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Split extension classifiers in the category of precrossed modules of commutative algebras

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Abstract: We construct an actor of a precat^1-algebra and then by using the natural equivalence between the categories of precat^1-algebras and that of precrossed modules, we construct the split extension classifier of the corresponding precrossed module, which gives rise to the representability of actions in the category of precrossed modules of commutative algebras under certain conditions.

Key words: Precrossed module, action, split extension classifier, actor

1. Introduction

In an algebraic category, obstruction theory of the objects depends on the representability of actions in the category. Representability of actions in semiabelian categories was investigated in [6]. A different study of this problem in the categories of interest was given in [9] with a combinatorial approach. The same was given for modified categories of interest in [8]. The definition of split extension classifier (object that represents actions) is formulated in [5] for semiabelian categories in terms of categorical notions of internal object action and semidirect product. Categories of interest are semiabelian categories. As an application of [7], in this special case these notions coincide with the ones given in [15]. An analogous situation exists in the case of modified category of interest defined in [8] and categories equivalent to them.

Many well-known categories of algebraic structures such as precat^1-algebras (Lie algebras, Leibniz algebras, associative algebras, associative commutative algebras) and commutative Von Neumann rings are modified categories of interest that satisfy all axioms of a category of groups with operations in [16] except one, which is replaced by a new axiom; these categories satisfy as well two additional axioms introduced in [15].

The category of precrossed modules of commutative algebras (which is equivalent to a modified category of interest, namely, the category of precat^1-algebras) was introduced in [16]. For related works, especially in higher dimensions, see [1, 2, 13]. Moreover, the (pre)crossed modules of commutative algebras were adapted to the computer environment in [4, 14]. The notion of crossed modules of commutative algebras can be thought of as a generalization of commutative algebras. For any algebra C, we have the crossed module $C \xrightarrow{id} C$ and

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so the category of commutative algebras is a full subcategory of crossed modules of commutative algebras. The same is true for precrossed modules, namely, $C \longrightarrow 0$ is a precrossed module, for any commutative algebra C. Naturally, it will be important to investigate the representability of actions, in other words, investigate the existence and construction of split extension classifiers in the category of precrossed modules.

Accordingly, for a given precrossed module $\mathcal{C} : C_1 \xrightarrow{d} C_0$, we found a condition under which we construct an actor of the corresponding precat^1-algebra $(C_1 \rtimes C_0, s, t)$ by using the general construction of universal strict general actor of an object given in [8]. Then, applying the equivalence of the categories **Precat^1-Comm** $\simeq$ **PXComm** of precat^1-algebras and precrossed modules, respectively, we carry the construction of an actor of $(C_1 \rtimes C_0, s, t)$ to the category of precrossed modules, which is a split extension classifier of the precrossed module $\mathcal{C} : C_1 \xrightarrow{d} C_0$ under certain conditions. Therefore, we found a new example of a category and individual objects there with representable actions. This problem is stated in [6] (Problem 2).

In order to achieve our goals the paper is organized as follows: in section 2, we give some needed notions from the literature and introduce the notions such as multipliers and generalized crossed multipliers of a precrossed module. In section 3, we construct an actor of a precat^1-algebra and consequently, in section 4, we construct the split extension classifier of a precrossed module.

2. Preliminaries

In this section we will recall some basic definitions and properties of precrossed modules of commutative algebras needed in the rest of the paper. Additionally, we define new notions such as multipliers and generalized crossed multipliers of a precrossed module and give some related properties. We also will recall the notion of the modified category of interest, and some related definitions and results from [8]. Finally, we will give the construction of a universal strict general actor of a precat^1 algebra (C, s^C, t^C) by using the general construction given for modified categories of interest in [8].

2.1. Precrossed modules of commutative algebras

Let k be a commutative ring with unit. All algebras in the present work will be over k and associative, commutative.

Definition 2.1 *Let C be an algebra. A k-linear map $f\colon C \to C$ satisfying $f(c * c') = f(c) * c'$ is called a multiplier of C.*

The set of all multipliers of C is denoted by $\mathcal{M}(C)$.

$\mathcal{M}(C)$ is not commutative, in general. If $C^2 = C$ or $\mathrm{Ann}C = 0$ then $\mathcal{M}(C)$ is commutative. See [3], for details.

Let C_1 and C_0 be algebras. Recall that an action of C_0 on C_1 is a k-linear map $C_0 \times C_1 \to C_1, (c_0, c_1) \mapsto c_0 \blacktriangleright c_1$ such that

$$\begin{aligned} c_0 \blacktriangleright (c_1 * c_1') &= (c_0 \blacktriangleright c_1) * c_1', \\ (c_0 * c_0') \blacktriangleright c_1 &= c_0 \blacktriangleright (c_0' \blacktriangleright c_1), \end{aligned}$$

for all $c_0, c_0' \in C_0$, $c_1, c_1' \in C_1$.

Example 2.2 *Let C be an algebra with $C^2 = C$ or $AnnC = 0$. Then the map*

$$\begin{array}{ccc} \mathcal{M}(C) \times C & \to & C \\ (f, c) & \mapsto & f(c) \end{array}$$

defines an action of $\mathcal{M}(C)$ on C.

