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Classification of metallic shaped hypersurfaces in real space forms

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Abstract: We define the notion of a metallic shaped hypersurface and give the full classification of metallic shaped hypersurfaces in real space forms. We deduce that every metallic shaped hypersurface in real space forms is a semisymmetric hypersurface.

Key words: Hypersurface, real space form, metallic means family, pseudosymmetric hypersurface, semisymmetric hypersurface

1. Introduction

The generalized secondary Fibonacci sequence (see [5]) is given by the relation

$$G(n+1) = pG(n) + qG(n-1), \quad n \geq 1,$$

where $G(0) = a$, $G(1) = b$, p and q are real numbers. If $p = q = 1$, then we obtain secondary Fibonacci sequence. If the limit

$$x = \lim_{n\to\infty} \frac{G(n+1)}{G(n)}$$

exists then it is a root of the equation

$$x^2 - px - q = 0; \tag{1.1}$$

see [4]. Let p and q be two integers. The positive solution of equation (1.1) is called a *member of the metallic means family* (briefly MMF) [4]. The positive solution of the above equation is

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

These numbers are called (p, q)-*metallic numbers* [4]. For the special values of p and q, we have the following (see [5]):

i) For $p = q = 1$ we obtain $\sigma_G = \frac{1+\sqrt{5}}{2}$, which is the *golden mean*;

ii) For $p = 2$ and $q = 1$ we obtain $\sigma_{Ag} = 1 + \sqrt{2}$, which is the *silver mean*;

iii) For $p = 3$ and $q = 1$ we obtain $\sigma_{Br} = \frac{3+\sqrt{13}}{2}$, which is the *bronze mean*;

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iv) For $p=1$ and $q=2$ we obtain $\sigma_{Cu}=2$, which is the *copper mean*;

v) For $p=1$ and $q=3$ we obtain $\sigma_{Ni}=\frac{1+\sqrt{13}}{2}$, which is the *nickel mean*.

Hence, de Spinadel [4] obtained a generalization of the golden mean as metallic means family or metallic proportions. The MMF has been used in describing fractal geometry and quasiperiodic dynamics (for more details see [11] and the references therein). Furthermore, El Naschie [9] obtained the relationships between the Hausdorff dimension of higher order Cantor sets and the golden mean or silver mean.

It is well known that golden mean has been used in constructions of buildings, music, paintings, etc.

In [2] and [10], Crasmareanu and Hretcanu introduced the notion of the golden structure on a manifold M. The golden structure on a manifold is a $(1,1)$-tensor field J on M satisfying the same equation as the golden ratio:

$$J^2 = J + I,$$

where I is the Kronecker tensor field of M.

In [3], Crasmareanu et al. defined the notion of a golden shaped hypersurface. A hypersurface M is called a *golden shaped hypersurface* if the shape operator A of M is a golden structure, i.e. $A^2 = A + I$, where I is the identity on the tangent bundle of M [3]. The full classification of the golden shaped hypersurfaces in real space forms was given in [3].

Similar to the golden structure on a manifold M, in [11], Hretcanu and Crasmareanu defined the metallic structure on a manifold M. The *metallic structure on a manifold* M is a $(1,1)$-tensor field J on M satisfying the equation

$$J^2 = pJ + qI,$$

where I is the Kronecker tensor field of M and p, q are positive integers [11].

In the present study, we define the notion of a metallic shaped hypersurface and our aim is to generalize the results of [3] to metallic shaped hypersurfaces in real space forms. We obtain the full classification of metallic shaped hypersurfaces in real space forms. We also deduce that every metallic shaped hypersurface in real space forms is a semisymmetric hypersurface. Similar to the figures given in [3], using Mathematica [17], we draw Figures 1, 2, and 3.

2. Metallic shaped hypersurfaces in real space forms

Let M be a hypersurface in the real space form $M^{n+1}(c)$ oriented by the unit normal vector field N. Denote the shape operator of M by A. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the principal curvatures of M. If M has constant principal curvatures then it is called an *isoparametric hypersurface* [1].

