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Hilbert series of the finite dimensional generalized Hecke algebras

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Abstract: It is known from the early results of Coxeter that the generalized Hecke algebras $H(Q_m, 3)$, $m \in \{2, 3, 4, 5\}$, are finite dimensional. In this paper we compute the Hilbert series of these finite-type group algebras.

Key words: Braid group, generalized Hecke algebras, Hilbert series

1. Introduction

The braid group B_{n+1} admits the following classical presentation given by Artin [1]:

$$B_{n+1} = \left\langle x_1, x_2, \dots, x_n \left| \begin{array}{l} x_i x_j = x_j x_i \text{ if } |i - j| \geq 2 \\ x_{i+1} x_i x_{i+1} = x_i x_{i+1} x_i \text{ if } 1 \leq i \leq n - 1 \end{array} \right. \right\rangle.$$

Elements of B_{n+1} are words expressed in the generators x_1, x_2, \dots, x_n and their inverses. The braid monoid MB_{n+1} has the same presentation as B_{n+1} .

The *generalized Hecke algebras* [5] are defined as the quotients

$$H(Q, n + 1) = \mathbb{C}[B_{n+1}] / (Q(b_i); i = 1, \dots, n)$$

of the group algebras of the braid group by the ideal generated by a polynomial $Q(b_i)$, having $Q(0) \neq 0$. If the degree of Q equals 3 we call them cubic Hecke algebras.

Coxeter [3] computed the cardinalities of the quotients of B_3 by relations $x_i^n = 1$, namely 24 for $n = 3$, 96 for $n = 4$, and 600 for $n = 5$. Then the algebras

$$H(Q_m, 3) = \left\langle b_1, b_2 : b_2 b_1 b_2 = b_1 b_2 b_1, b_1^m = 1, b_2^m = 1 \right\rangle$$

are finite dimensional of these dimensions, where $Q_m = x^m - 1$, $m \in \{2, 3, 4, 5\}$. For $n \geq 6$ these algebras are infinite dimensional. This motivated us to compute the Hilbert series in the finite dimensional case.

In [6] we constructed a linear system for the braid monoid B_{n+1} and computed the Hilbert series for the braid monoids MB_3 and MB_4 . In [7] we computed the Hilbert series of braid monoid MB_4 in band generators. In this paper we construct a similar kind of linear system to compute the Hilbert series.

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2. Hilbert series of the group algebras $H(Q_m, 3), m = 3, 4, 5$

Definition 2.1 [4] Let G be a finitely generated group and S be a finite set of generators of G . The word length $l_S(g)$ of an element $g \in G$ is the smallest integer n for which there exists $s_1, \dots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$.

Definition 2.2 [4] Let G be a finitely generated group and S be a finite set of generators of G . The growth function of the pair (G, S) associates to an integer $k \geq 0$ the number $a(k)$ of elements $g \in G$ such that $l_S(g) = k$ and the corresponding spherical growth series or the Hilbert series is given by $P_G(t) = \sum_{k=0}^{\infty} a(k)t^k$.

To get a canonical form of a word in an algebra the diamond lemma by Bergman [2] is extremely useful. To understand the notions of reductions and ambiguities we start with his terminology.

Let k be a commutative associative ring with unity, X a set, $\langle X \rangle$ the free monoid on X , and $k\langle X \rangle$ the free associative k -algebra on X . Let S be a set of pairs of the form $\sigma = (W_\sigma, f_\sigma)$, where $W_\sigma \in \langle X \rangle$ and $f_\sigma \in k\langle X \rangle$. For any $\sigma \in S$ and $A, B \in \langle X \rangle$, let $r_{A\sigma B} : k\langle X \rangle \rightarrow k\langle X \rangle$ be a k -module endomorphism such that this morphism sends $AW_\sigma B$ to $Af_\sigma B$ (r_* fixes all other elements of $\langle X \rangle$). The maps r_* are said to *reductions*. Let $\sigma, \tau \in S$ and $A, B, C \in \langle X \rangle - 1$ such that $W_\sigma = AB, W_\tau = BC$, and then ABC is said to an *ambiguity* of S . An element $a \in k\langle X \rangle$ is called *irreducible* (or *canonical*) if a involves none of the monomials $AW_\sigma B$; otherwise, a is called *reducible* (for more details see [2]).

If all relations in $k\langle X \rangle$ as a module are defined then we say that we have a complete set of relations in $k\langle X \rangle$. The diamond lemma [2] says that a set of relations is complete if all the ambiguities are solved. We call a complete set of relations in $H(Q_m, 3)$ a *complete presentation* of $H(Q_m, 3)$. The other names for the complete presentation being used are Gröbner bases, presentation with solvable ambiguities, rewriting systems, etc.

We made the terminology more understandable for the words in the braid monoids and for the algebra $H(Q_m, 3)$. Applying the above reductions we solve the ambiguities in $H(Q_m, 3)$. The solution of all the ambiguities gives us a complete set of relations in $H(Q_m, 3)$.