Definition 2.3 *Let $d\colon C_1 \to C_0$ be an algebra homomorphism with an action of C_0 on C_1 denoted by $c_0 \blacktriangleright c_1$, for all $c_0 \in C_0$, $c_1 \in C_1$. If*

$$d(c_0 \blacktriangleright c_1) = c_0 * d(c_1)$$

for all $c_0 \in C_0$, $c_1 \in C_1$, then the system $\mathcal{C}\colon C_1 \xrightarrow{d} C_0$ is called a precrossed module. Additionally, if

$$d(c_1) \blacktriangleright c_1' = c_1 * c_1'$$

for all $c_1, c_1' \in C_1$, then it is called a crossed module (the second condition is called a Peiffer identity).

Let $\mathcal{C}\colon C_1 \xrightarrow{d} C_0$ and $\mathcal{C}'\colon C_1' \xrightarrow{d'} C_0'$ be precrossed modules. A pair (μ_1, μ_0) consists of k-algebra homomorphisms $\mu_1\colon C_1 \to C_1'$, $\mu_0\colon C_0 \to C_0'$ satisfying $\mu_0 d = d'\mu_1$ and $\mu_1(c_0 \blacktriangleright c_1) = (\mu_0(c_0)) \blacktriangleright (\mu_1(c_1))$, for all $c_0 \in C_0$, $c_1 \in C_1$ is called a homomorphism from $\mathcal{C}$ to $\mathcal{C}'$. This gives rise to the category of precrossed modules whose objects are precrossed modules and morphisms are homomorphisms of precrossed modules. We denote this category by **PXComm**. Simultaneously, we have the category of crossed modules, which we denote here by **XComm**.

Examples 2.4 ***(i)*** *Any ideal I of an algebra C gives rise to an inclusion map $I \overset{inc.}{\hookrightarrow} C$, which is a crossed module with the action defined by the multiplication. Conversely, if $d\colon C_1 \to C_0$ is a crossed module, then $Im(d)$ is an ideal of C_0. In particular, $C \overset{id}{\to} C$ and $0 \overset{inc.}{\hookrightarrow} C$ are also crossed modules.*
(ii) *Let C be an algebra. Consider the map $\pi_1\colon C \times C \to C$ and the action of C on $C \times C$ defined by componentwise multiplication. Then $C \times C \overset{\pi_1}{\to} C$ is a precrossed module, which is not a crossed module.*
(iii) *Let C be an algebra satisfying $C^2 = C$ or $AnnC = 0$. Then $d\colon C \to \mathcal{M}(C), c \mapsto \varphi_c$ is a crossed module with the action defined in Example 2.2 where $\varphi_c\colon C \to C$ is defined by $\varphi_c(x) = c * x$, for all $x \in C$.*

A precrossed module $\mathcal{C}'\colon C_1' \xrightarrow{d'} C_0'$ is a precrossed submodule of the precrossed module $\mathcal{C}\colon C_1 \xrightarrow{d} C_0$ if C_1', C_0' are subalgebras of C_1, C_0 respectively, d' is the restriction of d, and the action of C_0' on C_1' is induced from the action of C_0 on C_1. In addition, if C_1', C_0' are ideals of C_1, C_0, respectively, $c_0 \blacktriangleright c_1' \in C_1'$, for all $c_0 \in C_0$, $c_1' \in C_1'$ and $c_0' \blacktriangleright c_1 \in C_1'$, for all $c_0' \in C_0'$, $c_1 \in C_1$ then the precrossed submodule $\mathcal{C}'\colon C_1' \xrightarrow{d'} C_0'$ is called an ideal of $\mathcal{C}\colon C_1 \xrightarrow{d} C_0$.

Definition 2.5 *Let $\mathcal{C}\colon C_1 \overset{d}{\to} C_0$ be a precrossed module. The pair (f, g) satisfying*

1. *$f \in \mathcal{M}(C_1)$, $g \in \mathcal{M}(C_0)$,*
2. *$df = gd$,*

3. $f(c_0 \blacktriangleright c_1) = c_0 \blacktriangleright (f(c_1)) = (g(c_0)) \blacktriangleright c_1$, *for all* $c_0 \in C_0$, $c_1 \in C_1$,

is called a multiplier of the precrossed module $\mathcal{C}$.

The set of all multipliers of a precrossed module $\mathcal{C}\colon C_1 \overset{d}{\to} C_0$ is denoted by $\mathcal{MUL}\,(\mathcal{C})$ and it is an algebra with the usual scalar multiplication, componentwise addition, and multiplication defined by

$$(f,g)(f',g') = (ff', gg'),$$

for all $(f,g),(f',g') \in \mathcal{MUL}\,(\mathcal{C})$, where ff' and gg' are compositions.

Now we will define the generalized multipliers of a precrossed module.

