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## Fiber product preserving bundle functors on fibered-fibered manifolds

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**Abstract:** We introduce the concept of modified vertical Weil functors on the category  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  of fibered-fibered manifolds with  $(m_1, m_2)$ -dimensional bases and their local fibered-fibered maps with local fibered diffeomorphisms as base maps. We then describe all fiber product preserving bundle functors on  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  in terms of modified vertical Weil functors.

**Key words:** Weil algebra, Weil functor, vertical Weil functor, Weil algebra bundle functor, modified vertical Weil functor, bundle functor, fiber product preserving bundle functor, natural transformation

### 1. Introduction

We assume that any manifold considered in this paper is Hausdorff, second countable, finite dimensional, without boundary, and smooth (i.e. of class  $C^\infty$ ). All maps between manifolds are assumed to be smooth (of class  $C^\infty$ ).

Let  $\mathcal{M}f$  be the category of manifolds and their local maps,  $\mathcal{M}f_m$  the category of  $m$ -dimensional manifolds and their local diffeomorphisms,  $\mathcal{FM}$  the category of fibered manifolds (surjective submersions between manifolds) and their local fibered maps,  $\mathcal{FM}_{m_1, m_2}$  the category of  $(m_1, m_2)$ -dimensional fibered manifolds (i.e. with  $m_1$ -dimensional bases and  $m_2$ -dimensional fibers) and their local fibered diffeomorphisms,  $\mathcal{FM}_m$  the category of fibered manifolds with  $m$ -dimensional bases and their local fibered maps with embeddings as base maps,  $\mathcal{F}_2\mathcal{M}$  the category of fibered-fibered manifolds (surjective fibered submersions between fibered manifolds with submersions between fibers) and their local fibered-fibered maps, and  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  the category of fibered-fibered manifolds with  $(m_1, m_2)$ -dimensional bases and their  $\mathcal{F}_2\mathcal{M}$ -maps with base maps being  $\mathcal{FM}_{m_1, m_2}$ -maps.

Thus, any  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object is of the form  $Y = ((p, \underline{p}) : (q : Y \rightarrow X) \rightarrow (q : \underline{Y} \rightarrow \underline{X}))$  (a surjective fibered submersion from an  $\mathcal{FM}$ -object  $q : Y \rightarrow X$  onto an  $\mathcal{FM}_{m_1, m_2}$ -object  $q : \underline{Y} \rightarrow \underline{X}$  inducing submersions between fibers). Any  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -map  $f : Y \rightarrow Y'$  is a system  $f = (f, \underline{f}) = (f, f_1, \underline{f}, f_2) : Y \rightarrow Y'$  of an  $\mathcal{FM}$ -map  $f = (f, f_1) : (q : Y \rightarrow X) \rightarrow (q' : Y' \rightarrow X')$  and an  $\mathcal{FM}_{m_1, m_2}$ -map  $\underline{f} = (\underline{f}, f_2) : (q : \underline{Y} \rightarrow \underline{X}) \rightarrow (q' : \underline{Y}' \rightarrow \underline{X}')$  with  $p' \circ f = \underline{f} \circ p$ .

A bundle functor  $F$  on  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  in the sense of [7] is a functor  $F : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  such that the value  $FY$  is a fibered manifold  $\pi_Y : FY \rightarrow Y$  for any  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object  $Y$  as above, the value  $Ff : FY \rightarrow FY'$

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of  $f : Y \rightarrow Y'$  is a fiber map covering  $f$  for any  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ -map  $f : Y \rightarrow Y'$ , and  $F i_U : FU \rightarrow \pi_Y^{-1}U$  is a diffeomorphism for the inclusion map  $i_U : U \rightarrow Y$  of an open subset  $U$  of  $Y$ . The definitions of bundle functors on  $\mathcal{M}f$ ,  $\mathcal{M}f_m$ ,  $\mathcal{F}\mathcal{M}$ ,  $\mathcal{F}\mathcal{M}_{m_1,m_2}$ ,  $\mathcal{F}\mathcal{M}_m$  are quite similar (we replace  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$  by  $\mathcal{M}f$  or  $\mathcal{M}f_m$  or  $\mathcal{F}\mathcal{M}$  or  $\mathcal{F}\mathcal{M}_{m_1,m_2}$  or  $\mathcal{F}\mathcal{M}_m$ ). A bundle functor  $F$  on  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$  is fiber product preserving if for any  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ -objects  $Y_1 = ((Y_1 \rightarrow X_1) \rightarrow (\underline{Y} \rightarrow \underline{X}))$  and  $Y_2 = ((Y_2 \rightarrow X_2) \rightarrow (\underline{Y} \rightarrow \underline{X}))$  we have  $F(Y_1 \times_{\underline{Y}} Y_2) = FY_1 \times_{\underline{Y}} FY_2$  modulo  $(Fpr_1, Fpr_2)$ , where  $pr_i : Y_1 \times_{\underline{Y}} Y_2 \rightarrow Y_i$  are the usual projections. We remark that  $Y_1 \times_{\underline{Y}} Y_2 = ((Y_1 \times_{\underline{Y}} Y_2 \rightarrow (X_1 \times_{\underline{X}} X_2)^o) \rightarrow (\underline{Y} \rightarrow \underline{X}))$ , where  $(X_1 \times_{\underline{X}} X_2)^o$  is the (open) image of  $Y_1 \times_{\underline{Y}} Y_2 \rightarrow X_1 \times_{\underline{X}} X_2$ .

The vertical functor  $V$  on  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$  sends any  $Y$  as above into

$$VY := \bigcup_{\underline{y} \in \underline{Y}} (Tp)^{-1}(0_{\underline{y}}) \subset TY$$

(i.e into the usual vertical bundle of the  $\mathcal{F}\mathcal{M}$ -object  $p : Y \rightarrow \underline{Y}$ ). The vertical functor  $V$  on  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$  is a fiber product preserving bundle functor.

A natural transformation  $\eta : F \rightarrow F^1$  between bundle functors on  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$  is a family of maps  $\eta_Y : FY \rightarrow F^1Y$  for any  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ -manifold  $Y$  such that  $F^1f \circ \eta_Y = \eta_{Y^1} \circ Ff$  for any  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ -map  $f : Y \rightarrow Y^1$ . (One can show that then  $\eta_Y : FY \rightarrow F^1Y$  is a fibered map covering the identity map  $\text{id}_Y$  for any  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ -object  $Y$  [7].)

A Weil algebra is a finite dimensional real local commutative algebra  $A$  with unity (i.e.  $A = \mathbf{R}.1 \oplus N_A$ , where  $N_A$  is a finite dimensional ideal of nilpotent elements).

In [17], Weil introduced the concept of near  $A$ -point on a manifold  $M$  as an algebra homomorphism of the algebra  $C^\infty(M, \mathbf{R})$  of smooth functions on  $M$  into a Weil algebra  $A$ . The space  $T^A M$  of all near  $A$ -points on  $M$  is called a Weil bundle. Eck (see [4]), Luciano (see [10]), and Kainz and Michor (see [5]) proved independently that product preserving bundle functors  $G : \mathcal{M}f \rightarrow \mathcal{F}\mathcal{M}$  (i.e. satisfying  $G(M \times M_1) = GM \times GM_1$  for any  $\mathcal{M}f$ -objects  $M$  and  $M_1$ ) are the Weil functors  $T^A : \mathcal{M}f \rightarrow \mathcal{F}\mathcal{M}$  for Weil algebras  $A = G\mathbf{R}$  and that natural transformations  $\eta : G \rightarrow G_1$  between product preserving bundle functors on  $\mathcal{M}f$  are in bijection with the algebra homomorphisms  $\eta_{\mathbf{R}} : G\mathbf{R} \rightarrow G_1\mathbf{R}$  between the corresponding Weil algebras.

