then M is a hemi-slant submanifold [26].

(g) If $\mathcal{D}^\perp = \{0\} = \mathcal{D}^T$ and $k=2$, then M is a bi-slant submanifold [9].

Definition 3.2 *A submanifold M of a l.p.R. manifold $\bar{M}$ is called a* skew semi-invariant submanifold of order 1*, if M is a skew semi-invariant submanifold with $k=1$.*

In this case, we have

$$TM = \mathcal{D}^\perp \oplus \mathcal{D}^T \oplus \mathcal{D}^\theta \quad , \tag{3.4}$$

where $\mathcal{D}^\theta = \mathcal{D}^{\lambda_1}$ and λ_1 is constant. We say that a skew semi-invariant submanifold of order 1 is *proper*, if $\mathcal{D}^\perp \neq \{0\}$ and $\mathcal{D}^T \neq \{0\}$.

A slant submanifold M of a l.p.R. manifold $\bar{M}$ is characterized by

$$T^2U = \lambda U \tag{3.5}$$

such that $\lambda \in [0,1]$, where $U \in TM$. Details can be found in [21]. Moreover, if θ is the slant angle of M, then we have $\lambda = \cos^2\theta$.

Throughout this paper, the letters V and W will denote the vector fields of the anti-invariant distribution $\mathcal{D}^\perp$, U and Z will denote the vector fields of the slant distribution $\mathcal{D}^\theta$, and X and Y will denote the vector fields of the invariant distribution $\mathcal{D}^T$.

For the further study of skew semi-invariant submanifold of order 1 of a l.p.R. manifold, we need the following lemmas.

Lemma 3.3 *Let M be a proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $\bar{M}$. Then we have*

$$g(\nabla_V W, X) = -g(A_{FW}V, FX) \ , \tag{3.6}$$

$$g(\nabla_V Z, X) = -\csc^2\theta\{g(A_{NTZ}V, X) + g(A_{NZ}V, FX)\} \ , \tag{3.7}$$

$$g(\nabla_Z V, X) = -g(A_{FV}Z, FX) \ , \tag{3.8}$$

for $V, W \in \mathcal{D}^\perp, Z \in \mathcal{D}^\theta$, and $X \in \mathcal{D}^T$.

Proof Using (2.2) and (2.1), we have $g(\nabla_V W, X) = g(\bar{\nabla}_V FW, FX)$ for $V, W \in \mathcal{D}^\perp$ and $X \in \mathcal{D}^T$. Hence, using (2.3), we get (3.6). In a similar way, we have $g(\nabla_V Z, X) = g(\bar{\nabla}_V FZ, FX)$, where $Z \in \mathcal{D}^\theta$. Then, using (3.1) and (2.1), we obtain

$$g(\nabla_V Z, X) = g(\bar{\nabla}_V FTZ, X) + g(\bar{\nabla}_V NZ, FX) \ .$$

Hence, using (3.1) and (2.3), we arrive at

$$g(\nabla_V Z, X) = g(\bar{\nabla}_V T^2 Z, X) + g(\bar{\nabla}_V N(TZ), X) - g(A_{NZ}V, FX) \ .$$

With the help of (3.5), (2.2), and (2.3), we get (3.7). Similarly, one can obtain (3.8). □

Lemma 3.4 *Let M be a proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $\bar{M}$. Then we have*

$$g(\nabla_U Z, X) = -\csc^2\theta\{g(A_{NTZ}U, X) + g(A_{NZ}U, FX)\} \ , \tag{3.9}$$

$$g(\nabla_X Y, Z) = \csc^2\theta\{g(A_{NTZ}X, Y) + g(A_{NZ}X, FY)\} \ , \tag{3.10}$$

$$g(\nabla_X Y, V) = g(A_{FV}X, FY) \ , \tag{3.11}$$

for $X, Y \in \mathcal{D}^T$, $U, Z \in \mathcal{D}^\theta$, and $V \in \mathcal{D}^\perp$.