Now we define the notion of a metallic shaped hypersurface:

Definition 2.1 *M is called a metallic shaped hypersurface if the shape operator A is a metallic structure. Hence A satisfies*

$$A^2 = pA + qI, \tag{2.1}$$

where I is the identity on the tangent bundle of M. If $p=q=1$, then we obtain a golden shaped hypersurface (see [3]). If $p=2$ and $q=1$, then the hypersurface is called silver shaped; if $p=3$ and $q=1$, then it is called bronze shaped; if $p=1$ and $q=2$, then it is called copper shaped; and if $p=1$ and $q=3$, then it is called nickel shaped.

The principal curvatures of M are

$$\phi = \frac{p+\sqrt{p^2+4q}}{2} \text{ and } \Phi = \frac{p-\sqrt{p^2+4q}}{2}.$$

For metallic shaped hypersurfaces in $\mathbb{R}^{n+1}$, we have the following theorem:

Theorem 2.1 *The metallic shaped hypersurfaces of* $\mathbb{R}^{n+1}$ *are the hyperspheres*:

$$S^n(\frac{1}{\phi}) = S^n\left(\frac{2}{p+\sqrt{p^2+4q}}\right)$$

or

$$S^n(\frac{1}{|\Phi|}) = S^n\left(\frac{2}{\sqrt{p^2+4q}-p}\right).$$

Proof We shall discuss the following three cases:

i) If $\lambda_1 = ... = \lambda_n = \phi$ then M is totally umbilical and hence it is the hypersphere $S^n(\frac{1}{\phi}) = S^n\left(\frac{2}{p+\sqrt{p^2+4q}}\right) \subset \mathbb{R}^{n+1}$.

ii) If $\lambda_1 = ... = \lambda_n = \Phi$ then M is again totally umbilical and hence it is the hypersphere $S^n(\frac{1}{|\Phi|}) = S^n\left(\frac{2}{\sqrt{p^2+4q}-p}\right) \subset \mathbb{R}^{n+1}$.

iii) Assume that $\lambda_1 = ... = \lambda_k = \phi$ and $\lambda_{k+1} = ... = \lambda_n = \Phi$. It is known from [1] that the isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ are hyperspheres, hypercylinders, and hyperplanes. However, we have only hyperspheres in $\mathbb{R}^{n+1}$ since our principal curvatures are different from zero. This means that the hypersurface is totally umbilical. Hence, this case cannot occur.

Thus we get the result as required. □

The hyperbolical space in the upper half space model is defined by

$$\mathbb{H}^{n+1} = \left\{x \in \mathbb{R}^{n+2} : \left(x^1\right)^2 + ... + \left(x^{n+1}\right)^2 - \left(x^{n+2}\right)^2 = -1, \quad x^{n+2} > 0\right\}.$$

The isoparametric hypersurfaces M of $\mathbb{H}^{n+1}$ are given by Ryan in [14] as

I) $M = \left\{x \in \mathbb{H}^{n+1} : x^k = 0\right\}$, for $1 \leq k \leq n+1$ with $A = [0]$.

II) $M = \left\{x \in \mathbb{H}^{n+1} : x^k = r > 0\right\}$, for $1 \leq k \leq n+1$ with $A = \sqrt{c+1}I$, where $c = -\frac{1}{r^2} \in (-1, 0)$; then $r \in (1, \infty)$. In this case M is isometric to the hyperbola $H^n(c)$.

III) $M = \left\{x \in \mathbb{H}^{n+1} : x^{n+2} = x^{n+1} + 1\right\}$, with $A = I$. In this case M is isometric to $\mathbb{R}^n$.

IV) $M = \left\{x \in \mathbb{H}^{n+1} : \left(x^1\right)^2 + ... + \left(x^{n+1}\right)^2 = r^2\right\} = \mathbb{H}^{n+1} \cap S^n(r)$ with $A = \sqrt{c+1}I$, where $c = \frac{1}{r^2} > 0$. In this case M is isometric to $S^n(r)$.

