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## Defect polynomials and Tutte polynomials of some asymmetric graphs

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**Abstract:** We give explicit expressions of the Tutte polynomial of asymmetric complete flower graph and asymmetric incomplete flower graph. We then express these Tutte polynomials as generating functions and decode some valuable information about the asymmetric complete flower graph and asymmetric incomplete flower graph. Furthermore, we convert the Tutte polynomials into coboundary polynomials and give explicit expressions of the  $k$ -defect polynomials of these structures. Finally, we conclude that nonisomorphic graphs in this class have the same Tutte polynomials, the same chromatic polynomials, and the same defect polynomials.

**Key words:** Tutte polynomial, cycle graph, flower graph, coboundary polynomials,  $k$ -defect polynomials

### 1. Introduction

There are several polynomials associated with a graph  $G$ ; we refer the reader to [4] for a detailed background. Polynomials play an important role in the study of graphs as they encode various information about a graph. Chromatic polynomials of graphs are sometimes easy to compute. However, Tutte polynomials of such graphs seem harder to find, and if known they are complicated. For example, the chromatic polynomial of  $K_n$  is  $\lambda \prod_{i=1}^{n-1} (\lambda - i)$ , but the Tutte polynomial of the same structure as described by Tutte [10] and Welsh [11] is complicated. There are several methods that are used to compute the Tutte polynomial of a graph; just to sample a few methods, we refer to [1, 6].

The coboundary polynomial  $B(G; \lambda, S)$  of a graph  $G$  is a polynomial in two independent variables  $\lambda$  and  $S$ . It was originally defined and studied by Crapo [2] as a generating function in  $S$  as

$$B(G; \lambda, S) = \sum S^k \phi_k(G; \lambda),$$

where  $\phi_k(G; \lambda)$  is a polynomial in  $\lambda$ .  $\phi_k(G; \lambda)$  is called the  $k$ -defect polynomial of the graph.

Let  $G$  a vertex colored graph. An edge is called bad if it joins two vertices of the same color. The  $k$ -defect polynomial,  $\phi_k(G; \lambda)$ , counts the number of ways of coloring  $G$  with  $k$  bad edges. It is easily seen that the chromatic polynomial of a graph  $G$ ,  $\chi(G; \lambda)$ , is equal to  $\phi_0(G; \lambda)$ .

The coboundary polynomial of a graph  $G$ ,  $B(G; \lambda, S)$  can be obtained from the Tutte polynomial by the following transformation:

$$B(G; \lambda, S) = (S - 1)^r T(G; \frac{S + \lambda - 1}{S - 1}, S).$$

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Chromatic polynomials of graphs have been studied widely; we refer the reader to [3]. On the other hand, little is known about the properties of the  $k$ -defect polynomials. The importance of the  $k$ -defect polynomials should not be underestimated, since if the  $k$ -defect polynomials are known it is equivalent to finding the Tutte polynomial. In addition, the improper coloring is applicable in network theory and time tabling just as is the proper coloring.

Flower graphs form a class of graphs that is highly symmetric. Some classes of flower graphs have attractive simple formulas for the chromatic polynomial; see [8]. In this paper, we study the asymmetric complete flower graph and asymmetric incomplete flower graph.

### 2. Asymmetric flower graphs

In this section we give a definition and an example of an asymmetric complete flower graph and an asymmetric incomplete flower graph.

A graph  $G$  is called a *complete flower graph* if it has  $n$  vertices that form an  $n$ -cycle and  $n$  sets of  $m - 2$  vertices which form  $m$ -cycles around the  $n$  cycle so that each  $m$ -cycle uniquely intersects with the  $n$ -cycle on a single edge. This graph is denoted by  $F_{n,m}$ . It is clear that  $F_{n,m}$  has  $n(m - 1)$  vertices and  $nm$  edges. The  $m$ -cycles are called the *petals* and the  $n$ -cycle is called the center of  $F_{n,m}$ . The  $n$  vertices that form the center are all of degree 4 and all the other vertices have degree 2. Note that  $m \geq 2$  and  $n \geq 2$ .