Proof Let $U, Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$. Then, using (2.2), (2.1), and (3.1), we have

$$g(\nabla_U Z, X) = g(\bar{\nabla}_U FZ, FX) = g(\bar{\nabla}_U TZ, FX) + g(\bar{\nabla}_U NZ, FX) \ .$$

Again, using (2.1) and (2.3), we obtain

$$g(\nabla_U Z, X) = g(\bar{\nabla}_U FTZ, X) - g(A_{NZ}U, FX) \ .$$

Here, if we use (3.3)-(a) and (3.5), then we get

$$g(\nabla_U Z, X) = \cos^2\theta g(\nabla_U Z, X) + g(\bar{\nabla}_U NTZ, X) - g(A_{NZ}U, FX) \ .$$

After some calculation, we find (3.9). For the proof of (3.10), using (2.2), (2.1), and (3.1), we have

$$g(\nabla_X Y, Z) = g(\bar{\nabla}_X FY, FZ) = g(\bar{\nabla}_X FY, TZ) + g(\bar{\nabla}_X FY, NZ)$$

for $X, Y \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$. Again, using (2.1) and (2.3), we obtain

$$g(\nabla_X Y, Z) = g(\bar{\nabla}_X Y, FTZ) + g(h(X, FY), NZ) \ .$$

With the help of (3.3)-(a) and (3.5), we get

$$g(\nabla_X Y, Z) = \cos^2\theta g(\nabla_X Y, Z) + g(\bar{\nabla}_X Y, NTZ) + g(h(X, FY), NZ) \ .$$

By direct calculation, we find (3.10). In a similar way, we can obtain (3.11). □

Lemma 3.5 *Let M be a proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $\bar{M}$. Then we have*

$$g(\nabla_U Z, V) = \sec^2\theta\{g(A_{FV}U, TZ) + g(A_{NTZ}U, V)\} \ , \tag{3.12}$$

$$g(\nabla_X V, Z) = -\sec^2\theta\{g(A_{FV}X, TZ) + g(A_{NTZ}X, V)\} \ , \tag{3.13}$$

for $X \in \mathcal{D}^T$, $U, Z \in \mathcal{D}^\theta$, and $V \in \mathcal{D}^\perp$.

Proof For any $U, Z \in \mathcal{D}^\theta$ and $V \in \mathcal{D}^\perp$, using (2.2), (2.1), and (3.1), we have

$$g(\nabla_U Z, V) = g(\bar{\nabla}_U TZ, FV) + g(\bar{\nabla}_U NZ, FV) \ .$$

Hence, using (2.2) and (2.1), we obtain

$$g(\nabla_U Z, V) = g(h(U, TZ), FV) + g(\bar{\nabla}_U FNZ, V) \ .$$

Here, if we use (3.2) and (2.4), we get

$$g(\nabla_U Z, V) = g(A_{FV}U, TZ) + g(\bar{\nabla}_U tNZ, V) + g(\bar{\nabla}_U \omega NZ, V) \ .$$

With the help of (3.3)-(a), (3.3)-(c), (3.5), and (2.3), we arrive at

$$g(\nabla_U Z, V) = g(A_{FV}U, TZ) + g(\bar{\nabla}_U (1 - \cos^2\theta)Z, V) + g(A_{NTZ}U, V) \ .$$

By direct calculation, we find (3.12). On the other hand, for any $X \in \mathcal{D}^T$, $Z \in \mathcal{D}^\theta$, and $V \in \mathcal{D}^\perp$, using (2.2), (2.1) and (3.1), we have

$$g(\nabla_X V, Z) = g(\bar{\nabla}_X FV, FZ) = g(\bar{\nabla}_X FV, TZ) + g(\bar{\nabla}_X FV, NZ) \ .$$

Again, using (2.3), (2.1), and (3.2), we obtain

$$g(\nabla_X V, Z) = -g(A_{FV}X, TZ) + g(\bar{\nabla}_X V, tNZ) + g(\bar{\nabla}_X V, \omega NZ) \ .$$

Here, using (3.3)-(a) and (3.3)-(c), we get

$$g(\nabla_X V, Z) = -g(A_{FV}X, TZ) - g(\bar{\nabla}_X V, \sin^2\theta Z) - g(\bar{\nabla}_X V, NTZ) \ .$$

Hence, using (2.2), we arrive at

$$\cos^2\theta g(\nabla_X V, Z) = -g(A_{FV}X, TZ) - g(h(X,V), NTZ) \ .$$

According to direct calculation, we find (3.13). □

4. Warped product skew semi-invariant submanifolds of order 1 of a locally product Riemannian manifold

In this section, we consider a warped product submanifold of type $M = M_1 \times_f M_T$ in a l.p.R. manifold $\bar{M}$, where M_1 is a hemi-slant submanifold and M_T is an invariant submanifold. Then it is clear that M is a proper skew semi-invariant submanifold of order 1 of $\bar{M}$. Thus, by definition of hemi-slant submanifold and skew semi-invariant submanifold of order 1, we have

$$TM = \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus \mathcal{D}^T \ . \tag{4.1}$$

In particular, if $\mathcal{D}^\theta = \{0\}$, then M is a warped product semi-invariant submanifold [22]. If $\mathcal{D}^\perp = \{0\}$, then M is a warped product semi-slant submanifold [23].

