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r-ideals in commutative rings

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Abstract: In this article we introduce the concept of r-ideals in commutative rings (note: an ideal I of a ring R is called r-ideal, if $ab \in I$ and $\mathrm{Ann}(a) = (0)$ imply that $b \in I$ for each $a, b \in R$). We study and investigate the behavior of r-ideals and compare them with other classical ideals, such as prime and maximal ideals. We also show that some known ideals such as z°-ideals are r-ideals. It is observed that if I is an r-ideal, then so too is a minimal prime ideal of I. We naturally extend the celebrated results such as Cohen's theorem for prime ideals and the Prime Avoidance Lemma to r-ideals. Consequently, we obtain interesting new facts related to the Prime Avoidance Lemma. It is also shown that R satisfies property A (note: a ring R satisfies property A if each finitely generated ideal consisting entirely of zerodivisors has a nonzero annihilator) if and only if for every r-ideal I of R, $I[x]$ is an r-ideal in $R[x]$. Using this concept in the context of $C(X)$, we show that every r-ideal is a z°-ideal if and only if X is a ∂-space (a space in which the boundary of any zeroset is contained in a zeroset with empty interior). Finally, we observe that, although the socle of $C(X)$ is never a prime ideal in $C(X)$, the socle of any reduced ring is always an r-ideal.

Key words: r-ideal, pr-ideal, annihilator, property A, zerodivisor, uz-ring, z°-ideal, r-multiplicatively closed, almost P-space, ∂-space, socle

1. Introduction

Throughout this paper all rings are commutative with $1 \neq 0$. Let R be a ring. For $a \in R$ we define $\mathrm{Ann}_R(a) = \{r \in R : ra = 0\}$ (briefly, $\mathrm{Ann}(a)$) and a is said to be a regular (resp., zerodivisor) element if $\mathrm{Ann}(a) = (0)$ (resp., $\mathrm{Ann}(a) \neq (0)$). aR denotes the principal ideal generated by $a \in R$. If S is a subset of R and I is an ideal of R, then we define $(I : S) = \{a \in R : aS \subseteq I\}$, clearly $(0 : S) = \mathrm{Ann}(S)$. By $\mathrm{r}(R)$, $\mathrm{zd}(R)$, and $\mathrm{u}(R)$ we mean the set of all regular elements, zerodivisor elements, and unit elements of R, respectively. An ideal I of R is called a regular ideal if it contains at least a regular element, i.e. $I \cap \mathrm{r}(R) \neq \emptyset$. If I is an ideal of R, then $\mathrm{Min}(I)$ denotes the set of all minimal prime ideals of I and we use $\mathrm{Min}(R)$ instead of $\mathrm{Min}((0))$. Similarly, $\mathrm{Max}(R)$ (resp., $\mathrm{Spec}(R)$) denotes the set of all maximal (resp., prime) ideals of R. For each $a \in R$, P_a (resp., M_a) is the intersection of all minimal prime (resp., maximal) ideals containing a. We use $\mathrm{rad}(R)$ (resp., $\mathrm{Jac}(R)$) instead of P_0 (resp., M_0). A proper ideal I of R is called a z°-ideal (resp., z-ideal) if for each $a \in I$ we have $P_a \subseteq I$ (resp., $M_a \subseteq I$). Equivalently, I is a z°-ideal if $a \in I$, $b \in R$, and $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ imply that $b \in I$. For more information about the aforementioned ideals in general commutative rings we refer the reader to [[2], [8], [26]]. If S is a subset of R, then an element $a \in S$ is called a

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von Neumann regular element if there exists $b \in S$ such that $a = a^2b$. Whenever we say a ring R or a subset of R is von Neumann regular, it means that all of their elements are von Neumann regular. An ideal I in a ring R is called a pure ideal if for each $a \in I$ there exists $b \in I$ such that $a = ab$. Let us also recall the following properties: A ring R satisfies a) property A: if each finitely generated (briefly, $f.g.$) ideal $I \subseteq \mathrm{zd}(R)$ has nonzero annihilator; b) annihilator condition (briefly, a.c.): if for each $f.g.$ ideal I of R there exists an element $b \in R$ with $\mathrm{Ann}(I) = \mathrm{Ann}(b)$; c) strong annihilator condition (briefly, s.a.c.): if for each $f.g.$ ideal I of R there exists an element $b \in I$ with $\mathrm{Ann}(I) = \mathrm{Ann}(b)$. We refer the reader to [[1], [2], [18], [25]] for the necessary background about the above concepts.

Let $C(X)$ (resp., $C^*(X)$) be the ring of (resp., bounded) real valued continuous functions on a Tychonoff space X. If $f \in C(X)$, then $Z(f) = \{x \in X : f(x) = 0\}$ is the zeroset of f and by $\mathrm{int}Z(f)$ we mean the interior of $Z(f)$. Recall that an ideal I of $C(X)$ is a z-ideal if $f \in I$, $g \in C(X)$, and $Z(f) = Z(g)$ imply that $g \in I$. It is known that if $f, g \in C(X)$, then $\mathrm{int}Z(f) = \mathrm{int}Z(g)$ if and only if $\mathrm{Ann}(f) = \mathrm{Ann}(g)$; see [[5]]. Hence, an ideal I in $C(X)$ is a z°-ideal if $f \in I$, $g \in C(X)$ and $\mathrm{int}Z(f) = \mathrm{int}Z(g)$ imply that $g \in I$; see [[7], [9]]. For more information about the ideals in $C(X)$, see [[7], [10], [12], [16]], and for details about topological spaces, see [[14], [16]].

In Section 2, we introduce r-ideals and pr-ideals in general commutative rings. It is shown that every z°-ideal is an r-ideal, and if I is an r-ideal of R and $P \in \mathrm{Min}(I)$, then P is an r-ideal, too. We also show in this section that the socle of every reduced ring is an r-ideal. In Section 3, we investigate the relations between r-ideals and prime ideals. We observe that every maximal r-ideal in a ring is a prime ideal. We show that in order for every prime r-ideal of a ring R to be minimal prime, it is necessary and sufficient that the classical ring of quotients of R be a von Neumann regular ring. Finally, we naturally extend the celebrated results such as Cohen's theorem for prime ideals and the Prime Avoidance Lemma to r-ideals. In Section 4, we observe that whenever I is an ideal of a ring R and $I[x]$ is an r-ideal, then trivially I is also an r-ideal, but the converse may not be true. In this section, we prove a ring R satisfies property A if and only if for every r-ideal I of R, $I[x]$ is an r-ideal in $R[x]$. Section 5 is devoted to the investigation of r-ideals in $C(X)$. We show that every r-ideal is a z°-ideal if and only if X is a ∂-space. It is observed that every ideal in $C(X)$ is an r-ideal if and only if X is an almost P-space. Using some appropriate facts in $C(X)$, we answer some natural questions in general. By giving several examples, we compare and contrast r-ideals with some well-known ideals, such as z-ideals and z°-ideals.

2. r-ideals

Our aim in this section is to study the r-ideals in commutative rings. We begin with the following definition.

Definition 2.1 *A proper ideal I in a ring R is called an r-ideal (resp., pr-ideal), if $ab \in I$ with $\mathrm{Ann}(a) = (0)$ implies that $b \in I$ (resp., $b^n \in I$, for some $n \in \mathbb{N}$), for each $a, b \in R$.*

Let I be an ideal of R and S be a multiplicatively closed (briefly, m.c.) subset in R. Clearly, $I_S = \{x \in R : sx \in I \text{ for some } s \in S\}$ is an ideal of R containing I. Now we call an ideal I an s-ideal if $I = I_S$, for some m.c. subset S of R. In case $S = \mathrm{r}(R)$, each s-ideal is an r-ideal. Recall that if $S = \mathrm{r}(R)$, then the ring $S^{-1}R$ is called the classical ring of quotients of R, which is denoted by $Q(R)$. Let $\varphi : R \to Q(R)$ be the natural homomorphism. For each ideal $\mathcal{J}$ in $Q(R)$, we put $\varphi^{-1}[\mathcal{J}] = \mathcal{J}^c$. Clearly, $\mathcal{J}^c$ is an ideal of R and it is called the contraction of $\mathcal{J}$ in R. For the details of the concept of contraction, see [[3]].