V) $M = \left\{x \in \mathbb{H}^{n+1} : \left(x^1\right)^2 + ... + \left(x^{k+1}\right)^2 = r^2\right\} = \mathbb{H}^{n+1} \cap S^k(r)$ for $1 \leq k \leq n$ with $A = \lambda I_k \oplus \frac{1}{\lambda} I_{n-k}$, where $\lambda = \frac{\sqrt{r^2+1}}{r} > 0$.

For metallic shaped hypersurfaces in $\mathbb{H}^{n+1}$, we can state the following theorem:

Theorem 2.2 *The metallic shaped hypersurfaces of* $\mathbb{H}^{n+1}$ *are as follows*:

i) For $1 \leq k \leq n+1$ *and* $q-1<p$, $M=\left\{x \in \mathbb{H}^{n+1}: x^{k}=r\right\}$, *which is isometric to the hyperbola* $H^{n}(c)$, $c=-\frac{1}{r^{2}}=\frac{p^{2}+2q-2-p\sqrt{p^{2}+4q}}{2} \in(-1,0)$.

ii) $M=\left\{x \in \mathbb{H}^{n+1}:\left(x^{1}\right)^{2}+\ldots+\left(x^{n+1}\right)^{2}=r^{2}\right\}=\mathbb{H}^{n+1} \cap S^{n}(r)$;

in this case either

a) M is isometric to $S^{n}(r)$, *where* $c=\frac{1}{r^{2}}=\frac{p^{2}+2q-2+p\sqrt{p^{2}+4q}}{2}$ *or*

b) M is isometric to $S^{n}(r)$, *where* $c=\frac{1}{r^{2}}=\frac{p^{2}+2q-2-p\sqrt{p^{2}+4q}}{2}$, $q-1>p$ *and* $q \neq 1$.

Proof Cases (I) and (III) in Ryan's classification can not occur since p and q are positive integers. Hence, we consider the following cases:

i) For $1 \leq k \leq n+1$ with $A=\sqrt{c+1}I$, where $c=-\frac{1}{r^{2}} \in(-1,0)$, from (2.1) we have

$$c^{2}+c(2-2q-p^{2})+1-2q+q^{2}-p^{2}=0,$$

which gives us

$$c=\frac{p^{2}+2q-2 \mp p\sqrt{p^{2}+4q}}{2}.$$

Since $c=-\frac{1}{r^{2}} \in(-1,0)$, we assume that $c=\frac{p^{2}+2q-2-p\sqrt{p^{2}+4q}}{2}<0$. This gives us $q-1<p$. Hence, from Ryan's classification (II), we obtain (i).

ii) Similar to the above case, for $A=\sqrt{c+1}I$, with $c=\frac{1}{r^{2}}>0$ we have

$$c=\frac{p^{2}+2q-2 \mp p\sqrt{p^{2}+4q}}{2}.$$

Since $c=\frac{p^{2}+2q-2+p\sqrt{p^{2}+4q}}{2}>0$, we only check the positiveness of $\frac{p^{2}+2q-2-p\sqrt{p^{2}+4q}}{2}$. This gives us $q-1>p$. Hence, we obtain ii) (a) and (b).

iii) For $1 \leq k \leq n$ with $A=\lambda I_{k} \oplus \frac{1}{\lambda} I_{n-k}$, where $\lambda=\frac{\sqrt{r^{2}+1}}{r}>0$ we get $\lambda=\frac{1-q}{p}<0$. Since $\lambda>0$, this case cannot occur.

This proves the theorem. $\square$

The isoparametric hypersurfaces M of $\mathbb{S}^{n+1}:=\mathbb{S}^{n+1}(1)$ are given by Ryan in [13] and [14] as:

i) M is umbilical and $M=\left\{x \in \mathbb{S}^{n+1}: x^{n+2}=\sqrt{1-r^{2}}\right\}$ for $r \in(0,1)$ with $A=\frac{\sqrt{1-r^{2}}}{r} I$. In this case, M is isometric to $S^{n}(r)$.

ii) M is the generalized Clifford torus $M=S^{m}(r_{1}) \times S^{n-m}(r_{2})$ with $r_{1}^{2}+r_{2}^{2}=1$ and $1 \leq m<n$. In this case, $r_{1}=\frac{1}{\sqrt{1+\lambda_{1}^{2}}}$ and $r_{2}=\frac{1}{\sqrt{1+\lambda_{2}^{2}}}$ with $\lambda_{1} \lambda_{2}=-1$ (see [8] and [3]).