Removing a petal  $p$  of  $F_{n,m}$  is to take one  $m$ -cycle and delete all the vertices of degree 2 and their adjacent edges. We define an *incomplete flower graph with  $i$  petals* to be a complete flower graph with  $n - i$  petals removed for  $i \in \{1, 2, \dots, n - 1\}$ . This graph is denoted by  $F_{n,m}^i$ . It should be noted that the positions of petals removed may be relevant and  $F_{n,m} = F_{n,m}^n$ . Thus, we have several nonisomorphic graphs represented by  $F_{n,m}^i$ . An *asymmetric complete flower graph* is a flower graph with center  $C_n$  and  $n$  petals of different sizes,  $m_k$  where  $k \in \{1, 2, \dots, n\}$ . If an asymmetric complete flower graph has  $j_k$  petals of size  $m_k$  it is denoted by  $F_{n,m_1j_1, m_2j_2, \dots, m_tj_t}$  where  $j_1 + j_2 + \dots + j_t = n$ . An *asymmetric incomplete flower graph with  $i$  petals* is isomorphic to an asymmetric complete flower graph with  $n - i$  petals missing. If an asymmetric incomplete flower graph has  $i$  petals where  $j_k$  petals are of size  $m_k$  it is denoted by  $F_{n,m_1j_1, m_2j_2, \dots, m_tj_t}^i$  where  $j_1 + j_2 + \dots + j_t = i$ .

The diagrams in Figure are examples of an asymmetric complete flower graph and an asymmetric incomplete flower graph.

### 3. Tutte polynomial

The Tutte polynomial  $T(G; x, y)$  of a graph  $G$ , is a polynomial in two independent variables  $x$  and  $y$  and is defined as

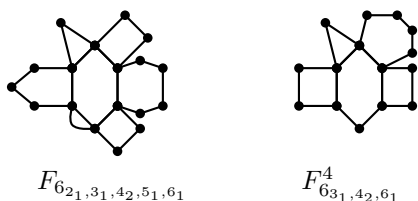


Figure. Asymmetric complete flower graph and asymmetric incomplete flower graph.

$$T(G; x, y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)} (y-1)^{|X|-r(X)},$$

where  $r(X)$  denotes the rank of the graph  $(V, X)$ ,  $V$  the set of vertices, and  $X$  the set of edges. Recall that  $G \setminus e$  is the graph obtained by deleting an edge  $e$  of  $G$  and  $G/e$  is the graph obtained by contracting an edge  $e$  of  $G$ . The Tutte polynomial can also be computed using the following deletion and contraction formula:

T1.  $T(I; x, y) = x$  and  $T(L; x, y) = y$  where  $I$  is an isthmus and  $L$  is a loop.

T2. If  $e$  is an edge of the graph  $G$  and  $e$  is neither a loop nor an isthmus, then

$$T(G; x, y) = T(G \setminus e; x, y) + T(G/e; x, y).$$

T3. If  $e$  is a loop or an isthmus of the graph  $G$ , then

$$T(G; x, y) = T(e; x, y)T(G/e; x, y).$$

The following theorem follows from the deletion and contraction formula of the Tutte polynomials.

**Theorem 3.1** *Let  $t_n$  be a tree on  $n$  vertices and let  $C_n$  be an  $n$ -cycle. Then*

$$(i) \quad T(t_n; x, y) = x^{n-1}.$$

$$(ii) \quad T(C_n; x, y) = y + \sum_{q=1}^{n-1} x^q.$$

Some well-known evaluations of the Tutte polynomial are given in the following theorems; we refer to [5, 7, 9].

**Theorem 3.2** *Let  $G$  be a graph of order  $n$ . Then the chromatic polynomial of  $G$  follows:*

$$\chi(G; \lambda) = (-1)^{r(G)} \lambda^{n-r(G)} T(G; 1-\lambda, 0).$$

**Theorem 3.3** *If  $G = (V, E)$  is a 2-connected graph, then each of the following statements holds:*

(a)  $T(G; -2, 0)$  corresponds to the number of Eulerian orientations.

(b)  $T(G; -1, -1)$  corresponds to the dimension of the bicycle space of binary codes.

(c)  $T(G; 0, -1)$  corresponds to the characteristic function of Eulerian graphs.

(d)  $T(G; 0, 0)$  corresponds to the characteristic function of the empty graph.

(e)  $T(G; 0, 2)$  corresponds to the number of orientations of a bridgeless graph  $G$  such that each edge is contained in an oriented cycle.

(f)  $T(G; 1, 0)$  corresponds to the number of acyclic orientations with one fixed source vertex.

(g)  $T(G; 1, 2)$  corresponds to the number of connected spanning subgraphs.

(h)  $T(G; 2, 0)$  corresponds to the number of acyclic orientations of  $G$ .

(i)  $T(G; 2, 1)$  corresponds to the number of acyclic subgraphs.

(j)  $T(G; 2, 2)$  corresponds to the number of spanning subgraphs.