Proposition 2.2 *Let R be a ring and I be an ideal of R. Then the following statements are equivalent:*

a) I is an r-ideal.

b) $rR \cap I = rI$, for each $r \in \mathrm{r}(R)$.

c) $I = (I : r)$, for each $r \in \mathrm{r}(R) \setminus I$.

d) $I = \mathcal{J}^c$, where $\mathcal{J}$ is an ideal in $Q(R)$.

Proof It is evident. □

Recall that part (c) of the previous proposition is similar to this statement about prime ideals, which says that a proper ideal P of a ring R is prime if and only if $P = (P : a)$, for each $a \in R \setminus P$. We should remind the reader that part (b) of the previous proposition may not be true if I is a prime ideal. The reason that part (b) is valid for an r-ideal I is the fact $I \cap \mathrm{r}(R) = \emptyset$; this immediately implies that part (b) is trivially true for prime ideal P with $P \cap \mathrm{r}(R) = \emptyset$.

We observe several elementary properties concerning r-ideals in any ring R as follows:

Remark 2.3 *a) If $f : R \to S$ is an isomorphism, then $f[I]$ is an r-ideal in S whenever I is an r-ideal in R, and $f^{-1}[J]$ is an r-ideal in R whenever J is an r-ideal in S.*

b) The zero ideal is an r-ideal.

c) The intersection of any family of r-ideals is an r-ideal.

d) If I is an r-ideal, then $I \subseteq \mathrm{zd}(R)$.

e) Every r-ideal is a pr-ideal.

f) A prime ideal is an r-ideal if and only if it consists entirely of zerodivisors. Consequently, every minimal prime ideal is an r-ideal.

g) If I is an r-ideal, $S \subseteq R$ and $S \nsubseteq I$, then $(I : S)$ is an r-ideal. In particular, $\mathrm{Ann}(S)$ is always an r-ideal.

h) It is well known that if I is a minimal ideal of a reduced ring R, and then $I = eR = \mathrm{Ann}(1 - e)$, where $e \in R$ is an idempotent element, i.e. $e^2 = e$. Hence, by part (g), every minimal ideal in a reduced ring is an r-ideal.

i) Every pure ideal and also every von Neumann regular ideal is an r-ideal.

j) If R satisfies the s.a.c., and I is an ideal of R, then I is an r-ideal if and only if for every ideal of J and K of R, whenever $JK \subseteq I$ and $\mathrm{Ann}(J) = (0)$, then $K \subseteq I$.

*k) The product of two r-ideals is not necessarily an r-ideal; see Example [**?**].*

*l) The sum of two r-ideal is not necessarily an r-ideal; see Example [**?**].*

Remark 2.4 *It is well known that $\mathcal{I}^c\mathcal{J}^c \subseteq (\mathcal{I}\mathcal{J})^c$ and $\mathcal{I}^c + \mathcal{J}^c \subseteq (\mathcal{I} + \mathcal{J})^c$, where $\mathcal{I}$ and $\mathcal{J}$ are ideals of $Q(R)$. Now suppose that I and J are r-ideals of R; hence, by part (d) of Proposition [**?**], $I = \mathcal{I}^c$ and $J = \mathcal{J}^c$, for some ideals $\mathcal{I}$ and $\mathcal{J}$ in $Q(R)$. One can easily show that:*

a) IJ is an r-ideal in R if and only if $(\mathcal{I}\mathcal{J})^c \subseteq \mathcal{I}^c\mathcal{J}^c$ (in fact, $(\mathcal{I}\mathcal{J})^c = \mathcal{I}^c\mathcal{J}^c$).

b) $I + J$ is an r-ideal in R if and only if $(\mathcal{I} + \mathcal{J})^c \subseteq \mathcal{I}^c + \mathcal{J}^c$ (in fact, $(\mathcal{I} + \mathcal{J})^c = \mathcal{I}^c + \mathcal{J}^c$).

We need the following lemma in the sequel.

Lemma 2.5 *Let R be a ring and I be an ideal of R. Then:*

a) I is an r-ideal if and only if whenever J and K are ideals of R with $J \cap \mathrm{r}(R) \neq \emptyset$ and $JK \subseteq I$, then $K \subseteq I$.

b) If $I \subseteq \mathrm{zd}(R)$ is not an r-ideal, then there exist ideals J and K such that $J \cap \mathrm{r}(R) \neq \emptyset$, $I \subsetneq J, K$, and $JK \subseteq I$.

Proof a) It is evident.
b) Suppose that I is not an r-ideal. Then there exist $r \in \mathrm{r}(R)$, $x \in R$ with $rx \in I$ but $x \notin I$. Now put $J = (I : x)$ and $K = (I : J)$. Clearly, $r \in J \setminus I$, $J \cap \mathrm{r}(R) \neq \emptyset$, $x \in K \setminus I$, and $JK \subseteq I$. □

The proof of the following result is evident by the above lemma.

Proposition 2.6 *a) Let R be a ring and I be an ideal of R with $I \cap \mathrm{r}(R) \neq \emptyset$. If J and K are r-ideals of R such that $IJ = IK$ or $I \cap J = I \cap K$, then $J = K$.*

b) Let R be a ring and I and J be ideals of R with $J \cap \mathrm{r}(R) \neq \emptyset$. If IJ is an r-ideal of R, then $I = IJ$. In particular, I is an r-ideal.

In Remark [**?**], we observe that an intersection of r-ideals is an r-ideal. In the following proposition we show that the converse is also true for prime ideals in the finite case. The result may not be true for an infinite number of primes; take the intersection of nonzero prime ideals in $\mathbb{Z}$.

Proposition 2.7 *Suppose that $P_1, \cdots, P_n$ are prime ideals in a ring R, which are not comparable. If $\bigcap_{i=1}^n P_i$ is an r-ideal, then P_i is an r-ideal, for $i = 1, \cdots, n$.*

Proof Let $rx \in P_i$ with $\mathrm{Ann}(r) = (0)$ and take $y \in (\prod_{j \neq i} P_j) \setminus P_i$. Hence, $rxy \in \bigcap_{i=1}^n P_i$. Since $\bigcap_{i=1}^n P_i$ is an r-ideal, we infer that $xy \in \bigcap_{i=1}^n P_i$, and therefore $xy \in P_i$. This implies that $x \in P_i$, i.e., P_i is an r-ideal. □

It is well known that a ring R is a field if and only if $I = (0)$ is the only maximal ideal of R. However, we cannot extend this to domains by claiming that R is a domain if and only if $I = (0)$ is its only prime ideal. By trading off the prime ideals with the r-ideals, we get the next interesting fact.

Proposition 2.8 *Let R be a ring. Then the following statements are equivalent:*

a) R is a domain.

b) The zero ideal is the only r-ideal of R.

c) $\mathrm{Ann}(ab) = \mathrm{Ann}(a) \cup \mathrm{Ann}(b)$, for every $a, b \in R$.

Proof $(a \Rightarrow b)$ Let R be a domain and $(0) \neq I$ be a proper ideal of R. Hence, there exists $0 \neq a \in I$. By our hypothesis, we have $\mathrm{Ann}(a) = (0)$, so I is not an r-ideal (note: otherwise $1 \in I$, which is absurd).

$(b \Rightarrow c)$ We know that $\mathrm{Ann}(x)$ is an r-ideal, for each $0 \neq x \in R$. Hence, by our hypothesis, we have $\mathrm{Ann}(x) = (0)$, for each $0 \neq x \in R$. This immediately implies that $\mathrm{Ann}(ab) = \mathrm{Ann}(a) \cup \mathrm{Ann}(b)$, for each $a, b \in R$.