Let us have a few more words about the ambiguity and its solution. Let U, V , and w be nonempty words; then we denote $Uw \times_w wV$ by the word UwV . In a relation in $H(Q_m, 3)$ we place the equivalent words on the left-hand side that are greater in length-lexicographic ordering (we choose a natural order $b_1 < b_2 < \dots < b_n$ between the generators). For example, the words $b_2b_1b_2$ and $b_1b_2b_1$ are equivalent in the braid monoid MB_3 . Hence, we write $b_2b_1b_2 = b_1b_2b_1$ as the basic braid relation. Let Uw and wV be two words consisting of the left-hand sides of two relations in MB_3 ; then (defined as above, too) the word $W = Uw \times_w wV$ is an ambiguity. In this case W has two resolutions, namely $(Uw)V$ and $U(wV)$. If we apply finite reductions on $(Uw)V$ and $U(wV)$ and both give exactly the same word then we say that the ambiguity W is solvable. If $(Uw)V$ and $U(wV)$ differ by lexicographically then W gives a new relation in MB_3 .

For example, consider a word $b_2b_1b_2$ from the left-hand side of a relation in $H(Q_3, 3)$. Then a word $b_2b_1b_2b_1b_2$ is an ambiguity and it has two resolutions $b_2b_1(b_2b_1b_2)$ and $(b_2b_1b_2)b_1b_2$. Applying a reduction on the first we have $b_2b_1^2b_2b_1$ and on second we get $b_1b_2b_1^2b_2$. Hence, we have the relation

$$b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2 \tag{2.1}$$

in a complete presentation of $H(Q_3, 3)$. In this way we solve all ambiguities in $H(Q_m, 3)$. Funar [5] gave a complete presentation of $H(Q_3, 3)$:

$$H(Q_3, 3) = \langle b_1, b_2 : b_2b_1b_2 = b_1b_2b_1, b_1^3 = 1, b_2^3 = 1, b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2, b_2b_1^2b_2^2 = b_1^2b_2^2b_1, b_2^2b_1^2b_2 = b_1b_2^2b_1^2 \rangle.$$

As defined earlier (here in the form of relations), in a complete presentation of $H(Q_m, 3)$, a word containing a subword consisting of L.H.S. of any relation of $H(Q_m, 3)$ will be called a *reducible word* and a word that does not contain a subword consisting of L.H.S. of any relation will be called an *irreducible word*.

In general we denote by $B_*^{(m)}$ the set of reducible words and by $A_*^{(m)}$ the set of irreducible words in $H(Q_m, 3)$. The words $Ub_2 \times_2 b_2V$ and $Ub_2b_1 \times_{21} b_2b_1V$ denote the products Ub_2V and Ub_2b_1V , respectively.

Let us denote by $B_1^{(m)} = \{b_2b_1b_2\}$ and $B_{k+1}^{(m)} = \{b_2b_1^{k+1}b_2b_1\}$ the set of reducible words of $H(Q_m, 3)$. For the irreducible words we use the following notations: $A_k^{(m)}$ denote the set of irreducible words starting with b^k and $A_{2^k,1}^{(m)}$ denote the set of irreducible words starting with $b_2^k b_1$. The Hilbert series of $A_k^{(m)}$, $A_{2^k,1}^{(m)}$, and $H(Q_m, 3)$ are denoted by $P_k^{(m)}$, $P_{2^k,1}^{(m)}$, and $P_H^{(m)}(t)$, respectively, where

$$P_H^{(m)}(t) = 1 + P_1^{(m)} + P_2^{(m)}.$$

For the computations of the Hilbert series of $H(Q_3, 3)$, we have the following linear system for the irreducible words.

Proposition 2.3 *The following equalities hold for the Hilbert series of irreducible words in $H(Q_3, 3)$:*

- 1) $P_1^{(3)} = t + t^2 + (t + t^2)P_2^{(3)}$,
- 2) $P_2^{(3)} = t + t^2 + P_{2,1}^{(3)} + P_{2^2,1}^{(3)}$,
- 3) $P_{2,1}^{(3)} = tP_1^{(3)} - t^2P_2^{(3)} - t^3P_{2,1}^{(3)} - t^5 - t^6 - t^7$,
- 4) $P_{2^2,1}^{(3)} = tP_{2,1}^{(3)} - t^5$.

Proof 1) The set $A_1^{(3)}$ is decomposed as $A_1^{(3)} = \{b_1, b_1^2\} \cup (\{b_1, b_1^2\} \times A_2^{(3)})$. This implies Relation 1).

2) It follows immediately from $A_2^{(3)} = \{b_1, b_1^2\} \cup A_{2,1}^{(3)} \cup A_{2^2,1}^{(3)}$.