For metallic shaped hypersurfaces in $\mathbb{S}^{n+1}$, we can state the following theorem:

Theorem 2.3 *The metallic shaped hypersurfaces of* $\mathbb{S}^{n+1}$ *are as follows*:

i) M is umbilical and $M=\left\{x\in\mathbb{S}^{n+1}:x^{n+2}=\sqrt{\frac{p^2+p\sqrt{p^2+4q}+2q}{p^2+p\sqrt{p^2+4q}+2q+2}}\right\}$. *In this case M is isometric to* $S^n\left(\sqrt{\frac{2}{p^2+p\sqrt{p^2+4q}+2q+2}}\right)$.

ii) M is umbilical and $M=\left\{x\in\mathbb{S}^{n+1}:x^{n+2}=\sqrt{\frac{p^2-p\sqrt{p^2+4q}+2q}{p^2-p\sqrt{p^2+4q}+2q+2}}\right\}$. *In this case M is isometric to* $S^n\left(\sqrt{\frac{2}{p^2-p\sqrt{p^2+4q}+2q+2}}\right)$.

iii) M is the Clifford torus $S^m\left(\sqrt{\frac{2}{p^2+p\sqrt{p^2+4}+4}}\right)\times S^{n-m}\left(\sqrt{\frac{2}{p^2-p\sqrt{p^2+4}+4}}\right)$ *and* $q=1$.

Proof Using Ryan's classification for isoparametric hypersurfaces M^n of $\mathbb{S}^{n+1}$, when $\frac{\sqrt{1-r^2}}{r}=\phi$ we have (i). When $\frac{\sqrt{1-r^2}}{r}=|\Phi|$ we obtain (ii) since $p^2-p\sqrt{p^2+4q}+2q+2>0$. If $r_1=\frac{1}{\sqrt{1+\phi^2}}$ and $r_2=\frac{1}{\sqrt{1+\Phi^2}}$ with $\phi\Phi=-1$ then we get $q=1$ and $r_1^2=\frac{2}{p^2+p\sqrt{p^2+4}+4}$ and $r_2^2=\frac{2}{p^2-p\sqrt{p^2+4}+4}$. Hence, we obtain the Clifford torus $S^m\left(\sqrt{\frac{2}{p^2+p\sqrt{p^2+4}+4}}\right)\times S^{n-m}\left(\sqrt{\frac{2}{p^2-p\sqrt{p^2+4}+4}}\right)$. □

3. Applications

For some values of p,q we get the following corollaries:

For $p=q=1$ we can state:

Corollary 3.1 *[3] The golden shaped hypersurfaces of $\mathbb{R}^{n+1}$ are the hyperspheres:*

$$S^n(\frac{1}{\phi})=S^n\left(\frac{\sqrt{5}-1}{2}\right)$$

and

$$S^n(\frac{1}{\Phi})=S^n\left(\frac{\sqrt{5}+1}{2}\right).$$

Corollary 3.2 *[3] The golden shaped hypersurfaces of $\mathbb{H}^{n+1}$ are as follows:*

i) For $1\le k\le n+1$, $M=\left\{x\in\mathbb{H}^{n+1}:x^k=\sqrt{\frac{1+\sqrt{5}}{2}}\right\}$, *which is isometric to the hyperbola* $H^n\left(\frac{1-\sqrt{5}}{2}\right)$.

ii) $M=\left\{x\in\mathbb{H}^{n+1}:\left(x^1\right)^2+...+\left(x^{n+1}\right)^2=r^2\right\}=\mathbb{H}^{n+1}\cap S^n(r)$; *in this case M is isometric to* $S^n\left(\sqrt{\frac{\sqrt{5}-1}{2}}\right)$.