$(c \Rightarrow a)$ Let $ab = 0$, where $a, b \in R$. Then $R = \mathrm{Ann}(ab) = \mathrm{Ann}(a) \cup \mathrm{Ann}(b)$ implies that $1 \in \mathrm{Ann}(a) \cup \mathrm{Ann}(b)$. This means that $a = 0$ or $b = 0$, i.e. R is a domain. □

Remark 2.9 *We should remind the reader that part (d) of Proposition [**?**] is quite natural with regard to some known facts. For example, if Q is the quotient field of a domain R, the zero ideal of R, which is the only r-ideal of R, is the contraction of the only proper ideal of Q (i.e. (0)). We also note that whenever P is a* *prime ideal in a ring R and $S = R \setminus P$, then each prime ideal of $S^{-1}R$ is contracted to a prime ideal of R. Finally, if in a ring R, we take $S = \mathrm{r}(R)$, then the contractions of all proper ideals of $Q(R)$ are naturally r-ideals in R (note: proper ideals of $Q(R)$ are all r-ideals).*

In Example [**?**], we will observe that the sum of two r-ideals need not be an r-ideal. In the following result we show that the sum of two special annihilator ideals of a ring and also the sum of a minimal prime ideal and an annihilator ideal in a reduced ring are r-ideal.

Proposition 2.10 *a) Let R be a ring and $a, b \in R$ with $a + b = 1$. Then $I = \mathrm{Ann}(a) + \mathrm{Ann}(b)$ is an r-ideal.*

b) Let R be a reduced ring, $P \in \mathrm{Min}(R)$ and $e \in R$ be an idempotent element. Then $I = P + \mathrm{Ann}(e)$ is an r-ideal.

Proof a) Suppose that $xy \in I$ and $\mathrm{Ann}(x) = (0)$. Hence, there exist $r \in \mathrm{Ann}(a)$ and $s \in \mathrm{Ann}(b)$ such that $xy = r + s$. Clearly, $xyab = 0$, and since $\mathrm{Ann}(x) = (0)$, we infer that $yab = 0$. Consequently, $ya \in \mathrm{Ann}(b)$ and $yb \in \mathrm{Ann}(a)$. Therefore, $y = y(a + b) = ya + yb$, i.e., $y \in I$.

b) Let $rx \in I$ with $\mathrm{Ann}(r) = (0)$ and $x \in R$. Hence, $rx = a + b$, where $a \in P$ and $be = 0$. Clearly, there exists $y \notin P$ such that $ay = 0$. Therefore, $eyrx = 0$, we have $eyx = 0$, and hence $ex \in P$. Now $x = ex + (1 - e)x \in P + \mathrm{Ann}(e) = I$, and therefore I is an r-ideal. □

If in the equality $a + b = 1$ of part (a) of the previous proposition, we replace 1 by R and a, b by two subsets A, B in R, then $\mathrm{Ann}(A) + \mathrm{Ann}(B)$ will be also an r-ideal.

In general, if R is a ring such that every ideal of R is an annihilator ideal (i.e. for every ideal I there exists $S \subseteq R$ such that $I = \mathrm{Ann}(S)$), then every ideal of R is an r-ideal. Also, if for any two ideals I and J in the ring R, there exists an ideal K such that $\mathrm{Ann}(I) + \mathrm{Ann}(J) = \mathrm{Ann}(K)$, then $\mathrm{Ann}(I) + \mathrm{Ann}(J)$ is an r-ideal. We should remind the reader that the latter case may happen in certain rings. In what follows we mention some examples. We recall that if X is an extremally disconnected space (i.e. every open subset of X has an open closure), then $C(X)$ has the above property; see [[6]]. In [[11]], the concepts of SA-ring and IN-ring are introduced and it is shown that these rings also satisfy the above property. We should also emphasize that in contrast with the latter fact the sum of two r-ideals is not necessarily an r-ideal in general; we refer the reader to Example 5.14 in this regard. However, it is worthwhile to remind the reader that any direct summand of an r-ideal is always an r-ideal (i.e. if $I = J \oplus K$, and I is an r-ideal, then so too are J and K).

Remark 2.11 *In contrast to the latter fact the summand of prime ideals may not be prime. To see this, take a von Neumann regular ring that is not a finite direct product of fields, and then take a prime ideal P that is not f.g. (note: von Neumann regular rings that are not a finite direct product of fields cannot be Noetherian; hence, by Cohen's theorem, it contains a prime ideal that is not f.g.), and notice that all of its f.g. subideals are direct summands, which are not prime ideal.*

Recall that the socle of a ring R, which is denoted by $\mathrm{soc}(R)$, is the sum of all minimal ideals of R. We also recall that the socle of a reduced ring R is of the form $\mathrm{soc}(R) = \oplus_{i \in A} e_i R$, where $\{e_i : i \in A\}$ is the set of idempotents of R; see [[23]]. By the following proposition we observe that the sum of principal ideals generated by idempotents is an r-ideal, from which the socle of a reduced ring is an r-ideal. We know that the socle plays an important role in the structure theory of rings, especially in the context of noncommutative rings and

$C(X)$. For details about the socle in general rings, see [[23]], and for a topological characterization of the socle of $C(X)$, see [[22]].

Proposition 2.12 *Let R be a ring, and $\{e_i : i \in A\}$ is a set of idempotents of R. Then $I = \sum_{i\in A} e_i R$ is an r-ideal.*

Proof Let $rx \in I$, where $x \in R$ and $\mathrm{Ann}(r) = (0)$. We are to show that $x \in I$. Since $I = \sum_{i\in A} e_i R$, we infer that $rx = \sum_{k=1}^{n} e_{i_k} r_{i_k}$ for some $i_1, \cdots, i_n \in A$ and $r_{i_1}, \cdots, r_{i_k} \in R$. Let us put $y = \prod_{k=1}^{n}(1 - e_{i_k})$. It is manifest that $rxy = 0$, and hence $xy = 0$. On the other hand, there exists $s \in I$ such that $y = 1 - s$. Therefore, $x(1 - s) = 0$, so $x = xs \in I$. □

Corollary 2.13 *Let R be a reduced ring. Then $\mathrm{soc}(R)$ is an r-ideal. In particular, there exists an ideal $\mathcal{J}$ of $Q(R)$ such that $\mathrm{soc}(R) = \mathcal{J}^c$.*

It is interesting that in $C(X)$, where X is an infinite topological space, the socle of $C(X)$ is an r-ideal that is not prime; see [[4], [15]].

Remark 2.14 *Let M be a projective R-module, where R is a von Neumann regular ring. Then M is isomorphic to a direct sum of countably generated r-ideals. To see this, we note that by a celebrated theorem of Kaplansky $M = \oplus_{i\in A} M_i$, where each M_i is a countably generated submodule of M. Since M is a regular module (i.e. each cyclic submodule of M is a direct summand), we infer that each $M_i = \oplus_{n=1}^{\infty} x_n R$ is regular too. Hence, by [[[20]], Lemma 2], we conclude that $M_i \cong \oplus_{n=1}^{\infty} e_n R$, where each e_n is idempotent. Now by Proposition [**?**], each M_i is isomorphic to an r-ideal, and we are done.*

We recall that in the ring $C(X)$, the sum of two minimal prime ideals is either a prime ideal or all of $C(X)$; see [[16]]. In contrast to this fact, the sum of two minimal prime ideals in general is not necessarily an r-ideal; see also the next example.

Example 2.15 *Let $R = \frac{F[x,y]}{xyF[x,y]}$, where F is a field. Then $P = \frac{xF[x,y]}{xyF[x,y]}$ and $Q = \frac{yF[x,y]}{xyF[x,y]}$ are minimal prime ideals of R. Clearly, $P + Q \neq R$ and $(x + y) + xyF[x, y] \in P + Q$ is a regular element. Hence, $P + Q$ is not an r-ideal.*

The following is a counterpart of the well-known fact that Q is a primary ideal of a ring R if and only if $\sqrt{Q}$ is a prime ideal.

Proposition 2.16 *Let R be a ring and I be an ideal of R. Then I is a pr-ideal if and only if $\sqrt{I}$ is an r-ideal.*

Proof Suppose that I is a pr-ideal and $ab \in \sqrt{I}$ with $\mathrm{Ann}(a) = (0)$. Then there exists $n \in \mathbb{N}$ such that $a^n b^n \in I$. Clearly, $\mathrm{Ann}(a^n) = (0)$, so there exists $m \in \mathbb{N}$ such that $b^{nm} \in I$ and therefore $b \in \sqrt{I}$. Conversely, we assume that $ab \in I$ with $\mathrm{Ann}(a) = (0)$. Since $ab \in \sqrt{I}$ we infer that $b \in \sqrt{I}$ and so there exists $n \in \mathbb{N}$ such that $b^n \in I$. □

As we observed in the previous proposition, whenever $\sqrt{I}$ is an r-ideal, then I is an pr-ideal. In the following example, we show that $\sqrt{I}$ may be an r-ideal where I may not be an r-ideal. This example also shows that a pr-ideal is not necessarily an r-ideal.