3) The decomposition of $A_{2,1}^{(3)}$ is given by

$$A_{2,1}^{(3)} = (\{b_2\} \times A_1^{(3)}) \setminus \left((B_1^{(3)} \times_2 A_2^{(3)}) \cup (B_2^{(3)} \times_{21} A_{2,1}^{(3)}) \cup \{b_2b_1^2b_2^2, b_2b_1^2b_2^2b_1, b_2b_1^2b_2^2b_1^2\} \right) \text{ and hence we have 3)).}$$

4) It follows from $A_{2^2,1}^{(3)} = (\{b_2\} \times A_{2,1}^{(3)}) \setminus \{b_2^2b_1^2b_2\}$. □

For the computation of Hilbert series the complete presentation of the algebra is very important. By the diamond lemma [2], the set of relations is complete in the complete presentation. Here we give the complete presentation of $H(Q_4, 3)$ in the following:

Proposition 2.4 *A complete presentation of $H(Q_4, 3)$ is given as*

$$H(Q_4, 3) = \langle b_1, b_2 : b_2b_1b_2 = b_1b_2b_1, b_1^4 = 1, b_2^4 = 1, b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2, b_2b_1^3b_2b_1 = b_1b_2b_1^2b_2^2, \\ b_2b_1^2b_2^2b_1^2 = b_1^2b_2^2b_1^2b_2, b_2b_1^2b_2^3 = b_1^3b_2^3b_1, b_2b_1^3b_2^3 = b_1^3b_2^3b_1, b_2b_1^3b_2^2b_1^3 = b_1^3b_2^2b_1^3b_2, b_2b_1^3b_2^2b_1^2b_2 = b_1b_2b_1^3b_2^2b_1^2, \\ b_2^2b_1^2b_2^2b_1 = b_1b_2^2b_1^2b_2^2, b_2^2b_1^3b_2^2b_1 = b_1b_2^2b_1^3b_2^2, b_2^3b_1^2b_2 = b_1b_2^3b_1^2, b_2^3b_1^3b_2 = b_1b_2^3b_1^3 \rangle.$$

(An outline of the proof of Proposition 2.4 is given in the Appendix.)

Now we develop a linear system for the irreducible words in $H(Q_4, 3)$.

Proposition 2.5 *The following equalities hold for the Hilbert series of irreducible words in $H(Q_4, 3)$:*

- 1) $P_1^{(4)} = t + t^2 + t^3 + (t + t^2 + t^3)P_2^{(4)}$,
- 2) $P_2^{(4)} = t + t^2 + t^3 + P_{2,1}^{(4)} + P_{2,2,1}^{(4)} + P_{2,3,1}^{(4)}$,
- 3) $P_{2,1}^{(4)} = tP_1^{(4)} - t^2P_2^{(4)} - t^3P_{2,1}^{(4)} - t^4P_{2,2,1}^{(4)} - t^6 - 3t^7 - 4t^8 - 6t^9 - 4t^{10} - t^{11}$,
- 4) $P_{2,2,1}^{(4)} = tP_{2,1}^{(4)} - t^7 - t^8 - t^9$,
- 5) $P_{2,3,1}^{(4)} = tP_{2,2,1}^{(4)} - t^6 - 2t^7 - t^8$.

Proof 1) The set $A_1^{(4)}$ is decomposed as $A_1^{(4)} = \{b_1, b_1^2, b_1^3\} \cup (\{b_1, b_1^2, b_1^3\} \times A_2^{(4)})$. This gives Relation 1).

2) It follows immediately from $A_2^{(4)} = \{b_2, b_2^2, b_2^3\} \cup A_{2,1}^{(4)} \cup A_{2,2,1}^{(4)} \cup A_{2,3,1}^{(4)}$.