To ease notation in this paper, if  $G_1$  and  $G_2$  are disjoint graphs, the graph obtained by the operation of merging one vertex of  $G_1$  and one vertex of  $G_2$  will be denoted by  $G_1 \bullet G_2$ . For any flower graph  $F_{n_m}^i$ , the edges of a petal will be labeled in the following sequence:  $e_1, e_2, \dots, e_m, e_1$ , where  $e_1$  is the edge in both the center and the petal,  $e_i$  is adjacent to  $e_{i+1}$ , and  $e_m$  is adjacent to  $e_1$ . We are interested in the edge  $e_2$  to ease our clarification in the following lemmas and theorems. Note that  $e_2$  is an edge of the petal with one vertex in the center and if we delete  $e_2$  from  $F_{n_m}^i$  we get  $F_{n_m}^{i-1} \bullet P_{m-2}$  where  $P_{m-2}$  is a tree with  $m - 2$  edges, isomorphic to a path.

In the following lemmas and theorems, we shall use the edge  $e_2$  in the deletion and contraction formula when applied to a petal unless otherwise stated.

**Lemma 3.4** Let  $F_{n_m}^1$  be a flower graph with 1 petal of size  $m$  and center  $C_n$ . Then

$$T(F_{n_m}^1; x, y) = \left( \sum_{q=0}^{m-2} x^q \right) T(C_n; x, y) + yT(C_{n-1}; x, y).$$

**Proof** We use induction on  $m$  and the deletion and contraction method of the Tutte polynomial. Let  $m = 2$ ; thus, we have a flower graph with center  $C_n$  and a 2-cycle petal. Thus,  $F_{n_2}^1$  is just  $C_n$  with one pair of parallel edges. Let  $e$  be one of the parallel edges. If we delete  $e$  we get  $C_n$  and if we contract  $e$  we get  $C_{n-1}$  with a loop. Hence,

$$\begin{aligned} T(F_{n_m}^1; x, y) &= T(C_n; x, y) + yT(C_{n-1}; x, y) \\ &= \left( \sum_{q=0}^{m-2} x^q \right) T(C_n; x, y) + yT(C_{n-1}; x, y), \end{aligned}$$

therefore true for the base case. Assume it is true when  $m = k$  and  $k > 2$ . Now let  $m = k + 1$  and let  $e$  be an edge  $e_2$  of the petal  $P$ . By deletion and contraction method,

$$T(F_{n_{k+1}}^1; x, y) = T(F_{n_{k+1}}^1 \setminus e; x, y) + T(F_{n_{k+1}}^1 / e; x, y). \tag{3.1}$$

However,  $F_{n_{k+1}}^1 \setminus e$  is isomorphic to  $C_n \bullet P_{k-1}$ . Recall that  $P_{k-1}$  is a tree isomorphic to a path with  $k - 1$  edges. Thus, the  $k - 1$  edges of  $P_{k-1}$  are isthmuses in  $F_{n_{k+1}}^1 \setminus e$ . Hence,

$$T(F_{n_{k+1}}^1 \setminus e; x, y) = x^{k-1}T(C_n; x, y).$$

$F_{n_{k+1}}^1 / e$  is isomorphic to  $F_{n_k}^1$ . Thus,

$$T(F_{n_{k+1}}^1 / e; x, y) = T(F_{n_k}^1; x, y).$$

By induction hypothesis,

$$T(F_{n_{k+1}}^1/e; x, y) = \left( \sum_{q=0}^{k-2} x^q \right) T(C_n; x, y) + yT(C_{n-1}; x, y).$$

If we substitute in Equation 3.1 we get the required result. □

Recalling the following notation,  $F_{n_{m_1j_1}, m_2j_2, \dots, m_tj_t}^i$  is an asymmetric incomplete flower graph with center  $C_n$  and  $i$  petals where  $j_k$  petals are of size  $m_k$ . The following lemma gives the recursive method for the Tutte polynomial of an asymmetric incomplete flower graph with center  $C_n$  and  $i$  petals,  $F_{n_{m_1j_1}, m_2j_2, \dots, m_tj_t}^i$ .

**Lemma 3.5** *Let  $G$  be an asymmetric incomplete flower graph with  $i$  petals,  $F_{n_{m_1j_1}, \dots, m_tj_t}^i$ , where  $i \in \{1, 2, \dots, n - 1\}$ . If we pick one petal, say of size  $m_t$ , then*

$$T(G; x, y) = \left[ \sum_{q=0}^{m_t-2} x^q \right] T(F_{n_{m_1j_1}, \dots, m_tj_{t-1}}^{i-1}; x, y) + yT(F_{(n-1)_{m_1j_1}, \dots, m_tj_{t-1}}^{i-1}; x, y).$$