On the other hand, since M_1 is a hemi-slant submanifold, by the equation (3.2) of [26], the normal bundle of $T^\perp M_1$ of M_1 is decomposed as $T^\perp M_1 = F(\mathcal{D}^\perp) \oplus N(\mathcal{D}^\theta) \oplus \mu$. Thus, we also have

$$T^\perp M = F(\mathcal{D}^\perp) \oplus N(\mathcal{D}^\theta) \oplus \mu \ , \tag{4.2}$$

since $\mathcal{D}^T$ is an invariant distribution, where μ is the orthogonal complementary distribution of $F(\mathcal{D}^\perp) \oplus N(\mathcal{D}^\theta)$ in $T^\perp M$ and it is an invariant subbundle of $T^\perp M$ with respect to F.

Remark 4.1 *From Theorem 3.1 of [22], we know that there is no proper warped product semi-invariant submanifold of type $M_T \times_f M_\perp$ of a l.p.R. manifold $\bar{M}$ such that M_T is an invariant submanifold and $M_\perp$ is an anti-invariant submanifold of $\bar{M}$. On the other hand, from Theorem 3.1 of [23] or Theorem 3.3 of [4], we know that there is no proper warped product submanifold in the form $M_T \times_f M_\theta$ of a l.p.R. manifold $\bar{M}$ such that M_θ is a proper slant submanifold and M_T is an invariant submanifold of $\bar{M}$. Thus, we conclude that there is no warped product skew semi-invariant submanifold of order 1 in the form $M_T \times_f M_1$ of a l.p.R. manifold $\bar{M}$ such that M_1 is a hemi-slant submanifold and M_T is an invariant submanifold of $\bar{M}$.*

We now present an example of warped product semi-invariant submanifold of order 1 of type $M_1 \times_f M_T$ in a l.p.R. manifold.

Example 4.2 *Consider the locally product Riemannian manifold $\mathbb{R}^{10} = \mathbb{R}^5 \times \mathbb{R}^5$ with the usual metric g and almost product structure F defined by*

$$F(\partial_i) = \partial_i, \quad F(\partial_j) = -\partial_j \ ,$$

where $i \in \{1, ..., 5\}, j \in \{6, ..., 10\}, \partial_k = \frac{\partial}{\partial x_k}$, and $(x_1, ..., x_{10})$ are natural coordinates of $\mathbb{R}^{10}$. Let M be a submanifold of $\bar{M} = (\mathbb{R}^{10}, g, F)$ given by

$$\phi(x, y, z, u, v) = (x + y,\ x - y,\ x\text{cos}u,\ x\text{sin}u,\ z,\ -z,\ x,\ \tfrac{2}{\sqrt{3}}y,\ x\text{cos}v,\ x\text{sin}v) \ ,$$

where $x > 0$.

Then we easily see that the local frame of TM is spanned by

$$\phi_x = \partial_1 + \partial_2 + \text{cos}u\partial_3 + \text{sin}u\partial_4 + \partial_7 + \text{cos}v\partial_9 + \text{sin}v\partial_{10} \ ,$$

$$\phi_y = \partial_1 - \partial_2 + \tfrac{2}{\sqrt{3}}\partial_8, \quad \phi_z = \partial_5 - \partial_6 \ ,$$

$$\phi_u = -x\text{sin}u\partial_3 + x\text{cos}u\partial_4, \quad \phi_v = -x\text{sin}v\partial_9 + x\text{cos}v\partial_{10} \ .$$

By direct calculation, we see that $\mathcal{D}^\theta = span\{\phi_x, \phi_y\}$ is a slant distribution with slant angle $\theta = \arccos \frac{1}{5}$ and $\mathcal{D}^\perp = span\{\phi_z\}$ is an anti-invariant distribution since $F(\phi_z)$ is orthogonal to TM. Moreover, $\mathcal{D}^T = span\{\phi_u, \phi_v\}$ is an invariant distribution. Thus, we conclude that M is a proper skew semi-invariant submanifold of order 1 of $\bar{M}$. Furthermore, one can easily see that $\mathcal{D}^\theta \oplus \mathcal{D}^\perp$ and $\mathcal{D}^T$ are integrable. If we denote the integral submanifolds of $\mathcal{D}^\theta, \mathcal{D}^\perp$, and $\mathcal{D}^T$ by $M_\theta, M_\perp$, and M_T, respectively, then the induced metric tensor of M is

$$\begin{aligned} ds^2 &= 5dx^2 + \tfrac{10}{3}dy^2 + 2dz^2 + x^2(du^2 + dv^2) \\ &= g_{M_\theta} + g_{M_\perp} + x^2 g_{M_T}. \end{aligned}$$

Thus, $M = (M_\theta \times M_\perp) \times_{x^2} M_T$ is a warped product skew semi-invariant submanifold of order 1 of $\bar{M}$ with warping function $f = x$.