3) The decomposition of $A_{2,1}^{(4)}$ is given by

$A_{2,1}^{(4)} = (\{b_2\} \times A_1^{(4)}) \setminus \left((B_1^{(4)} \times_2 A_2^{(4)}) \cup (B_2^{(4)} \times_{21} A_{2,1}^{(4)}) \cup (B_3^{(4)} \times_{21} A_{2,1}^{(4)}) \cup \{b_2b_1^2b_2^3, b_2b_1^2b_2^3b_1, b_2b_1^2b_2^3b_1^2, b_2b_1^2b_2^3b_1^3, b_2b_1^3b_2^3, b_2b_1^3b_2^3b_1, b_2b_1^3b_2^3b_1^2, b_2b_1^3b_2^3b_1^3, b_2b_1^2b_2^2b_1^2, b_2b_1^2b_2^2b_1^3, b_2b_1^2b_2^2b_1^4, b_2b_1^2b_2^2b_1^5, b_2b_1^2b_2^2b_1^6, b_2b_1^2b_2^2b_1^7, b_2b_1^2b_2^2b_1^8, b_2b_1^2b_2^2b_1^9, b_2b_1^2b_2^2b_1^{10}, b_2b_1^2b_2^2b_1^{11}, b_2b_1^2b_2^2b_1^{12}, b_2b_1^2b_2^2b_1^{13}, b_2b_1^2b_2^2b_1^{14}, b_2b_1^2b_2^2b_1^{15}, b_2b_1^2b_2^2b_1^{16}, b_2b_1^2b_2^2b_1^{17}, b_2b_1^2b_2^2b_1^{18}, b_2b_1^2b_2^2b_1^{19}, b_2b_1^2b_2^2b_1^{20}, b_2b_1^2b_2^2b_1^{21}, b_2b_1^2b_2^2b_1^{22}, b_2b_1^2b_2^2b_1^{23}, b_2b_1^2b_2^2b_1^{24}, b_2b_1^2b_2^2b_1^{25}, b_2b_1^2b_2^2b_1^{26}, b_2b_1^2b_2^2b_1^{27}, b_2b_1^2b_2^2b_1^{28}, b_2b_1^2b_2^2b_1^{29}, b_2b_1^2b_2^2b_1^{30}, b_2b_1^2b_2^2b_1^{31}, b_2b_1^2b_2^2b_1^{32}, b_2b_1^2b_2^2b_1^{33}, b_2b_1^2b_2^2b_1^{34}, b_2b_1^2b_2^2b_1^{35}, b_2b_1^2b_2^2b_1^{36}, b_2b_1^2b_2^2b_1^{37}, b_2b_1^2b_2^2b_1^{38}, b_2b_1^2b_2^2b_1^{39}, b_2b_1^2b_2^2b_1^{40}, b_2b_1^2b_2^2b_1^{41}, b_2b_1^2b_2^2b_1^{42}, b_2b_1^2b_2^2b_1^{43}, b_2b_1^2b_2^2b_1^{44}, b_2b_1^2b_2^2b_1^{45}, b_2b_1^2b_2^2b_1^{46}, b_2b_1^2b_2^2b_1^{47}, b_2b_1^2b_2^2b_1^{48}, b_2b_1^2b_2^2b_1^{49}, b_2b_1^2b_2^2b_1^{50}, b_2b_1^2b_2^2b_1^{51}, b_2b_1^2b_2^2b_1^{52}, b_2b_1^2b_2^2b_1^{53}, b_2b_1^2b_2^2b_1^{54}, b_2b_1^2b_2^2b_1^{55}, b_2b_1^2b_2^2b_1^{56}, b_2b_1^2b_2^2b_1^{57}, b_2b_1^2b_2^2b_1^{58}, b_2b_1^2b_2^2b_1^{59}, b_2b_1^2b_2^2b_1^{60}, b_2b_1^2b_2^2b_1^{61}, b_2b_1^2b_2^2b_1^{62}, b_2b_1^2b_2^2b_1^{63}, b_2b_1^2b_2^2b_1^{64}, b_2b_1^2b_2^2b_1^{65}, b_2b_1^2b_2^2b_1^{66}, b_2b_1^2b_2^2b_1^{67}, b_2b_1^2b_2^2b_1^{68}, b_2b_1^2b_2^2b_1^{69}, b_2b_1^2b_2^2b_1^{70}, b_2b_1^2b_2^2b_1^{71}, b_2b_1^2b_2^2b_1^{72}, b_2b_1^2b_2^2b_1^{73}, b_2b_1^2b_2^2b_1^{74}, b_2b_1^2b_2^2b_1^{75}, b_2b_1^2b_2^2b_1^{76}, b_2b_1^2b_2^2b_1^{77}, b_2b_1^2b_2^2b_1^{78}, b_2b_1^2b_2^2b_1^{79}, b_2b_1^2b_2^2b_1^{80}, b_2b_1^2b_2^2b_1^{81}, b_2b_1^2b_2^2b_1^{82}, b_2b_1^2b_2^2b_1^{83}, b_2b_1^2b_2^2b_1^{84}, b_2b_1^2b_2^2b_1^{85}, b_2b_1^2b_2^2b_1^{86}, b_2b_1^2b_2^2b_1^{87}, b_2b_1^2b_2^2b_1^{88}, b_2b_1^2b_2^2b_1^{89}, b_2b_1^2b_2^2b_1^{90}, b_2b_1^2b_2^2b_1^{91}, b_2b_1^2b_2^2b_1^{92}, b_2b_1^2b_2^2b_1^{93}, b_2b_1^2b_2^2b_1^{94}, b_2b_1^2b_2^2b_1^{95}, b_2b_1^2b_2^2b_1^{96}, b_2b_1^2b_2^2b_1^{97}, b_2b_1^2b_2^2b_1^{98}, b_2b_1^2b_2^2b_1^{99}, b_2b_1^2b_2^2b_1^{100} \}$ and hence we have 3).

4) It follows from $A_{2,2,1}^{(4)} = (\{b_2\} \times A_{2,1}^{(4)}) \setminus \{b_2^2b_1^2b_2^3b_1, b_2^2b_1^3b_2^3b_1, b_2^2b_1^3b_2^3b_1^2\}$.