Replacing (in the construction of  $V$ ) the tangent functor  $T$  by the Weil functor  $T^A$  corresponding to a Weil algebra  $A$  and  $0_{\underline{Y}}$  by the canonical section  $e_{\underline{Y}}$  of  $T^A \underline{Y}$ , one can define (in the same way) the vertical Weil functor  $V^A$  on  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ . Functor  $V^A$  is a fiber product preserving bundle functor on  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ , too.

In [11], for any homomorphism  $\mu : A \rightarrow B$  of Weil algebras, the author introduced the bundle functor  $T^\mu : \mathcal{F}\mathcal{M} \rightarrow \mathcal{F}\mathcal{M}$  and described all product preserving bundle functors on  $\mathcal{F}\mathcal{M}$  in terms of functors  $T^\mu$ . For the reader's convenience we present the construction of  $T^\mu$  in Section 2.

Replacing  $A$  (in the construction of  $V^A$ ) by  $\mu : A \rightarrow B$  as above, we can define the functor  $V^\mu$  on  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$  by

$$V^\mu Y = \bigcup_{\underline{y} \in \underline{Y}} (T^\mu p)^{-1}(e_{\underline{Y}}(\underline{y})) \subset T^\mu Y,$$

where  $Y$  in  $T^\mu Y$  denotes the fibered manifold  $Y = (q : Y \rightarrow X)$  and where  $e_{\underline{Y}} : \underline{Y} \rightarrow T^\mu \underline{Y}$  is the canonical section. Then  $V^\mu : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \rightarrow \mathcal{F}\mathcal{M}$  is a fiber product preserving bundle functor (the details in a more general setting can be found in Section 4).

A Weil algebra bundle functor on  $\mathcal{FM}_{m_1, m_2}$  is a bundle functor  $A$  on the category  $\mathcal{FM}_{m_1, m_2}$  such that  $A_z Z$  is a Weil algebra and  $A_z g : A_z Z \rightarrow A_{g(z)} Z_1$  is an algebra isomorphism for any  $\mathcal{FM}_{m_1, m_2}$ -map  $g : Z \rightarrow Z'$  and any point  $z \in Z$  (or shortly and more precisely  $A$  is a bundle functor on  $\mathcal{FM}_{m_1, m_2}$  into the category of all Weil algebra bundles and their algebra bundle maps).

Modifying the examples from [9], we have the following Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$ .

— The trivial Weil algebra bundle functor  $A$  on  $\mathcal{FM}_{m_1, m_2}$  given by  $AZ = Z \times A$  and  $Ag = g \times \text{id}_A$ , where  $A$  is a fixed Weil algebra.

— The Weil algebra bundle functor  $A$  on  $\mathcal{FM}_{m_1, m_2}$  given by  $AZ = (\wedge TZ)^0$  and  $Ag = \wedge Tg|_{(\wedge TZ)^0}$ , where  $\wedge TZ = (\wedge TZ)^0 \oplus (\wedge TZ)^1$  is the Grassmann algebra bundle of the tangent bundle  $TZ$  and  $(\wedge TZ)^0$  is the even degree subalgebra bundle.

— In the previous example we can replace the tangent functor  $T$  by an arbitrary vector bundle functor  $G$  on  $\mathcal{FM}_{m_1, m_2}$ .

— The Weil algebra bundle functor  $A$  on  $\mathcal{FM}_{m_1, m_2}$  given by  $AZ = J^r(Z, \mathbf{R})$  and  $Ag = J^r(g, \text{id}_{\mathbf{R}})$ .

— We can apply a fiber-wise tensor product to the above examples of Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$ .

A natural transformation between Weil algebra bundle functors  $A$  and  $A^1$  on  $\mathcal{FM}_{m_1, m_1}$  is an  $\mathcal{FM}_{m_1, m_2}$ -natural transformation  $\nu : A \rightarrow A^1$  of bundle functors such that  $\nu_z := (\nu_Z)_z : A_z Z \rightarrow A_z^1 Z$  is an algebra homomorphism for any  $\mathcal{FM}_{m_1, m_2}$ -object  $Z$  and any point  $z \in Z$ .

In the present paper, essentially extending the technique from [9, 12], we modify the above concept of the functors  $V^\mu$  on  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  as follows. First, given a natural transformation  $\mu : A \rightarrow B$  between Weil algebra bundle functors  $A$  and  $B$  on  $\mathcal{FM}_{m_1, m_2}$ , we define the bundle functor  $T^\mu$  on  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  by

$$T^\mu Y = \bigcup_{y \in Y} T_y^{\mu p(y)} Y$$

for any  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object  $Y$ , where  $Y$  on the right side denotes the fibered manifold  $q : Y \rightarrow X$ . By “restriction”, we define  $T^\mu : \mathcal{FM}_{m_1, m_2} \rightarrow \mathcal{FM}$ . Next, given an  $\mathcal{FM}_{m_1, m_2}$ -canonical section  $\sigma$  of  $T^\mu : \mathcal{FM}_{m_1, m_2} \rightarrow \mathcal{FM}$  (i.e. a system of sections  $\sigma : \underline{Y} \rightarrow T^\mu \underline{Y}$  for any  $\mathcal{FM}_{m_1, m_2}$ -object  $\underline{Y}$  such that  $T^\mu g \circ \sigma = \sigma \circ g$  for any  $\mathcal{FM}_{m_1, m_2}$ -map  $g : \underline{Y} \rightarrow \underline{Y}'$ ), we define the so-called modified vertical Weil functor  $V^{\mu, \sigma} : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  by

$$V^{\mu, \sigma} Y := \bigcup_{\underline{y} \in \underline{Y}} (T^\mu p)^{-1}(\sigma(\underline{y})) \subset T^\mu Y .$$

Then  $V^{\mu, \sigma} : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  is a fiber product preserving bundle functor (the details can be found in Section 3).

Thus, we have the category  $\text{MVW}_{m_1, m_2}$  of modified vertical Weil functors  $V^{\mu, \sigma} : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  (for all  $\mu$  and  $\sigma$  as above) and their natural transformations, and the obvious (forgetting, inclusion) functor

$$I : \text{MVW}_{m_1, m_2} \rightarrow \text{FPP}_{m_1, m_2} ,$$

where  $\text{FPP}_{m_1, m_2}$  is the category of fiber product preserving bundle functors on  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  and their natural transformations.

In Section 5, given a fiber product preserving bundle functor  $F$  on  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  we construct canonically a natural transformation  $\mu^F : A^F \rightarrow B^F$  of Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$  and an  $\mathcal{FM}_{m_1, m_2}$ -

canonical section  $\sigma^F$  of  $T^{\mu^F} : \mathcal{FM}_{m_1, m_2} \rightarrow \mathcal{FM}$ . Thus, we get the modified vertical Weil functor  $V^{\mu^F, \sigma^F}$  on  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ . In this way, we obtain functor

$$J : \text{FPP}_{m_1, m_2} \rightarrow \text{MVW}_{m_1, m_2} .$$

The main result of the present note is the following.

**Main result.** There is the equivalence

$$\text{FPP}_{m_1, m_2} \cong \text{MVW}_{m_1, m_2}$$

of categories. More precisely,  $I \circ J \cong \text{Id}_{\text{FPP}_{m_1, m_2}}$  and  $J \circ I \cong \text{Id}_{\text{MVW}_{m_1, m_2}}$ .

In [9], there are described all fiber product preserving bundle functors  $F$  of vertical type on  $\mathcal{FM}_m$  in terms of the so-called generalized vertical Weil functors  $V^A : \mathcal{FM}_m \rightarrow \mathcal{FM}$  corresponding to Weil algebra bundle functors  $A$  on  $\mathcal{M}f_m$ . In [12], we described also all fiber product preserving bundle functors  $F$  on  $\mathcal{FM}_m$  in a similar way. There is also another pure theoretical description of all fiber product preserving bundle functors on  $\mathcal{FM}_m$  by means of triples  $(A, H, t)$ ; see [8] (see also [6, 3]). Product preserving bundle functors on some categories over manifolds are considered in many papers, e.g., [13, 14, 15, 16]. Product preserving bundle functors on parameter dependent manifolds are studied in [1]. Natural operators to product preserving bundle functors are studied by many authors, e.g., [2].