**Proof** Let  $G$  be  $F_{n_{m_1j_1}, m_2j_2, \dots, m_tj_t}^i$ . We fix and consider one petal of  $G$  of size  $m_t$  and call this petal  $P$ . Let  $e$  be an edge  $e_2$  of petal  $P$ . Now by deletion and contraction formula of the Tutte polynomial,

$$T(G; x, y) = T(G \setminus e; x, y) + T(G/e; x, y). \tag{3.2}$$

Now  $G \setminus e$  is isomorphic to  $F_{n_{m_1j_1}, \dots, m_tj_{t-1}}^{i-1} \bullet P_{m_t-2}$  and the  $m_t - 2$  edges of  $P_{m_t-2}$  are isthmuses  $G \setminus e$ . Hence,

$$T(G \setminus e; x, y) = x^{m_t-2} T(F_{n_{m_1j_1}, \dots, m_tj_{t-1}}^{i-1}; x, y).$$

$G/e$  is isomorphic to  $F_{n_{m_1j_1}, \dots, m_tj_{t-1}, (m_t-1)_1}^i$ . Now petal  $P$  is of size  $m_t - 1$ . Repeat the deletion and contraction formula on petal  $P$  until this petal is removed and get  $T(G/e; x, y)$ . Then substitute back  $T(G \setminus e; x, y)$  and  $T(G/e; x, y)$  in Equation 3.2 to get the required result. □

**Lemma 3.6** *Let  $F_{n_{m_1j_1}, \dots, m_tj_t, m_{t+1}j_{t+1}}^i$  be an asymmetric flower graph with  $i$  petals, where  $i \in \{1, 2, \dots, n\}$  and let  $j_{t+1} = 1$ . Then*

$$\begin{aligned} & \sum_{\substack{\phi_1 + \dots + \phi_{t+1} = i+1-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \\ &= \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{\substack{\phi_1 + \dots + \phi_t = i-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \\ & \quad + \sum_{\substack{\phi_1 + \dots + \phi_t = i+1-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right]. \end{aligned}$$

**Proof** Since  $j_{t+1} = 1$ , then  $\phi_{t+1} \in \{0, 1\}$ . Thus the sum

$$\sum_{\substack{\phi_1 + \dots + \phi_{t+1} = i+1-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right]$$

can be split into two sums where one sum takes the case  $\phi_{t+1} = 1$  and the other sum takes the case  $\phi_{t+1} = 0$ .

If  $\phi_{t+1} = 1$  then  $\phi_1 + \dots + \phi_{t+1} = i + 1 - k$  implies  $\phi_1 + \dots + \phi_t = i - k$  and  $\binom{j_{t+1}}{\phi_{t+1}} = \binom{1}{1}$ . Hence,

$$\begin{aligned} & \sum_{\substack{\phi_1 + \dots + \phi_{t+1} = i+1-k \\ 0 \leq \phi_r \leq j_r \\ \phi_{t+1} = 1}} \left[ \prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \\ &= \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{\substack{\phi_1 + \dots + \phi_t = i-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right]. \end{aligned} \tag{3.3}$$

If  $\phi_{t+1} = 0$  then  $\phi_1 + \dots + \phi_{t+1} = i + 1 - k$  implies  $\phi_1 + \dots + \phi_t = i + 1 - k$  and  $\binom{j_{t+1}}{\phi_{t+1}} = \binom{1}{0}$ . Hence,

$$\sum_{\substack{\phi_1 + \dots + \phi_{t+1} = i+1-k \\ 0 \leq \phi_r \leq j_r \\ \phi_{t+1} = 0}} \left[ \prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] = \sum_{\substack{\phi_1 + \dots + \phi_t = i+1-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right]. \tag{3.4}$$

Hence the result if we sum both cases when  $\phi_{t+1} = 0$  and  $\phi_{t+1} = 1$ . □

The following theorem gives an explicit expression of the Tutte polynomial of an asymmetric incomplete flower graph.

**Theorem 3.7** *Let  $G$  be an asymmetric incomplete flower graph with  $i$  petals,  $F_{n_{m_1 j_1}, m_2 j_2, \dots, m_t j_t}^i$ , where  $i \in \{1, 2, \dots, n-1\}$ . Then*

$$T(G; x, y) = \sum_{k=0}^i y^k \left( \sum_{\substack{\phi_1 + \dots + \phi_t = i-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) \left( y + \sum_{q=1}^{n-1-k} x^q \right).$$

**Proof** We use induction on the number of petals  $i$ . Let  $i = 2$ , and let  $G$  be  $F_{n_{m_1}, m_2}^2$ , an asymmetric incomplete flower graph with 2 petals, one of size  $m_1$ , the other of size  $m_2$  and center  $C_n$ . Then, by Lemma 3.5,

$$T(G; x, y) = \left( \sum_{q=0}^{m_1-2} x^q \right) T(F_{n_{m_2}}^1; x, y) + yT(F_{(n-1)_{m_2}}^1; x, y). \tag{3.5}$$