Example 2.17 *Let S be a reduced ring with subring $\mathbb{Z}$ and $P \neq (0)$ be a minimal prime ideal in S with $P \cap \mathbb{Z} = (0)$. By [[[10]], Lemma 3.6], $Q = xP[x] \subseteq S[x]$ is a minimal prime ideal in $R = \mathbb{Z} + xS[x]$, and hence it is also an r-ideal. Now we consider $Q_n = x^n P[x]$ with $1 \neq n \in \mathbb{N}$. Clearly $\sqrt{Q_n} = Q$ is an r-ideal and by Proposition [?] we conclude that Q_n is a pr-ideal. We claim that Q_n is not an r-ideal. To see this, put $f(x) = x^{n-1}a$, where $0 \neq a \in P$ and $g(x) = x$. Thus, $f(x)g(x) = x^n a \in Q_n$. Now it is clear that* $\mathrm{Ann}(g) = (0)$ *and $f \notin Q_n$. Consequently, Q_n is not an r-ideal.*

Clearly, if I and J are r-ideals in a ring R, then IJ is a pr-ideal of R, but it may not be an r-ideal; for instance, in the previous example, the ideal Q is an r-ideal, while Q^2 is not an r-ideal (note: for a prime ideal P, P^2 is prime if and only if $P^2 = P$).

Using the previous proposition and Proposition [**?**], we have the next corollary.

Corollary 2.18 *Let R be a ring and I be an ideal of R. Then the following statements are equivalent:*

a) I is a pr-ideal.

b) $rR \cap \sqrt{I} = r\sqrt{I}$, for any $r \in \mathrm{r}(R)$.

c) $\sqrt{I} = \sqrt{(I : r)}$, for any $r \in \mathrm{r}(R) \setminus I$.

d) $I = \mathcal{J}^c$, where $\mathcal{J}$ is a primary ideal in $Q(R)$.

In the next section we will show that an r-ideal is not necessarily a z°-ideal; see part (d) of Remark [**?**]. In the following theorem, however, we observe that the converse holds.

Theorem 2.19 *a) Every z°-ideal in a ring R is an r-ideal.*

b) Every ideal consisting entirely of zerodivisors in a ring is contained in a prime r-ideal.

Proof a) Let I be a z°-ideal, $ab \in I$ and $\mathrm{Ann}(a) = (0)$. Clearly, $\mathrm{Ann}(b) = \mathrm{Ann}(ab)$. Since I is a z°-ideal, we conclude that $b \in I$.

b) It is evident. □

Let S be a m.c. subset of a reduced ring R. Clearly, $I = \sum_{a \in S} \mathrm{Ann}(a)$ is a z°-ideal, so by part (a) of the previous theorem, I is also an r-ideal.

We remind the reader that if I is a z°-ideal (resp., z-ideal) and $P \in \mathrm{Min}(I)$, then P is a z°-ideal (resp., z-ideal); see [[[8]], Theorem 1.16] (resp., see [[10], [26]]). The following is a similar result.

Theorem 2.20 *Let R be a ring and $P \in \mathrm{Min}(I)$, where I is an r-ideal of R. Then P is an r-ideal.*

Proof Suppose that $ab \in P$ and $\mathrm{Ann}(a) = (0)$. By [[[18]], Theorem 1.2], there exist $x \notin P$ and $n \in \mathbb{N}$ such that $x(ab)^n = xa^n b^n \in I$. Since $\mathrm{Ann}(a^n) = (0)$ and I is an r-ideal, we infer that $xb^n \in I \subseteq P$. Since $x \notin P$, we infer that $b^n \in P$ and therefore $b \in P$. □

We conclude this section with the following example and the proposition that follows it.

Example 2.21 *For two r-ideals I and J of R, with $J \supseteq I$, the ideal $\frac{J}{I}$ of $\frac{R}{I}$ may not be an r-ideal in $\frac{R}{I}$. To see this, suppose that $P \in \mathrm{Min}(R)$ and $M \in \mathrm{Max}(R)$ such that $P \subsetneqq M \subseteq \mathrm{zd}(R)$; for maximal ideals of this kind, see [[8]]. Clearly, P and M are r-ideals of R. However, $(0) \neq \frac{M}{P}$ and $\frac{R}{P}$ is a domain, so $\frac{M}{P}$ is not an r-ideal of $\frac{R}{P}$.*

Proposition 2.22 *Let I be an r-ideal in R contained in ideal J. If $\frac{J}{I}$ is an r-ideal in $\frac{R}{I}$, then J is also an r-ideal in R.*

Proof It is evident. □

3. r-ideals vs. prime ideals

This section is devoted to the relations between r-ideals and prime ideals and natural extensions of Cohen's theorem and the Prime Avoidance Lemma for r-ideals. We start with the following proposition.

Proposition 3.1 *Let R be a ring. Then every maximal r-ideal of R is a prime ideal.*

Proof Suppose that P is a maximal r-ideal of R, $xy \in P$ and $x \notin P$, and we are to show that $y \in P$. Clearly, $(P : x)$ is an r-ideal, $P \subseteq (P : x)$ and $y \in (P : x)$. Now by the maximality of P we have $P = (P : x)$. This implies that $y \in P$. □

Using [[[8]], Corollary 1.22], every maximal ideal consisting entirely of zerodivisors in a reduced ring with property A is a z°-ideal. In the following proposition we show that maximal r-ideals in reduced rings with property A are also z°-ideals.

Proposition 3.2 *Let R be a reduced ring with property A. Then every maximal r-ideal of R is a z°-ideal.*

Proof Suppose that P is a maximal r-ideal of R. Therefore, $P \subseteq \mathrm{zd}(R)$, and so by [[[8]], Proposition 1.21], there is a z°-ideal J such that $P \subseteq J$. By part (a) of Theorem [**?**], J is an r-ideal. Now the maximality of P implies that $P = J$. Hence, P is a z°-ideal. □

Recall that a nonzero ideal I in a ring R is called essential if for every nonzero ideal J of R we have $I \cap J \neq (0)$.

Proposition 3.3 *Let I be a nonzero r-ideal of a reduced ring R, which is not essential. Then there is a minimal prime ideal P containing I, which is a maximal r-ideal.*

Proof Since I is not an essential ideal, there is a nonzero ideal J of R such that $I \cap J = (0)$. Since R is reduced and $(0) \neq J$, we infer that there exists $P \in \mathrm{Min}(R)$ such that $J \nsubseteq P$ and hence there exists $x \in J \setminus P$. On the other hand, by Zorn's Lemma, there exists a maximal r-ideal N containing I such that $N \cap J = (0)$. Hence, $JN = (0)$; that is to say, $xN = (0) \subseteq P$. Now we conclude that $N \subseteq P$ and so $I \subseteq N = P$. (Note that N is a prime ideal by Proposition [**?**].) □

It is well known that every element of $Q(R)$ is either a unit or a zerodivisor. Motivated by this fact, we call a ring R a uz-ring if every element of R is either a unit or a zerodivisor. In this case, clearly $R = Q(R)$. For example, every von Neumann regular ring and any Artinian ring is a uz-ring. If R is a domain, then obviously R is a field if and only if R is a uz-ring. Clearly, a ring R is a field if and only if every ideal in R is prime. Similarly, R is a uz-ring if and only if every ideal in R is an r-ideal. More generally, we have the following result.

Proposition 3.4 *For any ring R the following statements are equivalent:*

a) R is a uz-ring.

b) Every essential ideal of R is an r-ideal.

c) Every principal ideal of R is an r-ideal.

d) Every prime ideal of R is an r-ideal.

e) Every maximal ideal of R is an r-ideal.

Proof It is evident. $\square$

The proof of the next result is similar to the proof of [[[8]], Proposition 1.26].