**Remark 1** *The category  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  has the same skeleton as the category of foliated fibered manifolds over foliated manifolds of dimension  $m_1 + m_2$  with leaves of dimension  $m_2$  and their morphisms covering local leaf preserving diffeomorphisms. Thus, our main result may be easily extended on fiber product preserving bundle functors in this category.*

## 2. Bundle functors $T^\mu$

We start with the following example (see [11]).

**Example 1** *Let  $\mu : A \rightarrow B$  be an algebra homomorphism between Weil algebras. If  $p : Y \rightarrow \underline{Y}$  is an  $\mathcal{FM}$ -object we put*

$$T^\mu Y = T^A \underline{Y}_{\mu_Y} \times_{T^B p} T^B Y = \{(u, v) \in T^A \underline{Y} \times T^B Y \mid \mu_Y(u) = T^B p(v)\}$$

*with the obvious projection on  $Y$ , where  $\mu_Y : T^A \underline{Y} \rightarrow T^B \underline{Y}$  is the natural transformation induced by  $\mu$ . If  $p' : Y' \rightarrow \underline{Y}'$  is another  $\mathcal{FM}$ -object and  $f : Y \rightarrow Y'$  is an  $\mathcal{FM}$ -map with the base map  $\underline{f} : \underline{Y} \rightarrow \underline{Y}'$  we define*

$$T^\mu f := T^A \underline{f} \times T^B f|_{T^\mu Y} : T^\mu Y \rightarrow T^\mu Y' .$$

*The correspondence  $T^\mu : \mathcal{FM} \rightarrow \mathcal{FM}$  is a product preserving bundle functor.*

*If  $\mu' : A' \rightarrow B'$  is another algebra homomorphism of Weil algebras and  $(\varphi, \psi)$  is a morphism  $\mu \rightarrow \mu'$  (i.e.  $\varphi : A \rightarrow A'$  and  $\psi : B \rightarrow B'$  are algebra homomorphisms with  $\mu' \circ \varphi = \psi \circ \mu$ ) we have the induced natural transformation  $(\varphi, \psi) : T^\mu \rightarrow T^{\mu'}$  defined by*

$$(\varphi, \psi)_Y := \varphi_Y \times \psi_Y|_{T^\mu Y} : T^\mu Y \rightarrow T^{\mu'} Y ,$$

*where  $\varphi_Y : T^A \underline{Y} \rightarrow T^{A'} \underline{Y}$  and  $\psi_Y : T^B Y \rightarrow T^{B'} Y$  are the natural transformations induced by algebra morphisms  $\varphi$  and  $\psi$  of Weil algebras.*

**3. The generalization of  $T^\mu$**

We can generalize the functors  $T^\mu$  as follows.

**Example 2** Let  $\mu : A \rightarrow B$  be a natural transformation between Weil algebra bundle functors  $A$  and  $B$  on  $\mathcal{FM}_{m_1, m_2}$ . If  $Y = ((p, \underline{p}) : (q : Y \rightarrow X) \rightarrow (q : \underline{Y} \rightarrow \underline{X}))$  is an  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object, we put

$$T^\mu Y := \bigcup_{y \in Y} T_y^{\mu_{p(y)}} Y$$

with the obvious projection  $\pi_Y^\mu : T^\mu Y \rightarrow Y$ , where  $Y$  on the right side denotes the fibered manifold  $q : Y \rightarrow X$  and where  $T^{\mu_{p(y)}}$  is the bundle functor on  $\mathcal{FM}$  (as in the previous section) corresponding to the algebra homomorphism  $\mu_{p(y)} : A_{p(y)}\underline{Y} \rightarrow B_{p(y)}\underline{Y}$  between Weil algebras (the restriction of the natural transformation  $\mu_{\underline{Y}} : A_{\underline{Y}} \rightarrow B_{\underline{Y}}$  to the fibers as indicated), where  $\underline{Y} = (q : \underline{Y} \rightarrow \underline{X})$  is the  $\mathcal{FM}_{m_1, m_2}$ -object. If  $Y' = ((p', \underline{p}') : (q' : Y' \rightarrow X') \rightarrow (q' : \underline{Y}' \rightarrow \underline{X}'))$  is another  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object and  $f : Y \rightarrow Y'$  is an  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -map determining (in an obvious way) an  $\mathcal{FM}_{m_1, m_2}$ -map  $\underline{f} : \underline{Y} \rightarrow \underline{Y}'$  (between the  $\mathcal{FM}_{m_1, m_2}$ -objects  $\underline{Y} = (q : \underline{Y} \rightarrow \underline{X})$  and  $\underline{Y}' = (q' : \underline{Y}' \rightarrow \underline{X}')$ ), we put  $T^\mu f = \bigcup_{y \in Y} T_y^\mu f : T^\mu Y \rightarrow T^\mu Y'$ , where  $T_y^\mu f : T_y^\mu Y \rightarrow T_{f(y)}^\mu Y'$  is the composition

$$T_y^{\mu_{p(y)}} Y \rightarrow T_{f(y)}^{\mu_{p(y)}} Y' \rightarrow T_{f(y)}^{\mu_{p(f(y))}} Y'$$

of the restriction  $T_y^{\mu_{p(y)}} f : T_y^{\mu_{p(y)}} Y \rightarrow T_{f(y)}^{\mu_{p(y)}} Y'$  of  $T^{\mu_{p(y)}} f : T^{\mu_{p(y)}} Y \rightarrow T^{\mu_{p(y)}} Y'$  to the fibers with the restriction  $(A_{p(y)}\underline{f}, B_{p(y)}\underline{f})_{f(y)} : T_{f(y)}^{\mu_{p(y)}} Y' \rightarrow T_{f(y)}^{\mu_{p(f(y))}} Y'$  of the natural transformation  $(A_{p(y)}\underline{f}, B_{p(y)}\underline{f})_{Y'} : T^{\mu_{p(y)}} Y' \rightarrow T^{\mu_{p(f(y))}} Y'$  induced (as in Example 1) by morphism  $(A_{p(y)}\underline{f}, B_{p(y)}\underline{f}) : \mu_{p(y)} \rightarrow \mu_{p(f(y))}$ , where  $Y = (q : Y \rightarrow X)$  and  $Y' = (q' : Y' \rightarrow X')$  are the fibered manifolds. Every fibered-fibered chart  $(U, \varphi)$  on  $Y$  induces chart

$$(T^\mu U, T^\mu \varphi) \text{ on } T^\mu Y$$

provided that we use the “translation” identification  $T^\mu(\mathbf{R}^{m_1, m_2, n_1, n_2}) \cong \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times T_{(0,0,0,0)}^{\mu(0,0)}(\mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}) (\cong \mathbf{R}^{n(m_1, m_2, n_1, n_2)})$ , where  $\mathbf{R}^{m_1, m_2, n_1, n_2}$  is the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object  $(pr_{\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}}, pr_{\mathbf{R}^{m_1}}) : (pr_{\mathbf{R}^{m_1} \times \mathbf{R}^{n_1}} : \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \rightarrow \mathbf{R}^{m_1} \times \mathbf{R}^{n_1}) \rightarrow (pr_{\mathbf{R}^{m_1}} : \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \rightarrow \mathbf{R}^{m_1})$  (the canonical projections) and where in  $T_{(0,0,0,0)}^{\mu(0,0)}$  we have the  $\mathcal{FM}$ -object  $\mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} = (pr_{\mathbf{R}^{m_1} \times \mathbf{R}^{n_1}} : \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \rightarrow \mathbf{R}^{m_1} \times \mathbf{R}^{n_1})$ . Thus, the correspondence  $T^\mu : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  is a bundle functor.