Let $\mathcal{D}^\theta$ and $\mathcal{D}^T$ be slant and invariant distributions on M, respectively. Then M is called $(\mathcal{D}^\theta, \mathcal{D}^T)$-*mixed totally geodesic* if $h(Z, X) = 0$, where $Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$ [20].

Before giving a necessary and sufficient condition for skew semi-invariant submanifold of order 1 to be a locally warped product, we recall Hiepko's result [14], (cf. [13], Remark 2.1): Let $\mathcal{D}_1$ be a vector subbundle in the tangent bundle of a Riemannian manifold M and let $\mathcal{D}_2$ be its normal bundle. Suppose that the two distributions are involutive. If we denote by M_1 and M_2 the integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, then M is locally isometric to warped product $M_1 \times_f M_2$ if the integral manifold M_1 is totally geodesic and the integral manifold M_2 is an extrinsic sphere; in other words, M_2 is a totally umbilical submanifold with a parallel mean curvature vector.

Theorem 4.3 *Let $M = M_1 \times_f M_T$ be a $(\mathcal{D}^\theta, \mathcal{D}^T)$-mixed totally geodesic proper skew semi-invariant submanifold of order 1 with integrable distribution $\mathcal{D}^T$ of a l.p.R. manifold $\bar{M}$. Then M is a locally warped product submanifold if and only if*

$$A_{FV}FX = -V[\sigma]X \ , \tag{4.3}$$

and

$$A_{NZ}FX + A_{NTZ}X = -Z[\sigma]\sin^2\theta X \tag{4.4}$$

for $X \in \mathcal{D}^T$, $Z \in \mathcal{D}^\theta$, $V \in \mathcal{D}^\perp$, *and a function* σ *defined on* M *such that* $Y[\sigma] = 0$ *for* $Y \in \mathcal{D}^T$.

Proof Let $M = M_1 \times_f M_T$ be a $(\mathcal{D}^\theta, \mathcal{D}^T)$-mixed totally geodesic warped product proper skew semi-invariant submanifold of order 1 with integrable distribution $\mathcal{D}^T$ of a l.p.R. manifold $\bar{M}$. Then, using (3.6) and (3.8), we have $g(A_{FV}W, FX) = 0$, and $g(A_{FV}Z, FX) = 0$ for any $V, W \in \mathcal{D}^\perp$, $Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$. Since A is self adjoint, we deduce that $A_{FV}FX$ has no components in TM_1. Therefore, $A_{FV}FX \in \mathcal{D}^T$. Thus, using (2.2), (2.1), and (1.1), for any $Y \in \mathcal{D}^T$, we obtain

$$g(A_{FV}FX, Y) = -g(\bar{\nabla}_Y FV, FX) = -g(\bar{\nabla}_Y V, X) = -g(\nabla_Y V, X) = -V(\ln f)g(X, Y) \ ,$$

which proves (4.3). Since M is $(\mathcal{D}^\theta, \mathcal{D}^T)$-mixed totally geodesic for any $Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$, we have $g(A_{NTZ}X, Z) = 0$. It means that $A_{NTZ}X$ has no components in $\mathcal{D}^\theta$. On the other hand, from Lemma 3.3 of [26], we know that $TZ \in \mathcal{D}^\theta$ for any $Z \in \mathcal{D}^\theta$. Thus, using this fact and (1.1), from (3.13), we get $g(A_{NTZ}X, V) = 0)$, that is, $A_{NTZ}X$ has no components in $\mathcal{D}^\perp$. Thus, from (4.1), we conclude that $A_{NTZ}X \in \mathcal{D}^T$. We also have $A_{NZ}X \in \mathcal{D}^T$. Then, for $X, Y \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$, using (1.1), from (3.10), we have

$$g(A_{NTZ}Y, X) + g(A_{NZ}FY, X) = -\sin^2\theta g(\nabla_X Z, Y) = -\sin^2\theta Z(\ln f)g(Y, X) \ .$$

This proves (4.4). Moreover, since $Y(\ln f) = 0$ for a warped product proper skew semi-invariant submanifold of order 1, we obtain $\sigma = \ln f$.