By Lemma 3.4,

$$T(F_{n_{m_2}}^1; x, y) = \left( \sum_{q=0}^{m_2-2} x^q \right) T(C_n; x, y) + yT(C_{n-1}; x, y),$$

$$T(F_{(n-1)_{m_2}}^1; x, y) = \left( \sum_{q=0}^{m_2-2} x^q \right) T(C_{n-1}; x, y) + yT(C_{n-2}; x, y).$$

Hence, if we substitute in Equation 3.5, we get

$$\begin{aligned} T(G; x, y) &= \left( \sum_{q=0}^{m_1-2} x^q \right) \left( \sum_{q=0}^{m_2-2} x^q \right) \left( y + \sum_{q=1}^{n-1} x^q \right) \\ &+ y \left( \sum_{q=0}^{m_1-2} x^q \right) \left( y + \sum_{q=1}^{n-2} x^q \right) + y \left( \sum_{q=0}^{m_2-2} x^q \right) \left( y + \sum_{q=1}^{n-2} x^q \right) + y^2 \left( y + \sum_{q=1}^{n-3} x^q \right) \\ &= \sum_{k=0}^2 y^k \left( \sum_{\substack{\phi_1+\dots+\phi_t=2-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^2 \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) \left( y + \sum_{q=1}^{n-1-k} x^q \right). \end{aligned}$$

Hence, it is true for an asymmetric incomplete flower graph with 2 petals. Now we assume it is true for  $i = l$  where  $l > 2$  and  $l < n - 1$ . Let  $i = l + 1$  where  $l + 1 \leq n - 1$ . Let  $G$  be  $F_{n_{m_1 j_1}, m_2 j_2, \dots, m_t j_t, m_{t+1}}^{l+1}$  such that it has one petal of size  $m_{t+1}$  more than  $F_{n_{m_1 j_1}, m_2 j_2, \dots, m_t j_t}^l$ . Then by Lemma 3.5,

$$T(G; x, y) = \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) T(F_{n_{m_1 j_1}, m_2 j_2, \dots, m_t j_t}^l; x, y) + yT(F_{(n-1)_{m_1 j_1}, m_2 j_2, \dots, m_t j_t}^l; x, y).$$

By induction hypothesis,

$$\begin{aligned} &\left( \sum_{q=0}^{m_{t+1}-2} x^q \right) T(F_{n_{m_1 j_1}, m_2 j_2, \dots, m_t j_t}^l; x, y) \tag{3.6} \\ &= \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{k=0}^l y^k \left( \sum_{\substack{\phi_1+\dots+\phi_t=l-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) T(C_{n-k}; x, y) \end{aligned}$$

and

$$\begin{aligned} &yT(F_{(n-1)_{m_1 j_1}, m_2 j_2, \dots, m_t j_t}^l; x, y) \tag{3.7} \\ &= y \sum_{k=0}^l y^k \left( \sum_{\substack{\phi_1+\dots+\phi_t=l-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) T(C_{n-1-k}; x, y). \end{aligned}$$



Hence,

$$\begin{aligned}
 T(G; x, y) & \tag{3.8} \\
 &= \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{k=0}^l y^k \left( \sum_{\substack{\phi_1+\dots+\phi_t=l-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) T(C_{n-k}; x, y) \\
 &+ y \sum_{k=0}^l y^k \left( \sum_{\substack{\phi_1+\dots+\phi_t=l-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) T(C_{n-1-k}; x, y).
 \end{aligned}$$

Let  $s$  be a fixed integer in the set  $\{0, 1, \dots, l\}$ . Then track the term  $y^s T(C_{n-s}; x, y)$  in Equation 3.6; it occurs when  $k = s$ . In Equation 3.7, the term  $y^s T(C_{n-s}; x, y)$  occurs when  $k - 1 = s$ . Thus, in Equation 3.8, we have the term  $y^s T(C_{n-s}; x, y)$  with coefficient

$$\begin{aligned}
 & \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{\substack{\phi_1+\dots+\phi_t=l-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \tag{3.9} \\
 &+ \sum_{\substack{\phi_1+\dots+\phi_t=l-(k-1) \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \\
 &= \sum_{\substack{\phi_1+\dots+\phi_{t+1}=i+1-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \text{ by Lemma 3.6,}
 \end{aligned}$$