5) It follows from $A_{2,3,1}^{(4)} = (\{b_2\} \times A_{2,2,1}^{(4)}) \setminus \{b_2^3b_1^2b_2, b_2^3b_1^2b_2^2, b_2^3b_1^3b_2, b_2^3b_1^3b_2^2\}$. □

Proposition 2.6 *A complete presentation of $H(Q_5, 3)$ is given by*

$H(Q_5, 3) = \langle b_1, b_2 : b_2b_1b_2 = b_1b_2b_1, b_1^5 = 1, b_2^5 = 1, R_i, 1 \leq i \leq 41 \rangle$, where the relations R_i are given by

- $R_1 : b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2, R_2 : b_2b_1^3b_2b_1 = b_1b_2b_1^2b_2^2, R_3 : b_2b_1^4b_2b_1 = b_1b_2b_1^2b_2^3,$
- $R_4 : b_2b_1^2b_2^4 = b_1^4b_2^2b_1, R_5 : b_2b_1^3b_2^4 = b_1^4b_2^3b_1, R_6 : b_2b_1^4b_2^4 = b_1^4b_2^4b_1,$
- $R_7 : b_2^4b_1^2b_2 = b_1b_2^5b_1^4, R_8 : b_2^4b_1^3b_2 = b_1b_2^5b_1^4, R_9 : b_2^4b_1^4b_2 = b_1b_2^5b_1^4,$
- $R_{10} : b_2b_1^2b_2^2b_1^3 = b_1^3b_2^3b_1^2b_2, R_{11} : b_2b_1^2b_2^3b_1^3 = b_1^3b_2^3b_1^2b_2, R_{12} : b_2b_1^3b_2^3b_1^4 = b_1^4b_2^3b_1^3b_2,$
- $R_{13} : b_2b_1^4b_2^3b_1^4 = b_1^4b_2^3b_1^3b_2, R_{14} : b_2b_1^3b_2^3b_1^4 = b_1^4b_2^3b_1^4b_2, R_{15} : b_2b_1^4b_2^3b_1^4 = b_1^4b_2^3b_1^4b_2,$
- $R_{16} : b_2^3b_1^2b_2^2b_1 = b_1b_2^2b_1^3b_2^3, R_{17} : b_2^3b_1^3b_2^2b_1 = b_1b_2^2b_1^3b_2^3, R_{18} : b_2^3b_1^4b_2^2b_1 = b_1b_2^2b_1^4b_2^3,$
- $R_{19} : b_2^3b_1^4b_2^3b_1 = b_1b_2^3b_1^4b_2^3, R_{20} : b_2b_1^2b_2^2b_1^2b_2^2 = b_1^2b_2^2b_1^2b_2^2b_1,$
- $R_{21} : b_2b_1^2b_2^3b_1^2b_2^3 = b_1^3b_2^3b_1^2b_2^3b_1, R_{22} : b_2b_1^3b_2^3b_1^3b_2^3 = b_1^3b_2^3b_1^3b_2^3b_1,$
- $R_{23} : b_2b_1^3b_2^3b_1^3b_2^3 = b_1^3b_2^3b_1^3b_2^3b_1, R_{24} : b_2b_1^4b_2^3b_1^2b_2 = b_1b_2b_1^3b_2^3b_1^3,$
- $R_{25} : b_2b_1^4b_2^3b_1^2b_2 = b_1b_2b_1^3b_2^3b_1^3, R_{26} : b_2^2b_1^2b_2^2b_1^2b_2 = b_1b_2^2b_1^2b_2^2b_1^2,$
- $R_{27} : b_2^2b_1^3b_2^2b_1^2b_2 = b_1b_2^2b_1^3b_2^2b_1^2, R_{28} : b_2^2b_1^4b_2^2b_1^2b_2 = b_1b_2^2b_1^4b_2^2b_1^2,$
- $R_{29} : b_2^2b_1^4b_2^2b_1^2b_2 = b_1b_2^2b_1^4b_2^2b_1^2, R_{30} : b_2^2b_1^2b_2^2b_1^2b_2 = b_1b_2^2b_1^2b_2^2b_1^2,$
- $R_{31} : b_2^2b_1^3b_2^2b_1^2b_2 = b_1b_2^2b_1^3b_2^2b_1^2, R_{32} : b_2^2b_1^3b_2^2b_1^2b_2 = b_1b_2^2b_1^3b_2^2b_1^2,$
- $R_{33} : b_2b_1^3b_2^2b_1^2b_2^2b_1 = b_1b_2b_1^3b_2^2b_1^2b_2^2, R_{34} : b_2b_1^3b_2^2b_1^2b_2^2b_1 = b_1^3b_2^2b_1^2b_2^2b_1^2b_2,$
- $R_{35} : b_2b_1^3b_2^2b_1^2b_2^2b_1 = b_1^3b_2^2b_1^2b_2^2b_1^2b_2, R_{36} : b_2b_1^4b_2^2b_1^2b_2^2b_1 = b_1b_2b_1^4b_2^2b_1^2b_2^2,$