If  $\mu' : A' \rightarrow B'$  is another natural transformation between Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$  and  $(\varphi, \psi)$  is a morphism  $\mu \rightarrow \mu'$  (i.e  $\varphi : A \rightarrow A'$  and  $\psi : B \rightarrow B'$  are natural transformations between Weil algebra bundle functors such that  $\mu' \circ \varphi = \psi \circ \mu$ ), we have the induced natural transformation  $(\varphi, \psi) : T^\mu \rightarrow T^{\mu'}$  given by

$$(\varphi, \psi)_Y = \bigcup_{y \in Y} (\varphi, \psi)_y : T^\mu Y \rightarrow T^{\mu'} Y,$$

where  $(\varphi, \psi)_y : T_y^{\mu_{p(y)}} Y \rightarrow T_y^{\mu'_{p(y)}} Y$  is the restriction of natural transformation  $(\varphi_{p(y)}, \psi_{p(y)})_Y : T^{\mu_{p(y)}} Y \rightarrow T^{\mu'_{p(y)}} Y$  induced (as in Example 1) by the morphism  $(\varphi_{p(y)}, \psi_{p(y)}) : \mu_{p(y)} \rightarrow \mu'_{p(y)}$ .

Clearly, any  $\mathcal{FM}_{m_1, m_2}$ -object  $\underline{Y} = (q : \underline{Y} \rightarrow \underline{X})$  can be treated as the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object  $\underline{Y} = ((id_{\underline{Y}}, id_{\underline{X}}) : (q : \underline{Y} \rightarrow \underline{X}) \rightarrow (q : \underline{Y} \rightarrow \underline{X}))$ . Similarly, any  $\mathcal{FM}_{m_1, m_2}$ -map  $f = (f, \underline{f}) : \underline{Y} \rightarrow \underline{Y}'$  between  $\mathcal{FM}_{m_1, m_2}$ -objects  $\underline{Y}$  and  $\underline{Y}' = (q' : \underline{Y}' \rightarrow \underline{X}')$  can be treated as the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  morphism  $f = (f, \underline{f}, f, \underline{f}) : \underline{Y} \rightarrow \underline{Y}'$  between the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -objects  $\underline{Y}$  and  $\underline{Y}'$ . Thus, for any natural transformation  $\mu : A \rightarrow B$  between Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$ , we have bundle functor  $T^\mu : \mathcal{FM}_{m_1, m_2} \rightarrow \mathcal{FM}$  (the “restriction” of  $T^\mu$  from Example 2).

**4. The bundle functors  $V^{\mu, \sigma}$**

We have the following general example of a fiber product preserving bundle functor on  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ .

**Example 3** Let  $\mu : A \rightarrow B$  be a natural transformation between Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$ . Suppose we have  $\mathcal{FM}_{m_1, m_2}$ -natural (canonical) section  $\sigma : \underline{Y} \rightarrow T^\mu \underline{Y}$  with respect to the bundle functor projection  $T^\mu \underline{Y} \rightarrow \underline{Y}$  for any  $\mathcal{FM}_{m_1, m_2}$ -object  $\underline{Y}$  (the naturality means that  $T^\mu \underline{f} \circ \sigma = \sigma \circ \underline{f}$  for any  $\mathcal{FM}_{m_1, m_2}$ -map  $\underline{f} : \underline{Y} \rightarrow \underline{Y}'$ ). If  $Y = ((p, \underline{p}) : (q : Y \rightarrow X) \rightarrow (q : \underline{Y} \rightarrow \underline{X}))$  is an  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object, we define

$$V^{\mu, \sigma} Y := (T^\mu p)^{-1}(im(\sigma)) = \bigcup_{\underline{y} \in \underline{Y}} (T^\mu p)^{-1}(\sigma(\underline{y})) \subset T^\mu Y$$

with the obvious projection onto  $Y$  (the restriction of  $\pi_Y^\mu$ ), where  $p : Y \rightarrow \underline{Y}$  is treated as the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -morphism  $(p, \underline{p}, id_{\underline{Y}}, id_{\underline{X}}) : Y \rightarrow \underline{Y}$  between the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -objects  $Y$  and  $\underline{Y} = ((id_{\underline{Y}}, id_{\underline{X}}) : (q : \underline{Y} \rightarrow \underline{X}) \rightarrow (q : \underline{Y} \rightarrow \underline{X}))$ . Since  $T^\mu p$  is a submersion (it can be observed in fibered-fibered charts),  $V^{\mu, \sigma} Y$  is a submanifold. If  $Y'$  is another  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object and  $f : Y \rightarrow Y'$  is an  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -map, then  $T^\mu f(V^{\mu, \sigma} Y) \subset V^{\mu, \sigma} Y'$ , and we define

$$V^{\mu, \sigma} f := T^\mu f|_{V^{\mu, \sigma} Y} : V^{\mu, \sigma} Y \rightarrow V^{\mu, \sigma} Y' .$$

One can see that  $V^{\mu, \sigma} : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  is a bundle functor. It will be called the modified vertical Weil functor (corresponding to  $(\mu, \sigma)$ ).

If  $\mu' : A' \rightarrow B'$  is another natural transformation between Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$ ,  $\sigma' : \underline{Y} \rightarrow T^{\mu'} \underline{Y}$  (for all  $\mathcal{FM}_{m_1, m_2}$ -objects  $\underline{Y}$ ) is another canonical section and  $(\varphi, \psi) : \mu \rightarrow \mu'$  is a morphism such that

$$\sigma' = (\varphi, \psi)_{\underline{Y}} \circ \sigma$$

for any  $\mathcal{FM}_{m_1, m_2}$ -object  $\underline{Y}$ , then  $(\varphi, \psi)_Y(V^{\mu, \sigma} Y) \subset V^{\mu', \sigma'} Y$ , and then we have the induced  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -natural transformation  $[\varphi, \psi] : V^{\mu, \sigma} \rightarrow V^{\mu', \sigma'}$  defined by

$$[\varphi, \psi]_Y := (\varphi, \psi)_{Y|V^{\mu, \sigma} Y}$$

for any  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object  $Y$ , where  $(\varphi, \psi)_Y : T^\mu Y \rightarrow T^{\mu'} Y$  is the natural transformation induced by  $(\varphi, \psi)$  (as in Example 2).

**Lemma 1** The bundle functor  $V^{\mu, \sigma} : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  is fiber product preserving.

**Proof** Let  $Z = (p : Z \rightarrow \underline{Z})$  be an  $\mathcal{FM}_{m_1, m_2}$ -object and  $z \in Z$ . Given an  $\mathcal{FM}$ -object  $N = (q : N \rightarrow \underline{N})$  we put

$$GN := V_z^{\mu, \sigma}(Z \times N)$$

with the obvious projection onto  $N$ , where  $Z \times N = ((pr_Z, pr_{\underline{Z}}) : (p \times q : Z \times N \rightarrow \underline{Z} \times \underline{N}) \rightarrow (p : Z \rightarrow \underline{Z}))$  is the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object. If  $N' = (q' : N' \rightarrow \underline{N}')$  is another  $\mathcal{FM}$ -object and  $f = (f, \underline{f}) : N \rightarrow N'$  is an  $\mathcal{FM}$ -map we define

$$Gf := V^{\mu, \sigma}(id_Z \times f)|_{GN} : GN \rightarrow GN',$$

where  $id_Z \times f = (id_Z \times f, id_{\underline{Z}} \times \underline{f}, id_Z, id_{\underline{Z}}) : Z \times N \rightarrow Z \times N'$  is the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -map. The correspondence  $G : \mathcal{FM} \rightarrow \mathcal{FM}$  is a bundle functor.