Corollary 3.3 *[3] The golden shaped hypersurfaces of $\mathbb{S}^{n+1}$ are as follows:*

i) M is umbilical and $M=\left\{x\in\mathbb{S}^{n+1}:x^{n+2}=\sqrt{\frac{3+\sqrt{5}}{5+\sqrt{5}}}\right\}$. *In this case M is isometric to* $S^n\left(\sqrt{\frac{2}{5+\sqrt{5}}}\right)$.

ii) M is umbilical and $M=\left\{x\in\mathbb{S}^{n+1}:x^{n+2}=\sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}}\right\}$. *In this case M is isometric to* $S^n\left(\sqrt{\frac{2}{5-\sqrt{5}}}\right)$.

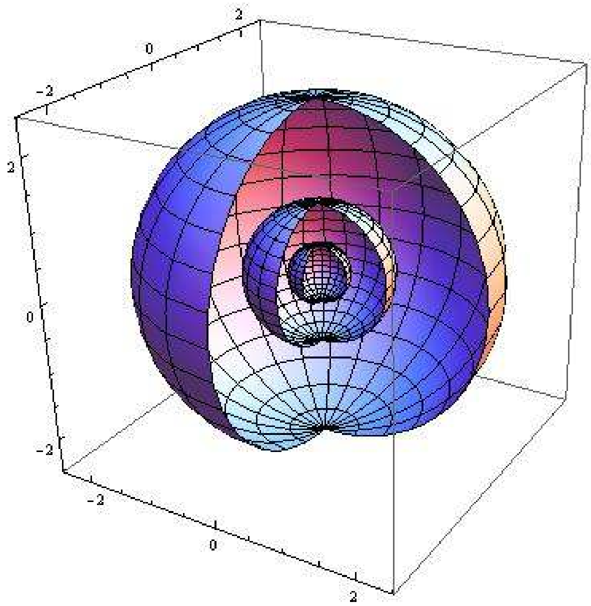

Figure 1. Silver shaped spheres $S^2\left(\sqrt{2}-1\right)$, $S^2\left(\sqrt{2}+1\right)$ in $\mathbb{R}^3$ and the unit sphere in center.

iii) M is the Clifford torus $S^m\left(\sqrt{\frac{2}{5+\sqrt{5}}}\right)\times S^{n-m}\left(\sqrt{\frac{2}{5-\sqrt{5}}}\right)$.

For $p=2$ and $q=1$ we have:

Corollary 3.4 *The silver shaped hypersurfaces of $\mathbb{R}^{n+1}$ are the hyperspheres:*

$$S^n(\frac{1}{\phi})=S^n\left(\sqrt{2}-1\right)$$

and

$$S^n(\frac{1}{|\Phi|})=S^n\left(\sqrt{2}+1\right).$$

Corollary 3.5 *The silver shaped hypersurfaces of $\mathbb{H}^{n+1}$ are as follows:*

i) For $1\leq k\leq n+1$, $M=\left\{x\in\mathbb{H}^{n+1}:x^k=\sqrt{\frac{\sqrt{2}+1}{2}}\right\}$, which is isometric to the hyperbola $H^n(2-2\sqrt{2})$.

ii) $M=\left\{x\in\mathbb{H}^{n+1}:\left(x^1\right)^2+...+\left(x^{n+1}\right)^2=r^2\right\}=\mathbb{H}^{n+1}\cap S^n(r)$; in this case M is isometric to $S^n\left(\sqrt{\frac{\sqrt{2}-1}{2}}\right)$.

Corollary 3.6 *The silver shaped hypersurfaces of $\mathbb{S}^{n+1}$ are as follows:*

i) M is umbilical and $M=\left\{x\in\mathbb{S}^{n+1}:x^{n+2}=\sqrt{\frac{3+2\sqrt{2}}{4+2\sqrt{2}}}\right\}$. In this case M is isometric to $S^n\left(\sqrt{\frac{1}{4+2\sqrt{2}}}\right)$.

ii) M is umbilical and $M=\left\{x\in\mathbb{S}^{n+1}:x^{n+2}=\sqrt{\frac{3-2\sqrt{2}}{4-2\sqrt{2}}}\right\}$. In this case M is isometric to $S^n\left(\sqrt{\frac{1}{4-2\sqrt{2}}}\right)$.