$$\begin{aligned}
 R_{37} : b_2b_1^4b_2^3b_1^3b_1^3 &= b_1^4b_2^3b_1^3b_2^3b_2, R_{38} : b_2b_1^4b_2^3b_1^3b_1 = b_1b_2b_1^4b_2^3b_1^3b_2^3, \\
 R_{39} : b_2^3b_1^3b_2^3b_1^3b_1 &= b_1b_2^3b_1^3b_2^3b_1^2, R_{40} : b_2^3b_1^3b_2^3b_1^3b_1 = b_1b_2^3b_1^3b_2^3b_1^2b_2^3, \\
 R_{41} : b_2b_1^4b_2^3b_1^3b_2^3b_1^2 &= b_1b_2b_1^4b_2^3b_1^3b_2^3b_1^2.
 \end{aligned}$$

(An outline of the proof of Proposition 2.6 is given in the Appendix.)

Now we develop a linear system for the irreducible words in $H(Q_5, 3)$.

Proposition 2.7 *The following equalities hold for the Hilbert series of irreducible words in $H(Q_5, 3)$:*

- 1) $P_1^{(5)} = t + t^2 + t^3 + t^4 + (t + t^2 + t^3 + t^4)P_2^{(5)}$,
- 2) $P_2^{(5)} = t + t^2 + t^3 + t^4 + P_{2.1}^{(5)} + P_{2^2.1}^{(5)} + P_{2^3.1}^{(5)} + P_{2^4.1}^{(5)}$,
- 3) $P_{2.1}^{(5)} = tP_1^{(5)} - t^2P_2^{(5)} - (t^3 + t^4 + t^5)P_{2.1}^{(5)} - t^7 - 3t^8 - 7t^9 - 11t^{10} - 17t^{11} - 19t^{12} - 21t^{13} - 20t^{14} - 19t^{15} - 14t^{16} - 7t^{17} - 2t^{18}$,
- 4) $P_{2^2.1}^{(5)} = tP_{2.1}^{(5)} - t^9 - t^{10} - t^{11} - 2t^{12} - 4t^{13} - 4t^{14} - 3t^{15} - t^{16}$,
- 5) $P_{2^3.1}^{(5)} = tP_{2^2.1}^{(5)} - t^8 - 2t^9 - 2t^{10} - 4t^{11} - 5t^{12} - 4t^{13} - 2t^{14}$,
- 6) $P_{2^4.1}^{(5)} = tP_{2^3.1}^{(5)} - t^7 - 2t^8 - 3t^9 - 3t^{10} - 3t^{11} - t^{12} - t^{13}$.

Proof 1) The set $A_1^{(5)}$ is decomposed as

$$A_1^{(5)} = \{b_1, b_1^2, b_1^3, b_1^4\} \cup (\{b_1, b_1^2, b_1^3, b_1^4\} \times A_2^{(5)}). \text{ This gives Relation 1).}$$

2) It follows immediately from

$$A_2^{(5)} = \{b_2, b_2^2, b_2^3, b_2^4\} \cup A_{2.1}^{(5)} \cup A_{2^2.1}^{(5)} \cup A_{2^3.1}^{(5)} \cup A_{2^4.1}^{(5)}.$$