Proposition 3.5 *Let R be a reduced ring. Then $Q(R)$ is a von Neumann regular ring if and only if every prime r-ideal of R is a minimal prime ideal.*

Proof Let $Q(R)$ be a von Neumann regular ring and P be a prime r-ideal of R that is not minimal prime, and seek a contradiction. Therefore, there exists $a \in P$ such that $\mathrm{Ann}_R(a) \subseteq P$. Hence, $\frac{a}{1} \in S^{-1}P$ and $\mathrm{Ann}_{Q(R)}(\frac{a}{1}) \subseteq S^{-1}P$. We conclude that $S^{-1}P \notin \mathrm{Min}(Q(R))$, which is a contradiction. Conversely, since R is reduced, by a well-known theorem of Kaplansky on characterization of von Neumann regular rings, it suffices to show that each prime ideal is a minimal prime ideal. To see this, we prove in fact that each maximal ideal is a minimal prime ideal. Let $\mathcal{M} \in \mathrm{Max}(Q(R))$; since $Q(R)$ is a uz-ring, we have $\mathcal{M} \subseteq \mathrm{zd}(Q(R))$, so $\mathcal{M}$ is a z°-ideal of $Q(R)$. Hence, $\mathcal{M}^c = \mathcal{M} \cap R$ is a prime z°-ideal of R and so it is a prime r-ideal of R, too. Now by our hypothesis we conclude that $\mathcal{M}^c \in \mathrm{Min}(R)$. Therefore, $\mathcal{M} \in \mathrm{Min}(Q(R))$. This implies that $Q(R)$ is a von Neumann regular ring. $\square$

In the following result we characterize the regularity of $Q(R)$ in terms of r-ideals of R. Recall that an ideal I is semiprime if $\sqrt{I} = I$.

Proposition 3.6 *Let R be a ring. Then:*

a) $Q(R)$ is a von Neumann regular ring if and only if every r-ideal of R is a semiprime ideal.

b) If $IJ = I \cap J$, where I and J are r-ideals of R, then $Q(R)$ is a von Neumann regular ring.

c) If every r-ideal of R is idempotent, then $Q(R)$ is a von Neumann regular ring.

Proof It is evident. $\square$

The following proposition is a counterpart of the celebrated Prime Avoidance Lemma for r-ideals; see [[21]] for recent work on this lemma. First we need the next definition.

Definition 3.7 *Let $B \subseteq \bigcup_{i\in I} A_i$, where B, A_is are subsets of a ring R. This inclusion is called irreducible if no A_i can be removed from the union.*

Theorem 3.8 *Let $I \subseteq \bigcup_{i=1}^n J_i$, where I and J_is are ideals of a ring R, be an irreducible inclusion. If J_1 is an r-ideal and the others have regular elements, then $I \subseteq J_1$.*

Proof Since $I \nsubseteq \bigcup_{i=2}^n J_i$, there exists $a \in I \setminus \bigcup_{i=2}^n J_i$. This implies that $a \in J_1$. Let $x \in I \bigcap (\bigcap_{i=2}^n J_i)$; clearly $x + a \notin \bigcup_{i=2}^n J_i$. Since $x + a \in I \subseteq \bigcup_{i=1}^n J_i$, we infer that $x \in J_1$. This implies that $I \bigcap (\bigcap_{i=2}^n J_i) \subseteq J_1$ and hence $I(\prod_{i=2}^n J_i) \subseteq J_1$. Since $(\prod_{i=2}^n J_i) \cap \mathrm{r}(R) \neq \emptyset$, by part (a) of Lemma [**?**], we conclude that $I \subseteq J_1$. $\square$

The following fact is an interesting variant of the Prime Avoidance Lemma.

Corollary 3.9 *Let $Q \subseteq \bigcup_{i=1}^n P_i$, where Q and P_is are ideals of a ring R, be an irreducible inclusion. If $P_1 \in \mathrm{Min}(R)$ and $P_i \cap \mathrm{r}(R) \neq \emptyset$, for all $i \geqslant 2$, then $Q \subseteq P_1$. Moreover, if Q is a prime ideal, then $Q = P_1$, i.e. $Q \in \mathrm{Min}(R)$.*

Proposition 3.10 *Let R be a reduced ring with $|\mathrm{Min}(R)| < \infty$ and $Q \subseteq \bigcup_{i=1}^{n} P_i$, where Q and P_i s are ideals of the ring R, be an irreducible inclusion. If $P_1 \in \mathrm{Min}(R)$ and P_i is an essential ideal for all $i \geqslant 2$, then $Q \subseteq P_1$. Moreover, if Q is a prime ideal, then $Q = P_1$, i.e. $Q \in \mathrm{Min}(R)$.*

Proof Since R is a Goldie ring (see [[[23]], Theorem 11.43]), we infer that each P_i contains a regular element for all $i \geqslant 2$; see [[[23]], Theorem 11.46]. Consequently, by the above corollary we are done. □

Definition 3.11 *Let R be a ring and S be a subset of R. We say that S is an r-multiplicatively closed (briefly, r-m.c.) set if $0 \notin S$, $1 \in S$, S contains at least a regular element $t \neq 1$, and $rx \in S$ for all regular elements $r \in S$ and all $x \in S$ (e.g., $S = R \setminus I$, where I is an r-ideal).*

We remind the reader that if S is a m.c. subset, then $S' = S \cup \mathrm{u}(R) \cup \{ux : u \in \mathrm{u}(R),\ x \in S\}$ is a m.c. subset containing all units. Clearly, if I is an ideal, then $I \cap S = \emptyset$ if and only if $I \cap S' = \emptyset$. Hence, for all practical purposes we may assume that whenever S is a m.c. subset, then $\mathrm{u}(R) \subseteq S$. Note that P is a prime ideal if and only if $S = R \setminus P$ is a m.c. set.

Similarly, let S be an r-m.c. subset and A be a m.c. subset containing a regular element (e.g., $A = \{r^n : n = 0, 1, 2, ...\}$, where $r \in \mathrm{r}(R)$); then $S' = S \cup A \cup \{ax : a \in A,\ x \in S\}$ is an r-m.c. subset. In particular, we may take A to be $\mathrm{r}(R)$. Hence, from now on we may assume that whenever S is an r-m.c. subset, then $\mathrm{r}(R) \subseteq S$ (note: if I is an r-ideal, then $S = R \setminus I$ naturally contains $\mathrm{r}(R)$). Therefore, I is an r-ideal of R if and only if $S = R \setminus I$ is an r-m.c. subset.

The following theorem is the counterpart of the celebrated theorem of IS Cohen for r-ideals.

Theorem 3.12 *Let I be an ideal of a ring R and S be an r-m.c. subset in R with $I \cap S = \emptyset$. Then there exists an r-ideal J such that $I \subseteq J$ and $J \cap S = \emptyset$.*

Proof Put $\mathcal{A} = \{K : K \text{ is an ideal of } R \text{ such that } I \subseteq K \text{ and } K \cap S = \emptyset\}$. Clearly, $\mathcal{A} \neq \emptyset$, and by Zorn's Lemma, $\mathcal{A}$ has a maximal element, namely J, with $I \subseteq J$ and $J \cap S = \emptyset$. We now claim that J is an r-ideal. Let $rx \in J$, $\mathrm{Ann}_(r) = (0)$, and $x \notin J$. We are to seek a contradiction. Clearly, $x \in (J : r)$ and so $J \subsetneqq (J : r)$. Now it is sufficient to show that $(J : r) \cap S = \emptyset$. To see this, let $t \in (J : r) \cap S$, and then $t \in S$ and $rt \in J$. Since $r \in \mathrm{r}(R) \subseteq S$, we infer that $rt \in S$, i.e. $rt \in J \cap S$, which is a contradiction. □

Definition 3.13 *Let S be a subset of a ring R. We say that S is an r-saturated m.c. subset if S is an r-m.c. Subset, and moreover, when $xy \in S$, then $x, y \in S$ for every $x, y \in R$.*

We should bring to the attention of the reader that whenever $\mathcal{A}$ is a set of r-ideals, then clearly $S = R \setminus \bigcup_{I \in \mathcal{A}} I$ is an r-saturated m.c. subset of R. In the following result we aim to show that every r-saturated m.c. subset of R is of the latter form, which is the counterpart of its corresponding fact for saturated m.c. sets.