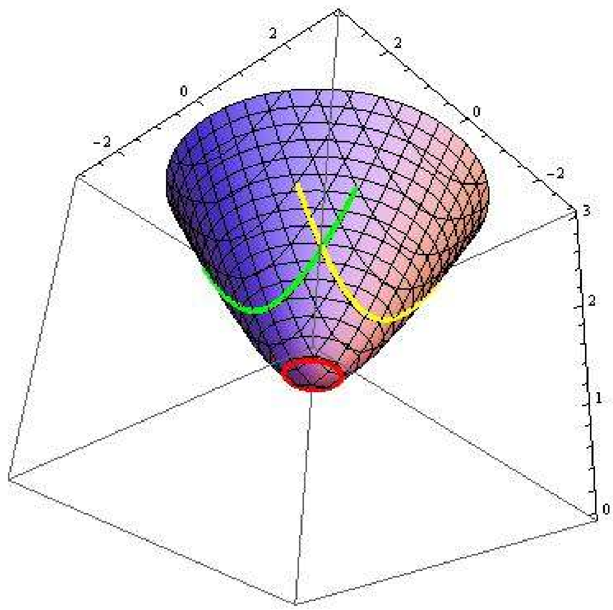

Figure 2. Silver shaped circle $S\left(\sqrt{\frac{\sqrt{2}-1}{2}}\right)$ and the hyperbolas $H(2-2\sqrt{2})$ in H^2.

iii) M is the Clifford torus $S^m\left(\sqrt{\frac{1}{4+2\sqrt{2}}}\right)\times S^{n-m}\left(\sqrt{\frac{1}{4-2\sqrt{2}}}\right)$.

For $p=3$ and $q=1$ we have:

Corollary 3.7 *The bronze shaped hypersurfaces of* $\mathbb{R}^{n+1}$ *are the hyperspheres*:

$$S^n(\frac{1}{\phi}) = S^n\left(\frac{\sqrt{13}-3}{2}\right)$$

and

$$S^n(\frac{1}{|\Phi|}) = S^n\left(\frac{\sqrt{13}+3}{2}\right).$$

Corollary 3.8 *The bronze shaped hypersurfaces of* $\mathbb{H}^{n+1}$ *are as follows*:

i) For $1 \leq k \leq n+1$, $M = \left\{x\in\mathbb{H}^{n+1} : x^k = \sqrt{\frac{3+\sqrt{13}}{6}}\right\}$, *which is isometric to the hyperbola* $H^n(\frac{3(3-\sqrt{13})}{2})$.

ii) $M = \left\{x\in\mathbb{H}^{n+1} : \left(x^1\right)^2+...+\left(x^{n+1}\right)^2 = \frac{\sqrt{13}-3}{6}\right\} = \mathbb{H}^{n+1}\cap S^n\left(\sqrt{\frac{\sqrt{13}-3}{6}}\right)$; *in this case M is isometric to* $S^n\left(\sqrt{\frac{\sqrt{13}-3}{6}}\right)$.

Corollary 3.9 *The bronze shaped hypersurfaces of* $\mathbb{S}^{n+1}$ *are as follows*:

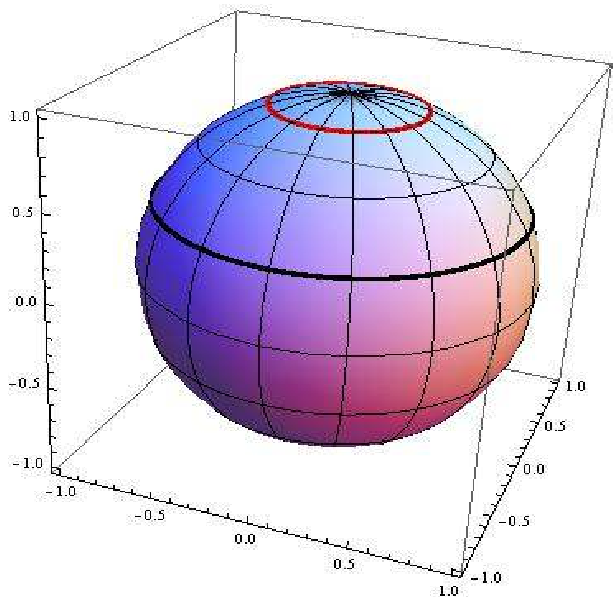

Figure 3. Silver shaped circles $S\left(\sqrt{\frac{1}{4+2\sqrt{2}}}\right)$ (red) and $S\left(\sqrt{\frac{1}{4-2\sqrt{2}}}\right)$ (black) in S^2.