hence true for any  $i \in \{1, 2, \dots, n - 1\}$ . □

In order to state our next result we need the following notation. We denote the coefficient of  $z^m$  a power series  $f(z)$  by  $[z^m]f(z)$ . Define

$$\begin{aligned}
 C_{i,k}(d) &= C_{i,k}(d; m_1, j_1, \dots, m_t, j_t) \\
 &= \sum_{\ell=0}^d \binom{i-k+d-\ell}{d-\ell} \sum_{\substack{\phi_1+\dots+\phi_t=i+1-k \\ (m_1-1)\varphi_1+\dots+(m_t-1)\varphi_t=\ell \\ 0 \leq \varphi_r \leq \phi_r \leq j_r}} (-1)^{\sum_{j=1}^t \varphi_j} \prod_{r=1}^t \binom{j_r}{\phi_r} \binom{\phi_r}{\varphi_r}.
 \end{aligned}$$

**Corollary 3.8** *Let  $G$  be an asymmetric incomplete flower graph with  $i$  petals,  $F_{n, m_1, j_1, m_2, j_2, \dots, m_t, j_t}^i$ , where  $i \in \{1, 2, \dots, n - 1\}$ . Then the coefficient of  $x^d y^k$  in  $T(G; x, y)$  is given by*

$$[x^d y^k]T(G; x, y) = C_{i,k}(d) + C_{i,k}(d - 1) - C_{i,k}(d - (n - k)).$$

**Proof** By Theorem 3.7, we have

$$[y^k]T(G; x, y) = \frac{1}{(1-x)^{i+1-k}} \sum_{\substack{\phi_1+\dots+\phi_t=i+1-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} (1-x^{m_r-1})^{\phi_r} \right] \\ + \frac{x}{(1-x)^{i+1-k}} \sum_{\substack{\phi_1+\dots+\phi_t=i-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} (1-x^{m_r-1})^{\phi_r} \right] (1-x^{n-1-k}),$$

which implies that  $[x^d y^k]T(G; x, y) = C_{i,k}(d) + C_{i,k}(d-1) - C_{i,k}(d-(n-k))$ , which completes the proof.  $\square$

**Theorem 3.9** Define  $C'_{i,k}(d, n) = C_{i,k}(d) + C_{i,k}(d-1) - C_{i,k}(d-(n-k))$ . Let  $G$  be an asymmetric incomplete flower graph with  $i$  petals,  $F_{n_{m_1 j_1}, m_2 j_2, \dots, m_t j_t}^i$ , where  $i \in \{1, 2, \dots, n-1\}$ . Then the  $\ell$ -defect polynomial of  $G$  is

$$\phi_\ell(G; \lambda) = \sum_{d,k \geq 0} \sum_{a=0}^{r-d} C'_{i,k}(d) (-1)^{r-d-a} (\lambda-1)^{d-(\ell-a-k)} \binom{r-d}{a} \binom{d}{\ell-a-k}.$$

**Proof** By Corollary 3.8, we have that

$$B(G; \lambda, S) = (S-1)^r T(G; 1 + \lambda/(S-1); S) = \sum_{d,k \geq 0} C'_{i,k}(d, n) (S-1)^{r-d} S^k (S + \lambda - 1)^d,$$

which implies that the coefficient of  $S^\ell$  in  $B(G; \lambda, S)$  is given by

$$\phi_\ell(G; \lambda) = \sum_{d,k \geq 0} \sum_{a=0}^{r-d} C'_{i,k}(d) (-1)^{r-d-a} (\lambda-1)^{d-(\ell-a-k)} \binom{r-d}{a} \binom{d}{\ell-a-k},$$

which completes the proof.  $\square$

We need the following lemmas before we give the explicit expression of the Tutte polynomial of an asymmetric complete flower graph.

**Lemma 3.10** Let  $G$  be  $F_{2_{m_1, m_2}}$ , an asymmetric complete flower graph with 2 petals of sizes  $m_1$  and  $m_2$ . Then

$$T(G; x, y) = \left( \sum_{q=0}^{m_1-2} x^q \right) T(F_{2_{m_2}}^1; x, y) + y^2 T(C_{m_2-1}; x, y).$$

**Proof** We consider the petal of size  $m_1$  and call this petal  $P$ , so we use deletion and contraction formula of the Tutte polynomial on the edges of petal  $P$  that are not in the center. Remove isthmuses and loops to get the required result.  $\square$

The following lemma gives the recursive method of the Tutte polynomial of an asymmetric complete flower graph.