Conversely, suppose that M is $(\mathcal{D}^\theta, \mathcal{D}^T)$-mixed totally geodesic proper skew semi-invariant submanifold of order 1 with integrable distribution $\mathcal{D}^T$ of a l.p.R. manifold $\bar{M}$ such that (4.3) and (4.4) hold. Using (3.6)$\sim$(3.9), (4.3), and (4.4), it is not difficult to see that $g(\nabla_{\hat{U}}\hat{V}, X) = 0$ for $\hat{U}, \hat{V} \in TM_1$ and $X \in \mathcal{D}^\perp$. It means that M_1 is totally geodesic in M. Let M_T be the integral manifold of $\mathcal{D}^T$ and h_T be the second fundamental form of M_T in M. Using (2.2), we have $g(h_T(X, Y), V) = g(\nabla_X Y, V)$ for $X, Y \in \mathcal{D}^T$ and $V \in \mathcal{D}^\perp$. Then (3.11) implies that $g(h_T(X, Y), V) = g(A_{FV}FY, X)$. Thus, using (4.3), we obtain

$$g(h_T(X, Y), V) = -V[\sigma]g(Y, X) \ . \tag{4.5}$$

Similarly, using (2.2), we have $g(h_T(X, Y), Z) = g(\nabla_X Y, Z)$ for $X, Y \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$. By (3.10), we obtain

$$g(h_T(X, Y), Z) = \csc^2\theta\{g(A_{NTZ}Y, X) + g(A_{NZ}FY, X)\} \ .$$

Using (4.4), we get

$$g(h_T(X, Y), Z) = -Z[\sigma]g(X, Y) \ . \tag{4.6}$$

Thus, for any $E = V + Z \in TM_1$, from (4.4) and (4.5), we arrive at

$$\begin{aligned} g(h_T(X, Y), E) &= g(h_T(X, Y), V) + g(h_T(X, Y), Z) \\ &= -\{V[\sigma] + Z[\sigma]\}g(X, Y). \end{aligned} \tag{4.7}$$

(4.7) states that M_T is totally umbilical in M. Let us denote the gradient of σ on $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by $grad^\perp\sigma$ and $grad^\theta\sigma$, respectively. From (4.7), we write

$$h_T(X,Y) = -\{grad^\perp\sigma + grad^\theta\sigma\}g(X,Y). \tag{4.8}$$

Thus, for any $E = V + Z \in TM_1$, we have

$$\begin{aligned} g(\nabla_X(grad^\perp\sigma + grad^\theta\sigma), E) &= g(\nabla_X grad^\perp\sigma, E) + g(\nabla_X grad^\theta\sigma, E) \\ &= \{Xg(grad^\perp\sigma, V) - g(grad^\perp\sigma, \nabla_X E)\} \\ &\quad +\{Xg(grad^\theta\sigma, Z) - g(grad^\theta\sigma, \nabla_X E)\} \\ &= X[V[\sigma]] - g(grad^\perp\sigma, \nabla_X E) + X[Z[\sigma]] - g(grad^\theta\sigma, \nabla_X E). \end{aligned}$$

At this point, since M_1 is totally geodesic in M, we have $g(A_{FV}X, TZ) = 0$ from (3.8). We have also $g(A_{NTZ}X, V) = 0$, since M is $(\mathcal{D}^\theta, \mathcal{D}^T)$-mixed totally geodesic. Thus, using these equations in (3.13), we get $g(\nabla_X V, Z) = -g(\nabla_X Z, V) = 0$. Using this fact, we obtain

$$g(\nabla_X(grad^\perp\sigma + grad^\theta\sigma), E) = X[V[\sigma]] - g(grad^\perp\sigma,\nabla_X Z) + X[Z[\sigma]] - g(grad^\theta\sigma, \nabla_X V)\ .$$

By direct calculation, we arrive at

$$\begin{aligned} g(\nabla_X(grad^\perp\sigma + grad^\theta\sigma), E) = &\{X[Z[\sigma]] - [X,Z][\sigma] + g(grad^\perp\sigma, \nabla_Z X)\} \\ &+\{X[V[\sigma] - [X,V][\sigma] + g(grad^\theta\sigma, \nabla_V X)\}. \end{aligned}$$

After some calculation, we get

$$g(\nabla_X(grad^\perp\sigma + grad^\theta\sigma), E) = \{Z[X[\sigma]] + g(grad^\perp\sigma, \nabla_Z X) + V[X[\sigma]] + g(grad^\theta\sigma, \nabla_V X)\}.$$

Since $X[\sigma] = 0$, from the last equation, we derive

$$g(\nabla_X(grad^\perp\sigma + grad^\theta\sigma), E) = -g(\nabla_Z grad^\perp\sigma, X) - g(\nabla_V grad^\theta\sigma, X)$$