Clearly, it is sufficient to show that  $G$  is product preserving. We have  $GN \subset T^{\mu z}(Z \times N)$ , where  $Z \times N = (q \times p : Z \times N \rightarrow \underline{Z} \times \underline{N})$  is the  $\mathcal{FM}$ -object. Moreover, we have

$$GN = \{\sigma(z)\} \times T^{\mu z}N$$

modulo the identification  $T^{\mu z}(Z \times N) = T^{\mu z}Z \times T^{\mu z}N$  ( $T^{\mu z} : \mathcal{FM} \rightarrow \mathcal{FM}$  is product preserving). We see that

$$Gf = T^{\mu z}(id_Z \times f)|_{GN}$$

(as  $A_z(id_Z) = id$  and  $B_z(id_Z) = id$ ). Thus,  $G$  is  $\mathcal{FM}$ -natural isomorphic with  $T^{\mu z}$ , and so  $G$  is product preserving, as well.  $\square$

## 5. The induced bundle functors $V^{\mu^F, \sigma^F}$

Now we are going to show that (conversely) any fiber product preserving bundle functor  $F : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  determines a natural transformation  $\mu^F : A^F \rightarrow B^F$  of Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$  and a canonical section  $\sigma^F : Y \rightarrow T^{\mu^F}Y$  for any  $\mathcal{FM}_{m_1, m_2}$ -object  $Y$ . We will use the following notation.

Given an  $\mathcal{FM}_{m_1, m_2}$ -object  $Y = (q : Y \rightarrow X)$  and a manifold  $N$  we have the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -objects

$$[Y \times N] := ((pr_Y, pr_X) : (q \times id_N : Y \times N \rightarrow X \times N) \rightarrow (q : Y \rightarrow X))$$

and

$$\langle Y \times N \rangle := ((pr_Y, id_X) : (q \circ pr_Y : Y \times N \rightarrow X) \rightarrow (q : Y \rightarrow X)),$$

where  $pr_Y : Y \times N \rightarrow Y$  and  $pr_X : X \times N \rightarrow X$  are the canonical projections. We have the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -map

$$[id_{Y \times N}] := (id_{Y \times N}, pr_X, id_Y, id_X) : [Y \times N] \rightarrow \langle Y \times N \rangle.$$

If  $N'$  is another manifold and  $f : N \rightarrow N'$  is a map we have  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -maps

$$[id_Y \times f] := (id_Y \times f, id_X \times f, id_Y, id_X) : [Y \times N] \rightarrow [Y \times N']$$

and

$$\langle id_Y \times f \rangle := (id_Y \times f, id_X, id_Y, id_X) : \langle Y \times N \rangle \rightarrow \langle Y \times N' \rangle.$$



If  $Y' = (q' : Y' \rightarrow X')$  is another  $\mathcal{FM}_{m_1, m_2}$ -object and  $\varphi = (\varphi, \underline{\varphi}) : Y \rightarrow Y'$  is an  $\mathcal{FM}_{m_1, m_2}$ -map we have  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -maps

$$[\varphi \times id_N] := (\varphi \times id_N, \underline{\varphi} \times id_N, \varphi, \underline{\varphi}) : [Y \times N] \rightarrow [Y' \times N]$$

and

$$\langle \varphi \times id_N \rangle := (\varphi \times id_N, \underline{\varphi}, \varphi, \underline{\varphi}) : \langle Y \times N \rangle \rightarrow \langle Y' \times N \rangle .$$

We have the following example.

**Example 4** Let  $F : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  be a bundle functor. If  $Y$  is an  $\mathcal{FM}_{m_1, m_2}$ -object we have fibered manifolds

$$A^F Y := F([Y \times \mathbf{R}]) \quad \text{and} \quad B^F Y := F(\langle Y \times \mathbf{R} \rangle)$$

with the projections  $A^F Y \rightarrow Y$  and  $B^F Y \rightarrow Y$  being the composition of the bundle functor projections with the canonical projection  $Y \times \mathbf{R} \rightarrow Y$ . We have the map

$$\mu_Y^F := F([id_{Y \times \mathbf{R}}]) : A^F Y \rightarrow B^F Y .$$

If  $Y'$  is another fibered manifold and  $\varphi : Y \rightarrow Y'$  is an  $\mathcal{FM}_{m_1, m_2}$ -map we have the induced maps

$$A^F \varphi := F([\varphi \times id_{\mathbf{R}}]) : A^F Y \rightarrow A^F Y' , \quad B^F \varphi := F(\langle \varphi \times id_{\mathbf{R}} \rangle) : B^F Y \rightarrow B^F Y' .$$

Since  $F$  is a functor,

$$B^F \varphi \circ \mu_Y^F = \mu_{Y'}^F \circ A^F \varphi .$$

Thus, we have the bundle functors  $A^F$  and  $B^F$  on  $\mathcal{FM}_{m_1, m_2}$  and the natural transformation  $\mu^F : A^F \rightarrow B^F$ .

If  $F' : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  is another bundle functor and  $\eta : F \rightarrow F'$  is a natural transformation we have the natural transformations  $\varphi^\eta : A^F \rightarrow A^{F'}$  and  $\psi^\eta : B^F \rightarrow B^{F'}$  given by

$$\varphi_Y^\eta := \eta_{[Y \times \mathbf{R}]} \quad \text{and} \quad \psi_Y^\eta := \eta_{\langle Y \times \mathbf{R} \rangle}$$

for any  $\mathcal{FM}_{m_1, m_2}$ -object  $Y$ . Since  $\eta$  is a natural transformation,

$$\mu^{F'} \circ \varphi^\eta = \psi^\eta \circ \mu^F .$$

Thus,  $(\varphi^\eta, \psi^\eta) : \mu^F \rightarrow \mu^{F'}$  is a morphism of natural transformations.

**Lemma 2** If  $F$  is fiber product preserving, then  $A^F$  and  $B^F$  are Weil algebra bundle functors and  $\mu^F : A^F \rightarrow B^F$  is a natural transformation of Weil algebra bundle functors.

If additionally  $F'$  is fiber product preserving, then  $(\varphi^\eta, \psi^\eta) : \mu^F \rightarrow \mu^{F'}$  is a morphism between natural transformations of Weil algebra bundle functors.

**Proof** Let  $Y$  be an  $\mathcal{FM}_{m_1, m_2}$ -object and  $y \in Y$  be a point. For any manifold  $N$  we define

$$F^{[y]} N := F_y([Y \times N]) \quad \text{and} \quad F^{\langle y \rangle} N := F_y(\langle Y \times N \rangle)$$

(the fibers over  $y$  of  $F([Y \times N] \rightarrow Y \times N \rightarrow Y)$  and of  $F(\langle Y \times N \rangle \rightarrow Y \times N \rightarrow Y)$  with the obvious projections onto  $N$ . If  $N'$  is another manifold and  $f : N \rightarrow N'$  is a map we have

$$F^{[y]}f := F([\text{id}_Y \times f])|_{F^{[y]}N} : F^{[y]}N \rightarrow F^{[y]}N'$$

and similarly

$$F^{\langle y \rangle}f := F(\langle \text{id}_Y \times f \rangle)|_{F^{\langle y \rangle}N} : F^{\langle y \rangle}N \rightarrow F^{\langle y \rangle}N' .$$

Thus, we have the bundle functors  $F^{[y]} : \mathcal{M}f \rightarrow \mathcal{F}\mathcal{M}$  and  $F^{\langle y \rangle} : \mathcal{M}f \rightarrow \mathcal{F}\mathcal{M}$ . Since  $F$  is fiber product preserving,  $F^{[y]}$  and  $F^{\langle y \rangle}$  are product preserving. Then  $F^{[y]}\mathbf{R}$  and  $F^{\langle y \rangle}\mathbf{R}$  are Weil algebras because of the well-known result concerning Weil functors. (If  $m : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  and  $+$  :  $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  are the multiplication and sum maps for  $\mathbf{R}$ , then  $F^{[y]}m : F^{[y]}\mathbf{R} \times F^{[y]}\mathbf{R} = F^{[y]}(\mathbf{R} \times \mathbf{R}) \rightarrow F^{[y]}\mathbf{R}$  and  $F^{[y]}(+)$  :  $F^{[y]}\mathbf{R} \times F^{[y]}\mathbf{R} = F^{[y]}(\mathbf{R} \times \mathbf{R}) \rightarrow F^{[y]}\mathbf{R}$  are the multiplication and sum maps in  $F^{[y]}\mathbf{R}$  (and similarly for  $F^{\langle y \rangle}$  instead of  $F^{[y]}$ ), where  $F^{[y]}(\mathbf{R} \times \mathbf{R}) = F^{[y]}\mathbf{R} \times F^{[y]}\mathbf{R}$  modulo the identification given by  $(F^{[y]}pr_1, F^{[y]}pr_2) : F^{[y]}(\mathbf{R} \times \mathbf{R}) \rightarrow F^{[y]}\mathbf{R} \times F^{[y]}\mathbf{R}$ , where  $pr_i : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  are the canonical projections.) Since  $A_y^F Y = F^{[y]}\mathbf{R}$  and  $B_y^F Y = F^{\langle y \rangle}\mathbf{R}$ ,  $A^F Y$  and  $B^F Y$  are Weil algebra bundles.