**Lemma 3.11** Let  $G$  be  $F_{n_{m_1j_1}, m_2j_2, \dots, m_tj_t}$ , an asymmetric complete flower graph. Then

$$T(G; x, y) = \left( \sum_{q=0}^{m_t-2} x^q \right) T(F_{n_{m_1j_1}, \dots, m_tj_{t-1}}^{n-1}; x, y) + yT(F_{(n-1)_{m_1j_1}, \dots, m_tj_{t-1}}; x, y).$$

**Proof** Similar to the proof of Lemma 3.5. □

The following theorem gives an explicit expression of the Tutte polynomial of an asymmetric complete flower graph.

**Theorem 3.12** Let  $G$  be the graph  $F_{n_{m_1j_1}, m_2j_2, \dots, m_tj_t}$ , an asymmetric complete flower graph. Then

$$T(G; x, y) = \left[ \sum_{k=0}^n y^k \sum_{\substack{\phi_1 + \dots + \phi_t = n-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \left( y + \left[ \sum_{q=1}^{n-1-k} x^q \right] \right) \right] - y^n.$$

**Proof** By induction on the number of vertices of the center  $n$ . The base case is  $n = 2$ . By Lemma 3.10,

$$T(F_{2_{m_1}, m_2}; x, y) = \left( \sum_{q=0}^{m_1-2} x^q \right) T(F_{2_{m_2}}^1; x, y) + y^2 T(C_{m_2-1}; x, y).$$

Applying Lemma 3.4 and then Theorem 3.1 part (ii), we get

$$\begin{aligned} & T(F_{2_{m_1}, m_2}; x, y) \\ &= \left( \sum_{q=0}^{m_1-2} x^q \right) \left( \sum_{q=0}^{m_2-2} x^q \right) T(C_2; x, y) + y \left( \sum_{q=0}^{m_1-2} x^q \right) T(C_1; x, y) + y^2 T(C_{m_2-1}; x, y) \\ &= \left( \sum_{q=0}^{m_1-2} x^q \right) \left( \sum_{q=0}^{m_2-2} x^q \right) (y + x) + y \left( \sum_{q=0}^{m_1-2} x^q \right) y + y^2 \left( y + \sum_{q=1}^{m_2-2} x^q \right). \end{aligned}$$

Rearranging, we get

$$\begin{aligned} & T(F_{2_{m_1}, m_2}; x, y) \\ &= \left( \sum_{q=0}^{m_1-2} x^q \right) \left( \sum_{q=0}^{m_2-2} x^q \right) \left( y + \sum_{q=1}^{2-1} x^q \right) + y^2 \left( \sum_{q=0}^{m_1-2} x^q \right) + y^2 \left( \sum_{q=0}^{m_2-2} x^q \right) + (y^3 - y^2) \\ &= \left[ \sum_{k=0}^1 y^k \sum_{\substack{\phi_1 + \dots + \phi_t = 2-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^2 \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \left( y + \left[ \sum_{q=1}^{1-k} x^q \right] \right) \right] + [y^3 - y^2]. \end{aligned}$$

Hence, it is true for an asymmetric complete flower with 2-cycle center. Assume it is true for  $n = l$  where  $l > 2$ .

Now consider  $n = l + 1$  and let  $G$  be  $F_{(l+1)_{m_1j_1}, m_2j_2, \dots, m_tj_t, m_{t+1}}$  such that  $G$  has one petal of size  $m_{t+1}$  more than  $F_{n_{m_1j_1}, m_2j_2, \dots, m_tj_t}^l$ . Then by Lemma 3.11,

$$T(G; x, y) = \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) T(F_{(l+1)_{m_1j_1}, \dots, m_tj_t}^l; x, y) + yT(F_{(l)_{m_1j_1}, \dots, m_tj_t}; x, y). \tag{3.10}$$

By Theorem 3.7,

$$\begin{aligned} & \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) T(F_{(l+1)m_1j_1, \dots, m_tj_t}^l; x, y) \\ &= \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{k=0}^l y^k \left( \sum_{\substack{\phi_1+\dots+\phi_t=l-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) \left( y + \sum_{q=1}^{l-k} x^q \right). \end{aligned} \tag{3.11}$$

By induction hypothesis,

$$\begin{aligned} & yT(F_{l m_1j_1, \dots, m_tj_t}; x, y) \\ &= y \left[ \sum_{k=0}^{l-1} y^k \sum_{\substack{\phi_1+\dots+\phi_t=l-k \\ =l-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \left( y + \left[ \sum_{q=1}^{l-1-k} x^q \right] \right) \right] + [y^{l+1} - y^l]. \end{aligned} \tag{3.12}$$