3) The decomposition of $A_{2.1}^{(5)}$ is given by

$$\begin{aligned}
 A_{2.1}^{(5)} = (\{b_2\} \times A_1^{(5)}) \setminus & \left((B_1^{(5)} \times_{21} A_2^{(5)}) \cup (B_2^{(5)} \times_{21} A_{2.1}^{(5)}) \cup (B_3^{(5)} \times_{21} A_{2.1}^{(5)}) \cup (B_4^{(5)} \times_{21} A_{2.1}^{(5)}) \cup \{b_2b_1^2b_1^4, \right. \\
 & b_2b_1^2b_1^4b_1, b_2b_1^2b_2^4b_1^2, b_2b_1^2b_2^4b_1^3, b_2b_1^2b_2^4b_1^4, b_2b_1^3b_1^4, b_2b_1^3b_2^4b_1, b_2b_1^3b_2^4b_1^2, b_2b_1^3b_2^4b_1^3, b_2b_1^3b_2^4b_1^4, b_2b_1^4b_1^4, \\
 & b_2b_1^4b_2^4b_1^2, b_2b_1^4b_2^4b_1^3, b_2b_1^4b_2^4b_1^4, b_2b_1^2b_2^3b_1^3, b_2b_1^2b_2^3b_1^4, b_2b_1^2b_2^3b_1^5, b_2b_1^2b_2^3b_1^6, b_2b_1^2b_2^3b_1^7, b_2b_1^2b_2^3b_1^8, b_2b_1^2b_2^3b_1^9, \\
 & b_2b_1^2b_2^3b_1^{10}, b_2b_1^2b_2^3b_1^{11}, b_2b_1^2b_2^3b_1^{12}, b_2b_1^2b_2^3b_1^{13}, b_2b_1^2b_2^3b_1^{14}, b_2b_1^2b_2^3b_1^{15}, b_2b_1^2b_2^3b_1^{16}, b_2b_1^2b_2^3b_1^{17}, b_2b_1^2b_2^3b_1^{18}, \\
 & b_2b_1^2b_2^3b_1^{19}, b_2b_1^2b_2^3b_1^{20}, b_2b_1^2b_2^3b_1^{21}, b_2b_1^2b_2^3b_1^{22}, b_2b_1^2b_2^3b_1^{23}, b_2b_1^2b_2^3b_1^{24}, b_2b_1^2b_2^3b_1^{25}, b_2b_1^2b_2^3b_1^{26}, \\
 & b_2b_1^2b_2^3b_1^{27}, b_2b_1^2b_2^3b_1^{28}, b_2b_1^2b_2^3b_1^{29}, b_2b_1^2b_2^3b_1^{30}, b_2b_1^2b_2^3b_1^{31}, b_2b_1^2b_2^3b_1^{32}, b_2b_1^2b_2^3b_1^{33}, b_2b_1^2b_2^3b_1^{34}, \\
 & b_2b_1^2b_2^3b_1^{35}, b_2b_1^2b_2^3b_1^{36}, b_2b_1^2b_2^3b_1^{37}, b_2b_1^2b_2^3b_1^{38}, b_2b_1^2b_2^3b_1^{39}, b_2b_1^2b_2^3b_1^{40}, b_2b_1^2b_2^3b_1^{41}, b_2b_1^2b_2^3b_1^{42}, \\
 & b_2b_1^2b_2^3b_1^{43}, b_2b_1^2b_2^3b_1^{44}, b_2b_1^2b_2^3b_1^{45}, b_2b_1^2b_2^3b_1^{46}, b_2b_1^2b_2^3b_1^{47}, b_2b_1^2b_2^3b_1^{48}, b_2b_1^2b_2^3b_1^{49}, b_2b_1^2b_2^3b_1^{50}, \\
 & b_2b_1^2b_2^3b_1^{51}, b_2b_1^2b_2^3b_1^{52}, b_2b_1^2b_2^3b_1^{53}, b_2b_1^2b_2^3b_1^{54}, b_2b_1^2b_2^3b_1^{55}, b_2b_1^2b_2^3b_1^{56}, b_2b_1^2b_2^3b_1^{57}, b_2b_1^2b_2^3b_1^{58}, \\
 & b_2b_1^2b_2^3b_1^{59}, b_2b_1^2b_2^3b_1^{60}, b_2b_1^2b_2^3b_1^{61}, b_2b_1^2b_2^3b_1^{62}, b_2b_1^2b_2^3b_1^{63}, b_2b_1^2b_2^3b_1^{64}, b_2b_1^2b_2^3b_1^{65}, b_2b_1^2b_2^3b_1^{66}, \\
 & b_2b_1^2b_2^3b_1^{67}, b_2b_1^2b_2^3b_1^{68}, b_2b_1^2b_2^3b_1^{69}, b_2b_1^2b_2^3b_1^{70}, b_2b_1^2b_2^3b_1^{71}, b_2b_1^2b_2^3b_1^{72}, b_2b_1^2b_2^3b_1^{73}, b_2b_1^2b_2^3b_1^{74}, \\
 & b_2b_1^2b_2^3b_1^{75}, b_2b_1^2b_2^3b_1^{76}, b_2b_1^2b_2^3b_1^{77}, b_2b_1^2b_2^3b_1^{78}, b_2b_1^2b_2^3b_1^{79}, b_2b_1^2b_2^3b_1^{80}, b_2b_1^2b_2^3b_1^{81}, b_2b_1^2b_2^3b_1^{82}, \\
 & b_2b_1^2b_2^3b_1^{83}, b_2b_1^2b_2^3b_1^{84}, b_2b_1^2b_2^3b_1^{85}, b_2b_1^2b_2^3b_1^{86}, b_2b_1^2b_2^3b_1^{87}, b_2b_1^2b_2^3b_1^{88}, b_2b_1^2b_2^3b_1^{89}, b_2b_1^2b_2^3b_1^{90}, \\
 & b_2b_1^2b_2^3b_1^{91}, b_2b_1^2b_2^3b_1^{92}, b_2b_1^2b_2^3b_1^{93}, b_2b_1^2b_2^3b_1^{94}, b_2b_1^2b_2^3b_1^{95}, b_2b_1^2b_2^3b_1^{96}, b_2b_1^2b_2^3b_1^{97}, b_2b_1^2b_2^3b_1^{98}, \\
 & b_2b_1^2b_2^3b_1^{99}, b_2b_1^2b_2^3b_1^{100} \}.
 \end{aligned}$$