Proposition 3.14 *Let S be an r-saturated m.c. subset of a ring R and*

$$\mathcal{A} = \{I : I \textit{ is an } r\textit{-ideal of } R \textit{ with } I \cap S = \emptyset\}.$$

Then $S = R \setminus \bigcup_{I \in \mathcal{A}} I$.

Proof Since $(0) \cap S = \emptyset$, we infer that $(0) \in \mathcal{A}$. This implies that $\mathcal{A} \neq \emptyset$ and it is manifest that $S \subseteq R \setminus \bigcup_{I \in \mathcal{A}} I$. Now suppose that $x \in R \setminus \bigcup_{I \in \mathcal{A}} I$ but $x \notin S$ and seek a contradiction. Since $xR \cap S = \emptyset$, by the previous theorem there exists an r-ideal I containing x such that $I \cap S = \emptyset$. Consequently, $I \in \mathcal{A}$. By our assumption x does not belong to any member of $\mathcal{A}$, whereas $x \in I \in \mathcal{A}$, which is the desired contradiction. □

Remark 3.15 *Let $R \subseteq T$ be rings. It is possible that J is an r-ideal of T, but $J \cap R = I$ is not an r-ideal of R. To see this, let $A = \mathbb{Z}$ and $T = \mathbb{Z} \times \mathbb{Z}$. Clearly, $\varphi : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $\varphi(x) = (x, 0)$ is a monomorphism. Then $R = \varphi(\mathbb{Z})$ is a domain. Also, it is clear that $J = \mathrm{Ann}((0,1))$ is a nonzero r-ideal in T. On the other hand, $R \subseteq J$, and hence $I = R = J \cap R$ is not an r-ideal in R.*

Definition 3.16 *Let R and T be rings with $R \subseteq T$. We say that R is essential in T, if $R \cap I \neq (0)$, for every nonzero ideal of T.*

For example, $C^*(X)$ is essential in $C(X)$. To see this, let I be an ideal in $C(X)$ and $0 \neq f \in I$, and clearly $0 \neq g = \frac{f}{1+f^2} \in I \cap C^*(X)$. More generally, R is essential in $Q(R)$.

In contrast to the fact in Remark [**?**], we have the following result.

Proposition 3.17 *Let $R \subseteq T$ be rings such that R is essential in T. If I is an r-ideal in T, then $I \cap R = J$ is an r-ideal in R.*

Proof Suppose that $r, x \in R$ and $rx \in J$ with $\mathrm{Ann}_R(r) = (0)$. We are to show that $x \in J$. Clearly, $rx \in I$. We claim that $\mathrm{Ann}_T(r) = (0)$. To see this, let $\mathrm{Ann}_T(r) \neq (0)$, and then by our hypothesis, we have $\mathrm{Ann}_T(r) \cap R \neq (0)$, so there exists $0 \neq y \in R$ such that $y \in \mathrm{Ann}_T(r)$, i.e. $yr = 0$. Consequently, we have $y \in \mathrm{Ann}_R(r)$, which is a contradiction. Thus, $x \in I$ and hence $x \in J$. □

4. r-ideals in polynomial rings

Let $R[x]$ denote the ring of polynomials with coefficients in R. If $f = \sum_{i=0}^{n} f_i x^i \in R[x]$, then the content of f, by definition, is the ideal of R generated by the coefficients of f and is denoted by $c(f)$, and the set of coefficients of f is denoted by $C(f)$, i.e. $C(f) = \{f_0, f_1, \ldots, f_n\}$. If I is an ideal of R then $I[x]$ is denoted by the set $\{f \in R[x] : C(f) \subseteq I\}$. Also let $R[[x]]$ be the ring of formal power series with coefficients in R. If $f = \sum_{i=0}^{\infty} f_i x^i \in R[[x]]$, then $C(f)$ is the sequence $\{f_n\}_{n \in \mathbb{N}}$.

Remark 4.1 *a) Let R be a reduced ring and $f \in R[x]$; then by [[[2]], Theorem 3.3], we have $\mathrm{Ann}(f) = \mathrm{Ann}(C(f))[x]$. Also, if $f \in R[[x]]$, then clearly $\mathrm{Ann}(f) = \mathrm{Ann}(C(f))[[x]]$.*

*b) If $I[x]$ is an r-ideal in $R[x]$, then I is an r-ideal in R. The converse is true if and only if R satisfies property A; see Theorem [**?**] (note: $R[x]$ and $C(X)$ have property A). We should also remind the reader that if $I = \mathrm{Ann}(a)$ with $0 \neq a \in R$, then $I[x]$ is an r-ideal in $R[x]$.*

*c) Let $I[[x]]$ be an r-ideal in $R[[x]]$, and then I is an r-ideal in R. The converse is true if R satisfies the c.a.c.; see Proposition [**?**]. It is also clear that if $I = \mathrm{Ann}(a)$ where $0 \neq a \in R$, then $I[[x]]$ is an r-ideal in $R[[x]]$.*

d) Let I be a semiprime ideal of a reduced ring R. Assume that $f,g \in R[[x]]$, where $f=\sum_{i=0}^{\infty} f_i x^i$ and $g=\sum_{i=0}^{\infty} g_i x^i$. Then one can easily show that $fg \in I[[x]]$ if and only if $f_n g_m \in I$, for $n,m=0,1,2,\cdots$.

e) If (I,x) is an r-ideal in $R[x]$, then I is an r-ideal in R. The converse is not true in general. For example, the ideal $I=(0)$ in R is an r-ideal, but $(I,x)=xR[x]$ is not an r-ideal in $R[x]$.

f) If $\mathcal{M} \in \mathrm{Max}(R[x])$, then by [[[19]], Theorem 150] there exists $f \in \mathcal{M}$ such that $\mathrm{Ann}_{R[x]}(f)=(0)$, so $\mathcal{M}$ is not an r-ideal. This implies that $R[x]$ is never a uz-ring.

g) If R satisfies property A, $f \in R[x]$ and $\mathrm{Ann}_{R[x]}(f)=(0)$, then by [[[18]], Theorem 2.6], there exists $a \in c(f)$ such that $\mathrm{Ann}_R(a)=(0)$, and hence $c(f)$ is not an r-ideal.

h) Let R be a uz-ring and $\mathcal{M} \in \mathrm{Max}(R[x])$, and then there is $f \in \mathcal{M}$ such that $\mathrm{Ann}_{R[x]}(f)=(0)$, by part (f). Whenever $I=c(f) \neq R$, then I is an r-ideal, whereas $I[x]$ is not an r-ideal.

In the following proposition we show that if I is an r-ideal in a reduced ring R, then $I[x]$ is an r-ideal in $R[x]$ if and only if R satisfies property A.

Theorem 4.2 *Let R be a ring. Then the following statements are equivalent:*

a) R satisfies property A.

b) I is an r-ideal in R if and only if $I[x]$ is an r-ideal in $R[x]$, for every ideal I of R.

Proof $(a \Rightarrow b)$ Let I be an r-ideal of R, $f,g \in R[x]$ and $fg \in I[x]$ with $\mathrm{Ann}_{R[x]}(g)=(0)$. Hence, by [[[2]], Proposition 3.5], we conclude that $c(g) \nsubseteq \mathrm{zd}(R)$. Therefore, there exists $r \in c(g)$ such that $\mathrm{Ann}_R(r)=(0)$. Clearly, $C(fg) \subseteq I$ and so $c(fg) \subseteq I$. Now by [[[17]], Theorem 28.1], we have $c(g)^{n+1}c(f)=c(g)^n c(fg)$, where n is the degree of f. This implies that $c(g)^{n+1}c(f) \subseteq I$. Since $r^{n+1} \in c(g)^{n+1}$, we infer that $r^{n+1}c(f) \subseteq I$. On the other hand, we have $\mathrm{Ann}_R(r^{n+1})=(0)$. Now we conclude that $c(f) \subseteq I$. Thus, $f \in I[x]$. The converse is evident.