i) M is umbilical and $M=\left\{x\in\mathbb{S}^{n+1}:x^{n+2}=\sqrt{\frac{11+3\sqrt{13}}{13+3\sqrt{13}}}\right\}$. In this case M is isometric to $S^n\left(\sqrt{\frac{2}{13+3\sqrt{13}}}\right)$.

ii) M is umbilical and $M=\left\{x\in\mathbb{S}^{n+1}:x^{n+2}=\sqrt{\frac{11-3\sqrt{13}}{13-3\sqrt{13}}}\right\}$. In this case M is isometric to $S^n\left(\sqrt{\frac{2}{13-3\sqrt{13}}}\right)$.

iii) M is the Clifford torus $S^m\left(\sqrt{\frac{2}{13+3\sqrt{13}}}\right)\times S^{n-m}\left(\sqrt{\frac{2}{13-3\sqrt{13}}}\right)$.

For $p=1$ and $q=2$ we can state:

Corollary 3.10 *The copper shaped hypersurfaces of $\mathbb{R}^{n+1}$ are the hyperspheres:*

$$S^n(\frac{1}{\phi})=S^n\left(\frac{1}{2}\right)$$

and

$$S^n(\frac{1}{|\Phi|})=S^n\left(1\right).$$

Corollary 3.11 *The copper shaped hypersurface of $\mathbb{H}^{n+1}$ is*

$M=\left\{x\in\mathbb{H}^{n+1}:\left(x^1\right)^2+...+\left(x^{n+1}\right)^2=\frac{1}{3}\right\}=\mathbb{H}^{n+1}\cap S^n(\frac{1}{\sqrt{3}})$; in this case M is isometric to $S^n(\frac{1}{\sqrt{3}})$.

Corollary 3.12 *The copper shaped hypersurfaces of $\mathbb{S}^{n+1}$ are as follows:*

i) M is umbilical and $M=\left\{x\in\mathbb{S}^{n+1}:x^{n+2}=\sqrt{\frac{2}{5}}\right\}$. In this case M is isometric to $S^n\left(\frac{1}{\sqrt{5}}\right)$.

ii) M is umbilical and $M=\left\{x\in\mathbb{S}^{n+1}:x^{n+2}=\frac{1}{\sqrt{2}}\right\}$. In this case M is isometric to $S^n\left(\frac{1}{\sqrt{2}}\right)$.

For $p=1$ and $q=3$ we have:

Corollary 3.13 *The nickel shaped hypersurfaces of* $\mathbb{R}^{n+1}$ *are the hyperspheres*:

$$S^n(\frac{1}{\phi}) = S^n\left(\frac{\sqrt{13}-1}{6}\right)$$

and

$$S^n(\frac{1}{|\Phi|}) = S^n\left(\frac{\sqrt{13}+1}{6}\right).$$

Corollary 3.14 *The nickel shaped hypersurfaces of* $\mathbb{H}^{n+1}$ *are as follows*:

i) $M = \left\{x \in \mathbb{H}^{n+1} : \left(x^1\right)^2 + ... + \left(x^{n+1}\right)^2 = \frac{5-\sqrt{13}}{6}\right\} = \mathbb{H}^{n+1} \cap S^n\left(\sqrt{\frac{5-\sqrt{13}}{6}}\right)$; *in this case* M *is isometric to* $S^n\left(\sqrt{\frac{5-\sqrt{13}}{6}}\right)$.

ii) $M = \left\{x \in \mathbb{H}^{n+1} : \left(x^1\right)^2 + ... + \left(x^{n+1}\right)^2 = \frac{5+\sqrt{13}}{6}\right\} = \mathbb{H}^{n+1} \cap S^n\left(\sqrt{\frac{5+\sqrt{13}}{6}}\right)$; *in this case* M *is isometric to* $S^n\left(\sqrt{\frac{5+\sqrt{13}}{6}}\right)$.