+ $6t^{13} + 3t^{14} + t^{15}$. Therefore, we have

$$\begin{aligned} P_H^{(5)}(t) &= 1 + P_1^{(5)} + P_2^{(5)} \\ &= 1 + 2t + 4t^2 + 7t^3 + 12t^4 + 18t^5 + 27t^6 + 38t^7 + 50t^8 + 59t^9 \\ &\quad + 67t^{10} + 70t^{11} + 68t^{12} + 59t^{13} + 48t^{14} + 34t^{15} + 21t^{16} + 10t^{17} \\ &\quad + 4t^{18} + t^{19}. \end{aligned}$$

□

Remark 2.9 Note that the degrees $d(P_H^{(m)})$ of the polynomial of the Hilbert series of $H(Q_m, 3)$, $m \in \{1, \dots, 5\}$ are 1, 3, 6, 11, 19. One can see that $d(P_H^{(m)})$ is related with Fibonacci numbers $F_m = F_{m-1} + F_{m-2}$ ($F_0 = 1$, $F_1 = 1$) by the relation $d(P_H^{(m)}) = F_{m+2} - 2$.

Remark 2.10 We believe that the result holds also when Q_m is an arbitrary polynomial of degree m , with $Q_m(0) \neq 0$, for $m = 3, 4, 5$. This is known to be true for $m = 3$ ([5]).

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Appendix

Proof of Proposition 2.4 (outline). We solve all the ambiguities to find out all the relations given in these propositions. The computations are too long, so we give here a few computations to find out the relations.

- Relation 2.1 has been explained before.
- The ambiguity $b_2b_1^2b_2b_1b_2$ has two resolutions $b_2b_1^2(b_2b_1b_2)$ and $(b_2b_1^2b_2b_1)b_2$. Applying reductions on both we get $b_2b_1^3b_2b_1$ and $b_1b_2b_1^2b_2^2$, respectively. We say that the ambiguity $b_2b_1^2b_2b_1b_2$ gives a relation $b_2b_1^3b_2b_1 = b_1b_2b_1^2b_2^2$.
- Similarly the ambiguity $b_2b_1b_2b_1^3b_2b_1$ gives a relation $b_1^2b_2b_1^2b_2^3 = b_1b_2^2b_1$. This relation is nonhomogeneous (i.e. the degrees are not same on both sides). By left multiplication to this relation by b_1^2 we make it homogeneous and get another relation, $b_2b_1^2b_2^3 = b_1^3b_2^2b_1$.
- The ambiguity $b_2b_1b_2b_1^2b_2^3$ gives a relation $b_1b_2b_1^3b_2^3 = b_2^3b_1$. Again this relation is not homogeneous. Left multiplication by b_1^3 gives $b_2b_1^3b_2^3 = b_1^3b_2^3b_1$.

By continuing solving the ambiguities we get new relations. In the process some ambiguities are solvable and give no new relations. Hence, solving all the ambiguities, we have the presentation of the Proposition 2.4.

Proof of Proposition 2.6 (outline). Proof of this proposition is much longer than that of Proposition 2.4. As above, here we give the proof of a few relations. Proofs of others are based on similar computations.

- R_1 and R_2 are the relations of $H(Q_4, 3)$.
- The ambiguity $b_2b_1^3b_2b_1b_2$ gives R_3 .
- $b_2b_1^4b_2b_1b_2$ gives a nonhomogeneous relation $b_1b_2b_1^2b_2^4 = b_2^2b_1$. Left multiplication by b_1^4 gives R_4 .
- $b_2b_1b_2b_1^2b_2^4$ gives $b_1b_2b_1^3b_2^4 = b_2^3b_1$. Left multiplication by b_1^4 gives R_5 .

- $b_2 b_1 b_2 b_1^3 b_2^4$ gives $b_1 b_2 b_1^4 b_2^4 = b_2^4 b_1$. Left multiplication by b_1^4 gives R_6 .
- $b_2 b_1^4 b_2^4 b_1 b_2$ gives $b_1^4 b_2^4 b_1^2 b_2 = b_2^2 b_1^4$. Left multiplication by b_1 gives R_7 .
- $b_2 b_1^2 b_2^4 b_1^2 b_2$ gives R_{12} .
- $b_2 b_1^3 b_2^4 b_1^2 b_2$ gives R_{13} .
- $b_2 b_1^3 b_2^4 b_1^3 b_2$ gives R_{15} .
- $b_2 b_1 b_2 b_1^4 b_2^2 b_1^4$ gives R_8 .
- $b_2 b_1 b_2 b_1^4 b_2^3 b_1^4$ gives R_9 and so on. By following this procedure and checking all the ambiguities and after a lot of computations we have all the relations. This process terminates when all the ambiguities are solvable and gives no further new relations.

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