$(b \Rightarrow a)$ Suppose, on the contrary, that R does not satisfy property A. We are to seek a contradiction. By [[[2]], Proposition 3.5], there exists $f \in R[x]$ such that $\mathrm{Ann}_{R[x]}(f)=(0)$ and $I=c(f) \subseteq \mathrm{zd}(R)$. Now by part (b) of Theorem [**?**], there exists a prime r-ideal P such that $I \subseteq P$, i.e. $c(f) \subseteq P$. Hence, $f \in P[x]$, while f is a regular element. Thus, $P[x]$ is not an r-ideal, which is the desired contradiction. □

Corollary 4.3 *Let R be a uz-ring. Then R satisfies property A if and only if $I[x]$ is an r-ideal in $R[x]$, for every ideal I of R.*

A ring R is said to have the finite (resp., countable) annihilator condition or briefly to have the f.a.c. (resp., the c.a.c.) if for every finite (resp., countable) subset S of R there exists an element $a \in S$ with $\mathrm{Ann}(S)=\mathrm{Ann}(a)$.

For example, the ring $\mathbb{Z}_{p^n}$, where p is a prime number and $n \in \mathbb{N}$, satisfies the f.a.c. To see this, let $a \in \mathbb{Z}_{p^n}$, and hence there exists $0 \leqslant r \leqslant n$, such that $a=p^r a_1$, with a_1 and p being relatively prime. One can easily show that $\mathrm{Ann}_{\mathbb{Z}_{p^n}}(a)=p^{n-r}\mathbb{Z}_{p^n}$. Now if $b=p^s b_1$, with $r \leqslant s$, then $\mathrm{Ann}(a,b)=\mathrm{Ann}(a)\cap\mathrm{Ann}(b)=p^{n-r}\mathbb{Z}_{p^n}\cap p^{n-s}\mathbb{Z}_{p^n}=p^{n-s}\mathbb{Z}_{p^n}=\mathrm{Ann}(b)$. More generally, if in a ring R, the set of all $\mathrm{Ann}(r)$, where $r \in R$, is a chain, then R satisfies the f.a.c. Clearly, if R is a finite ring, which satisfies the f.a.c., then R satisfies the c.a.c. Also, if F is a field, then $R=\frac{F[x]}{x^2F[x]}$ satisfies the c.a.c.

It is clear that if R satisfies the f.a.c., then it satisfies the s.a.c., and so it satisfies the a.c. A ring R may satisfy property A, but it may not satisfy a.c. and also f.a.c.; see [[[2]], Example 4.1].

Proposition 4.4 *Let R be a ring satisfying the f.a.c. (c.a.c.) and I be a semiprime ideal of R. Then I is an r-ideal in R if and only if $I[x]$ ($I[[x]]$) is an r-ideal in $R[x]$ ($R[[x]]$).*

Proof Let $f, g \in R[x]$ and $fg \in I[x]$ with $\mathrm{Ann}_{R[x]}(f) = (0)$. Thus, $\mathrm{Ann}_R(C(f)) = (0)$. By our hypothesis, there exists $a \in C(f)$ such that $\mathrm{Ann}_R(C(f)) = \mathrm{Ann}_R(a)$. Therefore, $\mathrm{Ann}_R(a) = (0)$. It is easy to show that $aC(g) \subseteq I$. Since I is an r-ideal in R, we infer that $C(g) \subseteq I$. This implies that $g \in I[x]$, i.e. $I[x]$ is an r-ideal in $R[x]$. The converse is evident. In case ($I[[x]]$), whenever R satisfies the c.a.c., the proof is similar. □

5. r-ideals in C(X)

In this section we will investigate the relations between r-ideals, z°-ideals, and z-ideals in $C(X)$. We characterize the topological spaces X for which r-ideals coincide with others. In this section, for the sake of brevity, $\mathrm{r}(C(X))$, $\mathrm{zd}(C(X))$, and $\mathrm{u}(C(X))$ are replaced by $\mathrm{r}(X)$, $\mathrm{zd}(X)$, and $\mathrm{u}(X)$. It is easy to see that $f \in C(X)$ is a regular element if and only if $\mathrm{int}Z(f) = \emptyset$; see also [[7]]. Let us recall the following definitions.

Definitions 5.1 *A topological space X is said to be:*

a) P-space if every prime ideal of $C(X)$ is a z-ideal.

b) F-space if finitely generated ideals of $C(X)$ are principal.

c) Almost P-space if every nonempty zeroset has a nonempty interior, or equivalently every z-ideal of $C(X)$ is a z°-ideal.

d) Quasi F-space if finitely generated ideals containing a nondivisor of 0 in $C(X)$ are principal, or equivalently the sum of two z°-ideals of $C(X)$ is a z°-ideal.

e) m-space if every prime z°-ideal of $C(X)$ is minimal prime ideal, or equivalently if for every zeroset Z in X there exists a zeroset F in X such that $Z \cup F = X$ with $\mathrm{int}Z \cap \mathrm{int}F = \emptyset$.

f) Quasi m-space if every prime z°-ideal of $C(X)$ is either a minimal prime or a maximal ideal.

g) W. almost P-space if for every two zerosets Z and F, with $\mathrm{int}Z \subseteq \mathrm{int}F$, there exists a zeroset E in X such that $Z \subseteq F \cup E$ and $\mathrm{int}E = \emptyset$.

h) ∂-space if for every zeroset Z in X there exists a zeroset F in X such that $\partial(Z) \subseteq F$ and $\mathrm{int}F = \emptyset$, where $\partial(Z) = Z \setminus \mathrm{int}Z$ is the boundary of Z.

For more details about P-spaces and F-spaces, see [[16]]. For almost P-spaces, see [[5], [24]]; for quasi F-spaces, see [[13]]; and for other spaces, see [[9]].

We cite the following facts from [[9]].

Proposition 5.2 *a) Every z-ideal $I \subseteq \mathrm{zd}(X)$ of $C(X)$ is a z°-ideal if and only if X is an almost P-space.*

b) Every prime z-ideal $P \subseteq \mathrm{zd}(X)$ of $C(X)$ is a z°-ideal if and only if X is a w. almost P-space.

c) Every prime ideal $P \subseteq \mathrm{zd}(X)$ of $C(X)$ is a z°-ideal if and only if X is a ∂-space.

Proposition 5.3 *For a topological space X the following statements are equivalent:*

a) X is an almost P-space.

b) Every ideal I of $C(X)$ is an r-ideal.

c) Every ideal $I \subseteq \mathrm{zd}(X)$ of $C(X)$ is an r-ideal.

Proof $(a \Leftrightarrow b)$ By [[[5]], Theorem 2.2] we know that X is an almost P-space if and only if $C(X)$ is a uz-ring. Therefore, every ideal in $C(X)$ is an r-ideal if and only if X is an almost P-space.

$(b \Rightarrow c)$ It is clear.

$(c \Rightarrow a)$ Suppose that $0 \neq f \in C(X)$ and $\mathrm{int}Z(f) = \emptyset$, and we are to show that $Z(f) = \emptyset$. Assume that $x \notin Z(f)$; therefore, there exist $g, h \in C(X)$ such that $x \in \mathrm{int}Z(g)$, $Z(f) \subseteq \mathrm{int}Z(h)$, and $Z(g) \cap Z(h) = \emptyset$. Now we put $I = fgC(X)$. Clearly, I is consisting entirely of zerodivisors, for $\mathrm{int}Z(fg) = \mathrm{int}Z(g) \neq \emptyset$. Thus, by our hypothesis, I is an r-ideal. Since $fg \in I$ and f is regular, we conclude that $g \in I$ and hence $g = fgk$ for some $k \in C(X)$. Now using $Z(f) \subseteq Z(g)$, we have $Z(f) = Z(f) \cap Z(g) \subseteq Z(h) \cap Z(g) = \emptyset$. This implies that $Z(f) = \emptyset$ and we are done. □

Proposition 5.4 *Every r-ideal of $C(X)$ is a z°-ideal if and only if X is a ∂-space.*

Proof The necessary is clear by part (c) of Proposition [**?**]. For sufficiency, the proof is similar to that of [[[9]], Theorem 4.4]. □

Let us remind the reader that in part (l) of Remark [**?**], we have noticed that the sum of two r-ideals is not necessarily an r-ideal. It is interesting to observe, in what follows, that in a ∂-space quasi F-space, the sum of r-ideals becomes an r-ideal.