If  $Y'$  is another  $\mathcal{F}\mathcal{M}_{m_1, m_2}$ -object and  $\varphi : Y \rightarrow Y'$  is an  $\mathcal{F}\mathcal{M}_{m_1, m_2}$ -map, we define

$$\tilde{\varphi}_N^{[y]} := F([\varphi \times \text{id}_N])|_{F^{[y]}N} : F^{[y]}N \rightarrow F^{[\varphi(y)]}N$$

and

$$\tilde{\varphi}_N^{\langle y \rangle} := F(\langle \varphi \times \text{id}_N \rangle)|_{F^{\langle y \rangle}N} : F^{\langle y \rangle}N \rightarrow F^{\langle \varphi(y) \rangle}N .$$

Then  $\tilde{\varphi}^{[y]} : F^{[y]} \rightarrow F^{[\varphi(y)]}$  and  $\tilde{\varphi}^{\langle y \rangle} : F^{\langle y \rangle} \rightarrow F^{\langle \varphi(y) \rangle}$  are natural transformations between product preserving bundle functors on  $\mathcal{M}f$ . Then  $\tilde{\varphi}_{\mathbf{R}}^{[y]} : F^{[y]}\mathbf{R} \rightarrow F^{[\varphi(y)]}\mathbf{R}$  and  $\tilde{\varphi}_{\mathbf{R}}^{\langle y \rangle} : F^{\langle y \rangle}\mathbf{R} \rightarrow F^{\langle \varphi(y) \rangle}\mathbf{R}$  are algebra homomorphisms, but  $\tilde{\varphi}_{\mathbf{R}}^{[y]} = A_y^F(\varphi) : A_y^F Y \rightarrow A_{\varphi(y)}^F Y'$  and  $\tilde{\varphi}_{\mathbf{R}}^{\langle y \rangle} = B_y^F(\varphi) : B_y^F Y \rightarrow B_{\varphi(y)}^F Y'$ .

Define

$$\tilde{\mu}_N^y := F([\text{id}_{Y \times N}])|_{F^{[y]}N} : F^{[y]}N \rightarrow F^{\langle y \rangle}N .$$

Then  $\tilde{\mu}^y : F^{[y]} \rightarrow F^{\langle y \rangle}$  is a natural transformation. Then  $\tilde{\mu}_{\mathbf{R}}^y : F^{[y]}\mathbf{R} \rightarrow F^{\langle y \rangle}\mathbf{R}$  is an algebra homomorphism, but  $\tilde{\mu}_{\mathbf{R}}^y = \mu_y^F : A_y^F Y \rightarrow B_y^F Y$  (the restriction of  $\mu_Y^F : A^F Y \rightarrow B^F Y$  to the fibers over  $y$ ).

We have proved that  $\mu^F : A^F \rightarrow B^F$  is a natural transformation between Weil algebra bundle functors on  $\mathcal{F}\mathcal{M}_{m_1, m_2}$ .

Define

$$\Phi_N^{[y]} := \eta_{[Y \times N]}|_{F^{[y]}N} : F^{[y]}N \rightarrow F'^{[y]}N$$

and

$$\Psi_N^{\langle y \rangle} := \eta_{\langle Y \times N \rangle}|_{F^{\langle y \rangle}N} : F^{\langle y \rangle}N \rightarrow F'^{\langle y \rangle}N .$$

Then  $\Phi^{[y]} : F^{[y]} \rightarrow F'^{[y]}$  and  $\Psi^{\langle y \rangle} : F^{\langle y \rangle} \rightarrow F'^{\langle y \rangle}$  are natural transformations of product preserving bundle functors on  $\mathcal{M}f$ . Then  $\Phi_{\mathbf{R}}^{[y]} : F^{[y]}\mathbf{R} \rightarrow F'^{[y]}\mathbf{R}$  and  $\Psi_{\mathbf{R}}^{\langle y \rangle} : F^{\langle y \rangle}\mathbf{R} \rightarrow F'^{\langle y \rangle}\mathbf{R}$  are algebra homomorphisms, but  $\Phi_{\mathbf{R}}^{[y]} = \varphi_y^\eta : A_y^F Y \rightarrow A_y^{F'} Y$  and  $\Psi_{\mathbf{R}}^{\langle y \rangle} = \psi_y^\eta : B_y^F Y \rightarrow B_y^{F'} Y$  (the restrictions of  $\varphi_Y^\eta : A^F Y \rightarrow A^{F'} Y$  and  $\psi_Y^\eta : B^F Y \rightarrow B^{F'} Y$  to the fibers over  $y$ ).

We have proved that  $(\varphi^\eta, \psi^\eta) : \mu^F \rightarrow \mu^{F'}$  is a morphism between natural transformations between Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$ .  $\square$

Let  $F : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  be a fiber product preserving bundle functor. We use the above notation.

**Example 5** Let  $Y = (q : Y \rightarrow X)$  be an  $\mathcal{FM}_{m_1, m_2}$ -object and  $y \in Y$  be a point. Similarly as in Section 2,  $Y$  is treated as the  $\mathcal{F}_2\mathcal{M}_{m_2, m_2}$ -object  $Y = ((id_Y, id_X) : (q : Y \rightarrow X) \rightarrow (q : Y \rightarrow X))$ . We define  $\sigma_1^F(y) : C_{q(y)}^\infty(X) \rightarrow A_y^F Y$  from the algebra  $C_{q(y)}^\infty(X)$  of germs at  $q(y)$  of smooth maps  $f : X \rightarrow \mathbf{R}$  by

$$\sigma_1^F(y)(f) := F(f^\square)(\theta_y) \in F_y([Y \times \mathbf{R}]) = A_y^F Y ,$$

where  $f^\square := ((id_Y, f \circ q), (id_X, f), id_Y, id_X) : Y \rightarrow [Y \times \mathbf{R}]$  and  $\theta_y \in F_y Y$  is the unique point. (Here and later, for simplicity of notations we write  $f$  instead of  $germ_z(f)$  if  $z$  is clear.) Since  $F$  is a functor, using the definition of the multiplication and sum in the Weil algebra  $A_y^F Y$ , one can standardly show that  $\sigma_1^F(y)$  is an algebra homomorphism (for example, if we apply  $F$  to the equality  $(fg)^\square = (id_Y \times m) \circ (id_Y, (f \circ q, g \circ q))$ , we obtain  $\sigma_1^F(y)(fg) = \sigma_1^F(y)(f) \cdot \sigma_1^F(y)(g)$ ). Then

$$\sigma_1^F(y) \in Hom(C_{q(y)}^\infty(X), A_y^F Y) = T_{q(y)}^{A_y^F Y} X .$$

Similarly, we define  $\sigma_2^F(y) : C_y^\infty(Y) \rightarrow B_y^F Y$  by

$$\sigma_2^F(y)(f) := F(f^{<>})(\theta_y) \in F_y(<Y \times \mathbf{R}>) = B_y^F Y ,$$

where  $f^{<>} = ((id_Y, f), id_X, id_Y, id_X) : Y \rightarrow <Y \times \mathbf{R}>$ . Then

$$\sigma_2^F(y) \in Hom(C_y^\infty(Y), B_y^F Y) = T_y^{B_y^F Y} Y .$$

Since  $F$  is a functor,  $\mu_y^F(\sigma_1^F(y)(f)) = \sigma_2^F(y)(f \circ q)$  for any  $f : X \rightarrow \mathbf{R}$  (we apply functor  $F$  to the equality  $id_{Y \times \mathbf{R}} \circ (id_Y, f \circ q) = (id_Y, f \circ q)$  with respective  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -maps and evaluate at  $\theta_y$ ), i.e.

$$(\mu_y^F)_X(\sigma_1^F(y)) = T_y^{B_y^F Y} q(\sigma_2^F(y)) .$$

Then

$$\sigma^F(y) := (\sigma_1^F(y), \sigma_2^F(y)) \in T_y^{\mu_y^F} Y = T_y^{\mu^F} Y .$$

Thus, we have defined canonical section  $\sigma^F : Y \rightarrow T^{\mu^F} Y$  for any  $\mathcal{FM}_{m_1, m_2}$ -object  $Y$ .