Hence, if we substitute Equation 3.11 and Equation 3.12 in Equation 3.10, we get

$$\begin{aligned} & T(G; x, y) \\ &= \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{k=0}^l y^k \left( \sum_{\substack{\phi_1+\dots+\phi_t=l-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) \left( y + \sum_{q=1}^{l-k} x^q \right) \\ &= y \left[ \sum_{k=0}^{l-1} y^k \sum_{\substack{\phi_1+\dots+\phi_t=l-k \\ =l-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \left( y + \left[ \sum_{q=1}^{l-1-k} x^q \right] \right) \right] + [y^{l+1} - y^l]. \end{aligned} \tag{3.13}$$

Let  $s$  be a fixed integer in  $\{0, 1, \dots, n-1\}$ . Track the term  $y^s \left( y + \left[ \sum_{q=1}^{l-1-s} x^q \right] \right)$  in Equation 3.11; it occurs when  $k = s$ . In Equation 3.12, the term  $y^s \left( y + \left[ \sum_{q=1}^{l-1-s} x^q \right] \right)$  occurs when  $k-1 = s$ . Thus, in Equation 3.13, we have the term  $y^s \left( y + \left[ \sum_{q=1}^{l-1-s} x^q \right] \right)$  with coefficient

$$\begin{aligned} & \left( \sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{\substack{\phi_1+\dots+\phi_t=l-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \\ &+ \sum_{\substack{\phi_1+\dots+\phi_t=l-(k-1) \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^t \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \\ &= \sum_{\substack{\phi_1+\dots+\phi_{t+1}=i+1-k \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \text{ by Lemma 3.6.} \end{aligned} \tag{3.14}$$

Hence,

$$T(G; x, y) = \left[ \sum_{s=0}^{(l+1)-1} y^s \sum_{\substack{\phi_1+\dots+\phi_t, \phi_{t+1} \\ = (l+1)-s \\ 0 \leq \phi_r \leq j_r}} \left[ \prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left( \sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \left( y + \left[ \sum_{q=1}^{(l+1)-1-s} x^q \right] \right) \right] + [y^{l+2}y^{l+1}],$$

therefore true for any asymmetric complete flower graph with center  $C_n$  where  $n > 1$ . □

By the proof of Corollary 3.8 and Theorem 3.12, we obtain the following result.

**Corollary 3.13** *Let  $G$  be the graph  $F_{n, m_{1j_1}, m_{2j_2}, \dots, m_{tj_t}}$ , an asymmetric complete flower graph. Then the coefficient of  $x^d y^k$  in  $T(G; x, y)$  is given by*

$$[x^d y^k]T(G; x, y) = C_{n,k}(d) + C_{n,k}(d-1) - C_{n,k}(d - (n-k)) - \delta_{k,n} \delta_{d,0},$$

where  $\delta_{a,b} = 1$  if  $a = b$  and  $\delta_{a,b} = 0$  otherwise.

Similar techniques as in the proof of Theorem 3.9, Corollary 3.13 lead to the following result.

**Theorem 3.14** *Define  $C''_{i,k}(d, n) = C_{i,k}(d) + C_{i,k}(d-1) - C_{i,k}(d - (n-k)) - \delta_{n,k} \delta_{d,0}$ , where  $\delta_{a,b} = 1$  if  $a = b$  and  $\delta_{a,b} = 0$  otherwise. Let  $G$  be an asymmetric incomplete flower graph with  $i$  petals,  $F_{n, m_{1j_1}, m_{2j_2}, \dots, m_{tj_t}}^i$ , where  $i \in \{1, 2, \dots, n-1\}$ . Then the  $\ell$ -defect polynomial of  $G$  is*

$$\phi_\ell(G; \lambda) = \sum_{d,k \geq 0} \sum_{a=0}^{r-d} C''_{i,k}(d) (-1)^{r-d-a} (\lambda-1)^{d-(\ell-a-k)} \binom{r-d}{a} \binom{d}{\ell-a-k}.$$

#### 4. Conclusion

Two nonisomorphic graphs are  $\chi$ -equivalent if they have the same chromatic polynomial. Two nonisomorphic graphs are Tutte-equivalent if they have the same Tutte polynomial. The following theorem highlights classes of graphs that are  $\chi$ -equivalent and Tutte-equivalent and that have the same  $k$ -defect polynomials.

**Theorem 4.1** *Let  $G_1$  and  $G_2$  be nonisomorphic asymmetric incomplete flower graphs represented by  $F_{n, m_{1j_1}, m_{2j_2}, \dots, m_{tj_t}}^i$ . Then*

(i)  $T(G_1; x, y) = T(G_2; x, y),$

(ii)  $\chi(G_1; \lambda) = \chi(G_2; \lambda),$

(iii)  $\phi_k(G_1; \lambda) = \phi_k(G_2; \lambda).$

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