Here, we know that $\nabla_Z grad^\perp\sigma, \nabla_V grad^\theta\sigma \in TM_1$, since M_1 is totally geodesic. Hence, we obtain $g(\nabla_X(grad^\perp\sigma + grad^\theta\sigma), E) = 0$. It means that $grad^\perp\sigma + grad^\theta\sigma$ is parallel in M. This fact and (4.8) imply that M_T is an extrinsic sphere. This completes the proof. □

5. A Chen-type inequality for warped product skew semi-invariant submanifolds of order 1

In this section, we give an inequality similar to Chen's inequality [11] for the squared norm of the second fundamental form in terms of the warping function for such submanifolds. We first give the following two lemmas for later use.

Lemma 5.1 *Let $M = M_1 \times_f M_T$ be a warped product proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $\bar{M}$. Then we have*

$$g(h(X,V), FW) = 0 \tag{5.1}$$

and

$$g(h(X,V), NZ) = 0\ , \tag{5.2}$$

for $X \in \mathcal{D}^T$, $Z \in \mathcal{D}^\theta$, and $V, W \in \mathcal{D}^\perp$.

Proof For any $V, W \in \mathcal{D}^{\perp}$ and $X \in \mathcal{D}^{T}$, using (2.2), (2.1), and (1.1), we get

$$g(h(X,V),FW) = g(\bar{\nabla}_V X, FW) = g(\bar{\nabla}_V FX, W) = g(\nabla_V FX, W) = V(\ln f)g(FX,W) = 0,$$

since $g(FX,W) = 0$. Hence, (5.1) follows. In a similar way, using (2.2), (2.1), (3.1), and (1.1), we have

$$\begin{aligned} g(h(X,V),NZ) &= g(\bar{\nabla}_V X, NZ) = g(\bar{\nabla}_V X, FZ) - g(\bar{\nabla}_V X, TZ) \\ &= g(\bar{\nabla}_V FX, Z) - g(\bar{\nabla}_V X, TZ) \\ &= g(\nabla_V FX, Z) - g(\nabla_V X, TZ) \\ &= V(\ln f)g(FX,Z) - V(\ln f)g(X,TZ) = 0, \end{aligned}$$

since $g(FX,Z) = 0$ and $g(X,TZ) = 0$. □

Lemma 5.2 *Let $M = M_1 \times_f M_T$ be a warped product proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $\bar{M}$. Then we have*

$$g(h(X,FY),FV) = -V(\ln f)g(X,Y) \tag{5.3}$$

and

$$g(h(X,Y),NZ) = TZ(\ln f)g(X,Y) \tag{5.4}$$

for $X, Y \in \mathcal{D}^{T}$, $Z \in \mathcal{D}^{\theta}$, and $V \in \mathcal{D}^{\perp}$.

Proof Using (2.2) and (2.1), we have

$$g(h(X,FY),FV) = g(\bar{\nabla}_X FY, FV) = g(\bar{\nabla}_X Y, V) = g(\nabla_X Y, V) = -g(\nabla_X V, Y)$$

for any $X, Y \in \mathcal{D}^{T}$ and $V \in \mathcal{D}^{\perp}$. Hence, using (1.1), we easily obtain (5.3). The last assertion (5.4) follows from Lemma 3.1-(ii) of [2] by using linearity. □

Theorem 5.3 *Let $M = M_1 \times_f M_T$ be a $(q+m)$-dimensional warped product proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $\bar{M}$ of dimension $2q+m$, where $dim(M_1) = q$ and $dim(M_T) = m$. Then M is $(\mathcal{D}^{\perp}, \mathcal{D}^{T})$-mixed totally geodesic; in other words, $h(\mathcal{D}^{\perp}, \mathcal{D}^{T}) = 0$.*

Proof Let $M = M_1 \times_f M_T$ be a $(q+m)$-dimensional warped product proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $\bar{M}$ of dimension $2q+m$. Then by the dimension argument in the hypothesis, the distribution μ involved in the definition of the normal bundle $T^{\perp}M$ of M vanishes. Therefore, from (4.2), we have $T^{\perp}M = F(\mathcal{D}^{\perp}) \oplus N(\mathcal{D}^{\theta})$. Thus, from (5.1) and (5.2), we deduce that $h(V,X) = 0$ for $X \in \mathcal{D}^{T}$ and $V \in \mathcal{D}^{\perp}$. It means that M is a $(\mathcal{D}^{\perp}, \mathcal{D}^{T})$-mixed totally geodesic. □