One can easily see that if  $F' : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  is another fiber product preserving bundle functor and  $\eta : F \rightarrow F'$  is a natural transformation, then

$$(\varphi^\eta, \psi^\eta)_Y \circ \sigma^F = \sigma^{F'}$$

for any  $\mathcal{FM}_{m_1, m_2}$ -object  $Y$ , where  $(\varphi^\eta, \psi^\eta)_Y : T^{\mu^F} Y \rightarrow T^{\mu^{F'}} Y$  is the natural transformation induced by the morphism  $(\varphi^\eta, \psi^\eta) : \mu^F \rightarrow \mu^{F'}$  (as in Example 2 for  $\mu = \mu^F$ ,  $\mu' = \mu^{F'}$ , and  $(\varphi, \psi) = (\varphi^\eta, \psi^\eta)$ ).

Thus, for any fiber product preserving bundle functor  $F : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  we have defined fiber product preserving bundle functor  $V^{\mu^F, \sigma^F} : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  (as in Example 3 for  $(\mu, \sigma) = (\mu^F, \sigma^F)$ ). Moreover, for any natural transformation  $\eta : F \rightarrow F'$  between fiber product preserving bundle functors on  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$  we have defined natural transformation  $[\varphi^\eta, \psi^\eta] : V^{\mu^F, \sigma^F} \rightarrow V^{\mu^{F'}, \sigma^{F'}}$  (as in Example 3 for  $(\varphi, \psi) = (\varphi^\eta, \psi^\eta)$ ).

**6. An equivalence  $F \cong V^{\mu^F, \sigma^F}$**

Let  $F : \mathcal{F}_2\mathcal{M}_{m_1, m_2} \rightarrow \mathcal{FM}$  be a fiber product preserving bundle functor. We prove that  $F \cong V^{\mu^F, \sigma^F}$ . We start with the following example.

**Example 6** Let  $Y = ((p, \underline{p}) : (q : Y \rightarrow X) \rightarrow (q : \underline{Y} \rightarrow \underline{X}))$  be an  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object. Let  $v \in F_y Y$ ,  $y \in Y$ . Define  $\Theta_Y^{1F}(v) : C_{q(y)}^\infty(X) \rightarrow A_{p(y)}^F \underline{Y}$  by

$$\Theta_Y^{1F}(v)(f) := Ff^\square(v) \in F_{p(y)}([\underline{Y} \times \mathbf{R}]) = A_{p(y)}^F \underline{Y},$$

where  $[\underline{Y} \times \mathbf{R}]$  is the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -object defined by the  $\mathcal{FM}_{m_1, m_2}$ -object  $\underline{Y} = (q : \underline{Y} \rightarrow \underline{X})$  (as in the previous section) and  $f^\square := ((p, f \circ q), (\underline{p}, f), id_{\underline{Y}}, id_{\underline{X}}) : Y \rightarrow [\underline{Y} \times \mathbf{R}]$ . Then  $\Theta_Y^{1F}(v)$  is an algebra homomorphism, i.e.

$$\Theta_Y^{1F}(v) \in Hom(C_{q(y)}^\infty(X), A_{p(y)}^F \underline{Y}) = T_{q(y)}^{A_{p(y)}^F \underline{Y}} X.$$

Define also  $\Theta_Y^{2F} : C_y^\infty(Y) \rightarrow B_y^F \underline{Y}$  by

$$\Theta_Y^{2F}(v)(f) = Ff^{<>}(v) \in F_{p(y)}(<\underline{Y} \times \mathbf{R}>) = B_{p(y)}^F \underline{Y},$$

where  $f^{<>} := ((p, f), \underline{p}, id_{\underline{Y}}, id_{\underline{X}}) : Y \rightarrow <\underline{Y} \times \mathbf{R}>$ . Then  $\Theta_Y^{2F}(v)$  is also an algebra homomorphism, i.e.

$$\Theta_Y^{2F}(v) \in Hom(C_y^\infty(Y), B_{p(y)}^F \underline{Y}) = T_y^{B_{p(y)}^F \underline{Y}} Y.$$

Since  $F$  is a functor,  $\mu_{p(y)}^F(\Theta_Y^{1F}(v)(f)) = \Theta_Y^{2F}(v)(f \circ q)$  for  $f : X \rightarrow \mathbf{R}$  (we apply functor  $F$  to the equality  $id_{\underline{Y} \times \mathbf{R}} \circ (p, f \circ q) = (p, f \circ q)$  with respective  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -morphisms and next evaluate at  $v$ ). Thus,  $(\mu_{p(y)}^F)_X(\Theta_Y^{1F}(v)) = T_{p(y)}^{B_{p(y)}^F \underline{Y}} q(\Theta_Y^{2F}(v))$ , i.e.

$$\Theta_Y^F(v) := (\Theta_Y^{1F}(v), \Theta_Y^{2F}(v)) \in T_y^{\mu_{p(y)}^F} Y = T_y^{\mu^F} Y.$$

Similarly, one can verify that  $T^{\mu^F} p(\Theta_Y^F(v)) = \sigma^F(p(y))$ , where  $p : Y \rightarrow \underline{Y}$  is treated as the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -map as in Example 3, i.e.

$$\Theta_Y^F(v) \in V_y^{\mu^F, \sigma^F} Y.$$

Thus, we have defined the natural transformation  $\Theta^F : F \rightarrow V^{\mu^F, \sigma^F}$ .

**Proposition 1** We have the  $\mathcal{F}_2\mathcal{M}_{m_1, m_2}$ -natural isomorphism  $\Theta^F : F \rightarrow V^{\mu^F, \sigma^F}$  (canonical in  $F$ ).

**Proof** Since  $F$  and  $V^{\mu^F, \sigma^F}$  are fiber product preserving, it suffices to show that

$$(\Theta_{[\underline{Y} \times \mathbf{R}]}^F)_{(y,0)} : F_{(y,0)}([\underline{Y} \times \mathbf{R}]) \rightarrow V_{(y,0)}^{\mu^F, \sigma^F}([\underline{Y} \times \mathbf{R}])$$

and

$$(\Theta_{\langle \underline{Y} \times \mathbf{R} \rangle}^F)_{(y,0)} : F_{(y,0)}(\langle \underline{Y} \times \mathbf{R} \rangle) \rightarrow V_{(y,0)}^{\mu^F, \sigma^F}(\langle \underline{Y} \times \mathbf{R} \rangle)$$

are diffeomorphisms for any  $\mathcal{FM}_{m_1, m_2}$ -object  $\underline{Y} = (q : \underline{Y} \rightarrow \underline{X})$  and any  $y \in \underline{Y}$ . One can easily see that the inverse diffeomorphisms are the restrictions of

$$(Hom(C_{(\underline{y},0)}^\infty(\underline{Y} \times \mathbf{R}), A_{\underline{y}}^F \underline{Y}), Hom(C_{(\underline{y},0)}^\infty(\underline{Y} \times \mathbf{R}), B_{\underline{y}}^F \underline{Y})) \rightarrow A_{\underline{y}}^F \underline{Y}$$

and

$$(Hom(C_{\underline{y}}^\infty(\underline{Y}), A_{\underline{y}}^F \underline{Y}), Hom(C_{(\underline{y},0)}^\infty(\underline{Y} \times \mathbf{R}), B_{\underline{y}}^F \underline{Y})) \rightarrow B_{\underline{y}}^F \underline{Y}$$

given by  $(w^1, w^2) \rightarrow w^1(pr_2)$  and  $(u^1, u^2) \rightarrow u^2(pr_2)$ , where  $pr_2 : \underline{Y} \times \mathbf{R} \rightarrow \mathbf{R}$  is the projection onto the second factor. □

**7. An equivalence**  $V^{\mu, \sigma} \cong V^{\mu^{V^{\mu, \sigma}}, \sigma^{V^{\mu, \sigma}}}$

Let  $\mu : A \rightarrow B$  be a natural transformation between Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$  and  $\sigma : Y \rightarrow T^\mu Y$  be a canonical section for any  $\mathcal{FM}_{m_1, m_2}$ -object  $Y = (q : Y \rightarrow X)$ .