Corollary 5.5 *Let X be a ∂-space. Then the following statements hold:*

a) I is an r-ideal in $C(X)$ if and only if it is a z°-ideal.

b) I is an r-ideal in $C(X)$ if and only if $\sqrt{I}$ is an r-ideal.

c) I is an r-ideal in $C(X)$ if and only if every minimal prime ideal of I is an r-ideal.

d) Every prime ideal in $C(X)$ is an r-ideal in $C(X)$ if and only if every prime ideal is a z°-ideal.

e) The sum of two r-ideals of $C(X)$ is an r-ideal if and only if X is a quasi F-space.

Since a ∂-space almost P-space is a P-space, the following corollary is immediate.

Corollary 5.6 *Let X be a ∂-space. Then the following statements are equivalent:*

a) X is a P-space.

b) Every ideal is an r-ideal in $C(X)$.

c) Every prime ideal is an r-ideal in $C(X)$.

Proposition 5.7 *Every prime r-ideal of $C(X)$ is a z°-ideal if and only if X is an m-space.*

Proof It is evident. □

Lemma 5.8 *Let X be an m-space. Then every r-ideal of $C(X)$ is a z-ideal.*

Proof Suppose that I is an r-ideal, $f, g \in C(X)$, $f \in I$, and $Z(f) = Z(g)$; we are to show that $g \in I$. By our hypothesis, there exists $0 \leq h \in C(X)$ such that $hf^{\frac{1}{3}} = 0$ and $\text{int}Z(h + f^{\frac{2}{3}}) = \emptyset$. Clearly, $f^{\frac{1}{3}}(h + f^{\frac{2}{3}}) = f \in I$. Since I is an r-ideal, we infer that $f^{\frac{1}{3}} \in I$ and hence $f^{\frac{2}{3}} \in I$. On the other hand, $Z(h) \cup Z(f^{\frac{2}{3}}) = Z(h) \cup Z(f) = Z(h) \cup Z(g) = X$ implies that $gh = 0$. Now we have $g(h + f^{\frac{2}{3}}) = gf^{\frac{2}{3}} \in I$. Hence, by our hypothesis, we conclude that $g \in I$. □

The following corollary is now evident.

Corollary 5.9 *Let X be an m-space, $f \in C(X)$ and $I = fC(X)$. Then the following statements are equivalent:*

a) $\text{int}Z(f) = Z(f)$.

b) I is an r-ideal.

c) I is a z-ideal.

d) I is a z°-ideal.

Using Proposition [?] and the fact that every almost P-space that is also a ∂-space is a P-space, the following corollary is now evident.

Corollary 5.10 *Let X be an almost P-space. Then the following statements are equivalent:*

a) X is a P-space.

b) Every r-ideal in $C(X)$ is a z-ideal.

c) Every r-ideal in $C(X)$ is a z°-ideal.

Theorem 5.11 *Every r-ideal in the class of all z-ideals of $C(X)$ is a z°-ideal if and only if X is w. almost P-space.*

Proof Let I be an r-ideal that is also a z-ideal. Assume that $\text{int}Z(f) \subseteq \text{int}Z(g)$ and $f \in I$, and we must show that $g \in I$. By definition of w. almost P-spaces, there exists $h \in C(X)$ such that $\text{int}Z(h) = \emptyset$ and $Z(f) \subseteq Z(gh)$. Since I is a z-ideal, we infer that $gh \in I$. Since I is an r-ideal we conclude that $g \in I$. Conversely, it suffices to show that every prime z-ideal consisting entirely of zerodivisors is a z°-ideal, by [[[9]], Theorem 4.2]. To this end, we just notice that every prime ideal consisting entirely of zerodivisors is an r-ideal. □

Let us recall that the socle of $C(X)$, denoted by $C_F(X)$, is of the form $C_F(X) = \{f \in C(X) : X \setminus Z(f)$ is a finite subset of $X\}$; see [[[22]], Proposition 3.3]. It is also shown that $C_F(X)$ is never a prime ideal in $C(X)$; see [[[4], Proposition 2.5] and [[15]]. One can easily show that $C_F(X)$ is a z°-ideal. Note that we have already shown (see Corollary [?]) that the socle of any reduced ring is an r-ideal.

Remark 5.12 *We should emphasize that $C_F(X)$ is an r-ideal, as we may present in a direct proof, in which we do not need to use Theorem [?] or Corollary [?]. Let $fg \in C_F(X)$, $\text{int}Z(f) = \emptyset$, and $g \in C(X)$. Clearly,* $\text{cl}(X \setminus Z(f)) = X$, *and hence*

$$X \setminus Z(g) \subseteq \text{cl}(X \setminus Z(g)) = \text{cl}(X \setminus Z(fg)) = X \setminus Z(fg).$$

Therefore, $X \setminus Z(g)$ is a finite subset of X, i.e. $g \in C_F(X)$.

One can easily see that other ideals in $C(X)$ of this kind, such as $C_K(X) = \{f \in C(X) : \text{cl}(X \setminus Z(f))$ is a compact subset of $X\}$, are r-ideals, too.

Remark 5.13 *Suppose that X is an almost P-space that is not P-space.*

a) $C(X)$ is a uz-ring but it is not a von Neumann regular ring.

b) Any r-ideal is not necessarily a pure ideal. For example, by [[[1]], Corollary 2.4] there exists $x \in X$ such that $M_x = \{f \in C(X) : \ f(x) = 0\}$ is not a pure ideal, while this ideal is an r-ideal. More generally, whenever A is regular closed in X, i.e. $\text{cl}(\text{int}(A)) = A$ (X is not necessarily an almost P-space), then $M_A = \{f \in C(X) : A \subseteq Z(f)\}$ is an r-ideal.

c) Any r-ideal is not necessarily a von Neumann regular ideal. Since X is not a P-space, there exists $f \in C(X)$ such that f is not a von Neumann regular element. Now ideal $I = fC(X)$ is not von Neumann regular ideal, while this ideal is an r-ideal.

d) Any r-ideal is not necessarily a z-ideal and so is not a z°-ideal either. Since X is not a P-space, there exists an ideal I in $C(X)$ such that it is not a z-ideal, while this ideal is an r-ideal.

It is well known that the sum of two prime ideals (z-ideals) in $C(X)$ is either $C(X)$ or is a prime ideal (z-ideal); see [[16]]. The next example shows that r-ideals do not have this property.

Example 5.14 *The sum of two r-ideals may not be an r-ideal. For example, we consider two ideals in $C(\mathbb{R})$, namely $M_{[0,\infty)} = \{f \in C(\mathbb{R}) : \ [0,\infty) \subseteq Z(f)\}$ and $M_{(-\infty,0]} = \{f \in C(\mathbb{R}) : \ (-\infty,0] \subseteq Z(f)\}$. Clearly, these ideals are z°-ideals and by part (a) of Theorem [?] are r-ideals. Now we put $f(x) = 0$ if $0 \le x$, $f(x) = x$ if $x < 0$, and $g(x) = 0$ if $x \le 0$, $g(x) = x$, if $0 < x$. Clearly, $f \in M_{[0,\infty)}$, $g \in M_{(-\infty,0]}$ and $f + g = i$, where $i \in C(\mathbb{R})$ is the identity function. Hence, $i \in M_{[0,\infty)} + M_{(-\infty,0]}$. On the other hand, $Z(i) = \{0\}$ implies $\text{int}Z(i) = \emptyset$, and so i is a regular element. Therefore, $M_{[0,\infty)} + M_{(-\infty,0]}$ is not an r-ideal.*

The next example shows that every ideal consisting of zerodivisors is not necessarily an r-ideal (even if it is a semiprime or even a z-ideal). Recall that every z-ideal in $C(X)$ is a semiprime ideal.

Example 5.15 *Any z-ideal consisting entirely of zerodivisors is not necessarily an r-ideal. For example, in $C(\mathbb{R})$ we consider $I = \{f \in C(\mathbb{R}) : [0,1] \cup \{2\} \subseteq Z(f)\}$. Clearly, I is a z-ideal consisting entirely of zerodivisors. Now suppose that $Z(g) = [0,1]$ and $Z(h) = \{2\}$, where $g, h \in C(\mathbb{R})$. It is obvious that $[0,1] \cup \{2\} = Z(g) \cup Z(h) = Z(gh)$, so $gh \in I$. Since $\text{int}Z(h) = \emptyset$ and $g \notin I$, we conclude that I is not an r-ideal.*

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