Let M be a $(k+n+m)$-dimensional warped product proper skew semi-invariant submanifold of order 1 of a $(2k+2n+m)$-dimensional l.p.R. manifold $\bar{M}$. We choose a canonical orthonormal basis $\{e_1, ..., e_m, \bar{e}_1, ..., \bar{e}_k, \tilde{e}_1, ..., \tilde{e}_n, e_1^*, ..., e_k^*, F\tilde{e}_1, ..., F\tilde{e}_n\}$ such that $\{e_1, ..., e_m\}$ is an orthonormal basis of $\mathcal{D}^{T}$, $\{\bar{e}_1, ..., \bar{e}_k\}$ is an orthonormal basis of $\mathcal{D}^{\theta}$, $\{\tilde{e}_1, ..., \tilde{e}_n\}$ is an orthonormal basis of $\mathcal{D}^{\perp}$, $\{e_1^*, ..., e_k^*\}$ is an orthonormal basis of $N\mathcal{D}^{\theta}$, and $\{F\tilde{e}_1, ..., F\tilde{e}_n\}$ is an orthonormal basis of $F\mathcal{D}^{\perp}$.

Remark 5.4 *From (2.1), we can observe that* $\{Fe_1, ..., Fe_m\}$ *is also an orthonormal basis of* $\mathcal{D}^T$. *On the other hand, with the help of the equations (3.5) and (3.6) of [26], we can see that* $\{\sec\theta T\bar{e}_1, ..., \sec\theta T\bar{e}_k\}$ *is also an orthonormal basis of* $\mathcal{D}^\theta$ *and* $\{\csc\theta N\bar{e}_1, ..., \csc\theta N\bar{e}_k\}$ *is also an orthonormal basis of* $N\mathcal{D}^\theta$.

We now state the main result of this section.

Theorem 5.5 *Let* $M = M_1 \times_f M_T$ *be a* $(k+n+m)$*-dimensional warped product proper skew semi-invariant submanifold of order* 1 *of a* $(2k+2n+m)$*-dimensional l.p.R. manifold* $\bar{M}$. *Then the squared norm of the second fundamental form of* M *satisfies*

$$\|h\|^2 \geq m\{\|\nabla^\perp(\ln f)\|^2 + \cot^2\theta\|\nabla^\theta(\ln f)\|^2\} \ , \tag{5.5}$$

where $m = dim(M_T)$, *and* $\nabla^\perp(\ln f)$ *and* $\nabla^\theta(\ln f)$ *are gradients of* $\ln f$ *on* $\mathcal{D}^\perp$ *and* $\mathcal{D}^\theta$, *respectively. If the equality case of (5.5) holds identically, then* M_1 *is a totally geodesic submanifold of* $\bar{M}$ *and* M *is a mixed totally geodesic. Moreover,* M_T *cannot be minimal.*

Proof In view of decomposition (4.1), the squared norm of the second fundamental form h can be decomposed as

$$\begin{aligned}\|h\|^2 = &\|h(\mathcal{D}^T,\mathcal{D}^T)\|^2 + \|h(\mathcal{D}^\theta,\mathcal{D}^\theta)\|^2 + \|h(\mathcal{D}^\perp,\mathcal{D}^\perp)\|^2 \\ &+2\|h(\mathcal{D}^T,\mathcal{D}^\perp)\|^2 + 2\|h(\mathcal{D}^T,\mathcal{D}^\theta)\|^2 + 2\|h(\mathcal{D}^\perp,\mathcal{D}^\theta)\|^2.\end{aligned}$$

By Theorem 5.3, M is $(\mathcal{D}^\perp,\mathcal{D}^T)$-mixed totally geodesic; thus we get

$$\begin{aligned}\|h\|^2 = &\|h(\mathcal{D}^T,\mathcal{D}^T)\|^2 + \|h(\mathcal{D}^\theta,\mathcal{D}^\theta)\|^2 + \|h(\mathcal{D}^\perp,\mathcal{D}^\perp)\|^2 \\ &+2\|h(\mathcal{D}^T,\mathcal{D}^\theta)\|^2 + 2\|h(\mathcal{D}^\perp,\mathcal{D}^\theta)\|^2,\end{aligned}$$

which can be written as follows:

$$\begin{aligned}\|h\|^2 = &\sum_{i,j=1}^{m}\sum_{a=1}^{n} g(h(e_i,e_j),F\tilde{e}_a)^2 + \sum_{i,j=1}^{m}\sum_{r=1}^{k} g(h(e_i,e_j),e_r^*)^2 \\ &+ \sum_{a,b,c=1}^{n} g(h(\tilde{e}_a,\tilde{e}_b),F\tilde{e}_c)^2 + \sum_{a,b=1}^{n}\sum_{r=1}^{k} g(h(\tilde{e}_a,\tilde{e}_b),e_r^*)^2 \\ &+ \sum_{r,s=1}^{k}\sum_{a=1}^{n} g(h(\bar{e}_r,\bar{e}_s),F\tilde{e}_a)^2 + \sum_{r,s,q=1}^{k} g(h(\bar{e}_r,\bar{e}_s),e_q^*)^2 \\ &+2\sum_{i=1}^{m}\sum_{r=1}^{k}\sum_{a=1}^{n} g(h(e_i,\bar{e}_r),F\tilde{e}_a)^2 + 2\sum_{i=1}^{m}\sum_{r,s=1}^{k} g(h(e_i,\bar{e}_r),e_s^*)^2 \\ &+2\sum_{r=1}^{k}\sum_{a,b=1}^{n} g(h(\bar{e}_r,\tilde{e}_a),F\tilde{e}_b)^2 + 2\sum_{r,s=1}^{k}\sum_{a=1}^{n} g(h(\bar{e}_r,\tilde{e}_a),e_s^*)^2.\end{aligned} \tag{5.6}$$

Here, using (5.1)∼(5.3) and Remark 5.4, we have

$$\sum_{i,j=1}^{m}\sum_{a=1}^{n} g(h(e_i,e_j),F\tilde{e}_a)^2 = \sum_{i,j=1}^{m}\sum_{a=1}^{n}(-\tilde{e}_a(\ln f)g(e_i,e_j))^2 \tag{5.7}$$

and

$$\sum_{i,j=1}^{m}\sum_{r=1}^{k} g(h(e_i,e_j),e_r^*)^2 = \sum_{i,j=1}^{m}\sum_{r=1}^{k} g(h(e_i,e_j),N\bar{e}_r)^2\csc^2\theta. \tag{5.8}$$

Moreover, using (5.4) in (5.8), we get

$$\sum_{i,j=1}^{m}\sum_{r=1}^{k} g(h(e_i,e_j),e_r^*)^2 = \sum_{i,j=1}^{m}\sum_{r=1}^{k}(T\bar{e}_r(\ln f)g(e_i,e_j))^2\csc^2\theta. \tag{5.9}$$

Using (5.7) and (5.9) in (5.6), we get

$$\|h\|^2 \geq m\|\nabla^{\perp}(\ln f)\|^2 + \sum_{i,j=1}^{m}\sum_{r=1}^{k}(T\bar{e}_r(\ln f)g(e_i,e_j))^2\csc^2\theta\ . \tag{5.10}$$

By Remark 5.4, we replace $\bar{e}_r$ by $\sec\theta T\bar{e}_r$ in the last term of (5.10) and using (3.5) we have

$$\begin{aligned}&\sum_{i,j=1}^{m}\sum_{r=1}^{k}(T\bar{e}_r(\ln f)g(e_i,e_j))^2\csc^2\theta\\ =&\sum_{i,j=1}^{m}\sum_{r=1}^{k}\cos^4\theta(\bar{e}_r(\ln f)g(e_i,e_j))^2\csc^2\theta = m\cot^2\theta\|\nabla^{\theta}(\ln f)\|^2.\end{aligned} \tag{5.11}$$

Thus, using (5.11) in (5.10), we find (5.5).

Next, if the equality case of (5.5) holds identically, then from (5.6) we have

$$h(\mathcal{D}^{\perp},\mathcal{D}^{\perp}) = 0, \quad h(\mathcal{D}^{\theta},\mathcal{D}^{\theta}) = 0, \quad h(\mathcal{D}^{\perp},\mathcal{D}^{\theta}) = 0 \tag{5.12}$$

and

$$h(\mathcal{D}^{T},\mathcal{D}^{\theta}) = 0. \tag{5.13}$$

Since M_1 is totally geodesic in M, from (5.12) it follows that M_1 is also totally geodesic in $\bar{M}$. On the other hand, Theorem 5.3 and the equation (5.13) imply that M is a mixed totally geodesic. Finally, if we suppose that M is minimal, then from (5.3) and (5.4) we conclude that $\|\nabla(\ln f)\| = 0$, which is a contradiction. □

Remark 5.6 *Theorem 5.5 coincides with Theorem 4.2 of [22] if* $\mathcal{D}^{\theta} = \{0\}$. *In other words, Theorem 5.5 is a generalization of Theorem 4.2 of [22].*

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