**Example 7** Let  $Y = (q : Y \rightarrow X)$  be an  $\mathcal{FM}_{m_1, m_2}$ -object.

Define  $\Phi_Y^{1(\mu, \sigma)} : AY \rightarrow V^{\mu, \sigma}([Y \times \mathbf{R}]) = A^{V^{\mu, \sigma}} Y$  as follows. Let  $y \in Y$ . Identifying  $\mathbf{R}$  with the  $\mathcal{FM}$ -object  $id_{\mathbf{R}} : \mathbf{R} \rightarrow \mathbf{R}$  we have the “product preserving”  $\mathcal{FM}$ -identification  $T^{\mu_y}(Y \times \mathbf{R}) = T^{\mu_y} Y \times T^{\mu_y} \mathbf{R}$ . Under this identification,  $V_y^{\mu, \sigma}([Y \times \mathbf{R}]) = \{\sigma(y)\} \times T^{\mu_y} \mathbf{R}$ . On the other hand,  $T^{\mu_y} \mathbf{R} = \{(b, (\mu_y)_{\mathbf{R}}(b)) \in T^{A_y} Y \times T^{B_y} Y \mid b \in T^{A_y} Y\}$  and  $T^{A_y} Y = A_y Y$  modulo the usual identification (from the theory of Weil functors). Thus,  $T^{\mu_y} \mathbf{R} = A_y Y$  modulo the clear identification. Then  $V_y^{\mu, \sigma} Y = \{\sigma(y)\} \times A_y Y$  modulo the identification, and we define

$$(\Phi_Y^{1(\mu, \sigma)})_y := (\sigma(y), id_{A_y Y})$$

(modulo the identification). Clearly, it is an algebra isomorphism. Thus, we have defined isomorphism  $\Phi_Y^{1(\mu, \sigma)} : AY \rightarrow A^{V^{\mu, \sigma}} Y$  of Weil algebra bundles, and we have the natural isomorphism  $\Phi^{1(\mu, \sigma)} : A \rightarrow A^{V^{\mu, \sigma}}$  of Weil algebra bundle functors on  $\mathcal{FM}_{m_1, m_2}$ .

Define  $\Phi_Y^{2(\mu, \sigma)} : BY \rightarrow V^{\mu, \sigma}(\langle Y \times \mathbf{R} \rangle) = B^{V^{\mu, \sigma}} Y$  as follows. Let  $y \in Y$ . We have the “product preserving”  $\mathcal{FM}$ -identification  $T^{\mu_y}(Y \times \mathbf{R}) = T^{\mu_y} Y \times T^{\mu_y} \mathbf{R}$ , where  $\mathbf{R} = (pt_{\mathbf{R}} : \mathbf{R} \rightarrow pt)$ , where  $pt$  is the one point manifold. Under this identification,  $V_y^{\mu, \sigma}(\langle Y \times \mathbf{R} \rangle) = \{\sigma(y)\} \times T^{\mu_y} \mathbf{R}$ . On the other hand,  $T^{\mu_y} \mathbf{R} = pt \times T^{B_y} Y = T^{B_y} Y = B_y Y$  modulo the clear identifications. Then  $B_y^{V^{\mu, \sigma}} Y := \{\sigma(y)\} \times B_y Y$  modulo the identifications, and we define

$$(\Phi_Y^{2(\mu, \sigma)})_y := (\sigma(y), id_{B_y Y})$$

modulo the identifications. It is an algebra isomorphism. Thus, we have defined isomorphism  $\Phi^{2(\mu,\sigma)} : B \rightarrow B^{V^{\mu,\sigma}}$  of Weil algebra bundle functors on  $\mathcal{FM}_{m_1,m_2}$ .

One can standardly verify that

$$\mu_Y^{V^{\mu,\sigma}} \circ \Phi_Y^{1(\mu,\sigma)} = \Phi_Y^{2(\mu,\sigma)} \circ \mu_Y,$$

i.e. that  $(\Phi^{1(\mu,\sigma)}, \Phi^{2(\mu,\sigma)}) : \mu \rightarrow \mu^{V^{\mu,\sigma}}$  is an isomorphism of Weil algebra bundle functors on  $\mathcal{FM}_{m_1,m_2}$ . One can also standardly verify that

$$(\Phi^{1(\mu,\sigma)}, \Phi^{2(\mu,\sigma)})_Y \circ \sigma = \sigma^{V^{\mu,\sigma}},$$

where  $(\Phi^{1(\mu,\sigma)}, \Phi^{2(\mu,\sigma)})_Y : T^\mu Y \rightarrow T^{\mu^{V^{\mu,\sigma}}} Y$  is the natural transformation induced by morphism  $(\Phi^{1(\mu,\sigma)}, \Phi^{2(\mu,\sigma)})$ .

Let

$$\mathcal{T}^{(\mu,\sigma)} := [\Phi^{1(\mu,\sigma)}, \Phi^{2(\mu,\sigma)}] : V^{\mu,\sigma} \rightarrow V^{\mu^{V^{\mu,\sigma}}, \sigma^{V^{\mu,\sigma}}}$$

be the induced natural transformation.

Thus, we have proved.

**Proposition 2** *There is the  $\mathcal{FM}_{m_1,m_2}$ -natural isomorphism  $\mathcal{T}^{(\mu,\sigma)} : V^{\mu,\sigma} \rightarrow V^{\mu^{V^{\mu,\sigma}}, \sigma^{V^{\mu,\sigma}}}$  (canonical in  $(\mu, \sigma)$ ).*

### 8. The main result

Let  $FPP_{m_1,m_2}$  be the category of fiber product preserving bundle functors  $F$  on  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$  and their natural transformations  $\eta : F \rightarrow F'$ . Let  $MVW_{m_1,m_2}$  be the category of modified vertical Weil functors on  $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$  of the form  $V^{\mu,\sigma}$  and their natural transformations of the form  $[\varphi, \psi]$  (see Example 3).

**Theorem 1** *We have the equivalence*

$$FPP_{m_1,m_2} \cong MVW_{m_1,m_2}$$

of categories. It means that there are functors  $J : FPP_{m_1,m_2} \rightarrow MVW_{m_1,m_2}$  and  $I : MVW_{m_1,m_2} \rightarrow FPP_{m_1,m_2}$  with  $J \circ I \cong Id_{MVW_{m_1,m_2}}$  and  $I \circ J \cong Id_{FPP_{m_1,m_2}}$ .

**Proof** We define  $J$  by  $J(F) := V^{\mu^F, \sigma^F}$  and  $J(\eta) := [\varphi^\eta, \psi^\eta]$ ; see Examples 4 and 5 (for  $(\mu^F, \sigma^F)$  and  $(\varphi^\eta, \psi^\eta)$ ) and then Example 3 (for the modified vertical Weil functors and their natural transformations). We define  $I$  to be the usual forgetting functor. The isomorphism  $I \circ J \cong Id$  is by Proposition 1. The isomorphism  $J \circ I \cong Id$  is by Proposition 2. □

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