Corollary 3.15 *The nickel shaped hypersurfaces of* $\mathbb{S}^{n+1}$ *are as follows*:

i) M *is umbilical and* $M = \left\{x \in \mathbb{S}^{n+1} : x^{n+2} = \sqrt{\frac{7+\sqrt{13}}{9+\sqrt{13}}}\right\}$. *In this case* M *is isometric to* $S^n\left(\sqrt{\frac{2}{9+\sqrt{13}}}\right)$.

ii) M *is umbilical and* $M = \left\{x \in \mathbb{S}^{n+1} : x^{n+2} = \sqrt{\frac{7-\sqrt{13}}{9-\sqrt{13}}}\right\}$. *In this case* M *is isometric to* $S^n\left(\sqrt{\frac{2}{9-\sqrt{13}}}\right)$.

4. Pseudosymmetric hypersurfaces

Let (M, g) be a Riemannian manifold and R denote the Riemann–Christoffel curvature tensor field of (M, g). Then the tensor fields $R \cdot R$ and $Q(g, R)$ are defined by

$$\left(R(X,Y) \cdot R\right)(X_1, X_2, X_3, X_4) = -R(R(X,Y)X_1, X_2, X_3, X_4)$$
$$-... - R(X_1, X_2, X_3, R(X,Y)X_4)$$

and

$$Q(g,R)(X_1, X_2, X_3, X_4; X, Y) = -R((X \wedge Y)X_1, X_2, X_3, X_4)$$
$$-... - R(X_1, X_2, X_3, (X \wedge Y)X_4),$$

respectively, where $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y;$$

see [6]. A Riemannian manifold (M, g) is called *pseudosymmetric* if and only if the condition

$$R \cdot R = L_R Q(g,R)$$

holds on the set U_R defined by

$$U_R = \left\{ p \in M : \left(R - \frac{\kappa}{n(n-1)} G \right) \neq 0 \text{ at } p \right\},$$

where L_R is some function on U_R and the $(0,4)$-tensor field G is defined by

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4),$$

κ is the scalar curvature, and X, Y, X_1, X_2, X_3, X_4 are vector fields on M [6]. If L_R is a constant then (M, g) is called a *pseudosymmetric manifold of constant type* (see [6] and [12]). If $R \cdot R = 0$ then (M, g) is called *semisymmetric* [15]. For the geometrical interpretations of pseudosymmetric hypersurfaces in the Euclidean space see [16].

In [7], Deszcz et al. proved the following lemma:

Lemma 4.1 *[7] Let (M, g) be a hypersurface immersed isometrically in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 3$. If at a point $p \in U_R \subset M$, the shape operator A of M satisfies the condition*

$$A^2 = \alpha A + \beta I_d,$$

$\alpha, \beta \in \mathbb{R}$, *then the relation*

$$R \cdot R = \left(\frac{\widetilde{\kappa}}{n(n-1)} - \beta \right) Q(g, R) \tag{4.1}$$

holds at p, where $\widetilde{\kappa}$ is the scalar curvature of the ambient space $N^{n+1}(c)$.

By the use of the above lemma and Definition 2.1 we can state the following corollary:

Corollary 4.1 *Every metallic shaped hypersurface of a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 3$, is a pseudosymmetric hypersurface of constant type.*

However, from Theorem 2.1, Theorem 2.2, and Theorem 2.3, it can be easily seen that metallic shaped hypersurfaces of a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 3$, are semisymmetric. Semisymmetic isoparametric hypersurfaces were studied by Ryan in [13] (see Remark on page 379). Hence, we obtain the following result:

Corollary 4.2 *Every metallic shaped hypersurface of a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 3$, is a semisymmetric hypersurface.*

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