Definition 2.6 *Let* $\mathcal{C} : C_1 \overset{d}{\longrightarrow} C_0$ *be a precrossed module. Consider the triples* $(\alpha, \partial, \alpha^1)$ *such that*

1. $\alpha(c_0 \blacktriangleright c_1) = c_0 \blacktriangleright \alpha(c_1) = (\partial(c_0)) * c_1$

2. $\alpha^1(c_0 \blacktriangleright c_1) = c_0 \blacktriangleright \alpha^1(c_1) = (\beta(c_0)) \blacktriangleright c_1$

3. $\beta d = d\alpha = d\alpha^1$,

for all $c_0 \in C_0$, $c_1 \in C_1$ *where* $\alpha, \alpha^1 \in \mathcal{M}(\mathcal{C}_1)$, $\partial : C_0 \longrightarrow C_1$ *is a crossed multiplier that is a* k*-linear map that satisfies* $\partial(c_0 * c_0') = c_0 \blacktriangleright \partial(c_0')$, *for all* $c_0, c_0' \in C_0$, *and* $\beta = d\partial$. *These kinds of triples will be called generalized multipliers of the precrossed module* $\mathcal{C}$ *and denoted by* $\mathcal{GMUL}\,(\mathcal{C})$.

By a direct calculation we have that β is a multiplier of C_0.

Example 2.7 *Let* $\mathcal{C} : C_1 \longrightarrow C_0$ *be a precrossed module. Fix an element* $c_1 \in C_1$. *Define* $\alpha_{c_1}(c_1') = c_1 * c_1'$, $\alpha^1_{c_1}(c_1') = d\,(c_1) \blacktriangleright c_1'$, $\partial_{c_1}(c_0) = c_0 \blacktriangleright c_1$, *for all* $c_0 \in C_0$, $c_1' \in C_1$. *Then* $(\alpha_{c_1}, \partial_{c_1}, \alpha^1_{c_1})$ *is a generalized crossed multiplier of the precrossed module* $\mathcal{C}$.

Consequently, $\mathcal{GMUL}\ (\mathcal{C})$ is nonempty

Proposition 2.8 *Let* $\mathcal{C} : C_1 \longrightarrow C_0$ *be a precrossed module and* $(\alpha, \partial, \alpha^1), (\delta, \partial', \delta^1) \in \mathcal{GMUL}\ (\mathcal{C})$ *where* C_0 *satisfies* $C_0 C_0 = C_0$ *or* $AnnC_0 = 0$. *Define*

$$(\alpha, \partial, \alpha^1)(\delta, \partial', \delta^1) = (\alpha\delta, \partial\partial', \alpha^1\delta^1)$$

where $\alpha\delta, \alpha^1\delta^1$ *are compositions and* $\partial\partial' = \alpha\partial'$. *Then this multiplication is commutative, i.e.* $\alpha\delta = \delta\alpha$, $\alpha^1\delta^1 = \delta^1\alpha^1$ *and* $\partial\partial' = \alpha\partial' = \delta\partial = \partial'\partial$.

Proof Suppose $Ann(C_0) = 0$. Let $x \in C_0$. We have $y \blacktriangleright \alpha(\partial'(x)) = \alpha(y \blacktriangleright \partial'(x)) = \alpha(\partial'(x * y)) = \alpha(x \blacktriangleright \partial'(y)) = \partial(x) * \partial'(y) = \partial'(y) * \partial(x) = \delta(y \blacktriangleright \partial(x)) = y \blacktriangleright \delta(\partial(x))$, for all $y \in C_0$. Then we have $\alpha\partial = \delta\partial$. Suppose $C_0C_0 = C_0$. Let $x \in C_0$. Then there exists $a, b \in C_0$ such that $a*b = x$. Then $\alpha(\partial'(x)) = \alpha(\partial'(a*b)) = a * \alpha(\partial'(b)) = \delta(a \blacktriangleright \partial(b)) = \delta(a \blacktriangleright \partial(b)) = \delta(\partial(a * b)) = \delta(\partial(x))$. Then we have $\alpha\partial = \delta\partial$, as required. By a similar way we have that $\alpha\delta = \delta\alpha, \alpha^1\delta^1 = \delta^1\alpha^1$. □

With this defined multiplication and usual scalar multiplication and addition, $\mathcal{GMUL}\ (\mathcal{C})$ is an algebra.

Remark 2.9 *Let* $\mathcal{C}\colon C_1 \overset{d}{\to} C_0$ *be a precrossed module that does not satisfy the Peiffer identity. Then we have at least two elements* $c_1, c_1' \in C_1$ *such that* $(d(c_1)) \blacktriangleright c_1' \neq c_1 * c_1'$. *Consider the triple* $(\alpha_{c_1}, d_{c_1}, \alpha^1_{c_1}) \in \mathcal{GMUL}\ (\mathcal{C})$. *Then we have* $\alpha_{c_1} \neq \alpha^1_{c_1}$.

Remark 2.10 *Definition of generalized crossed multipliers of precrossed modules is deduced from the multipliers of the semidirect product of precat*^{1}*-algebras constructed in Section 3.*

We will finish this subsection by recalling the category of precat^1-algebras. Details can be found in [10].

Let C be an algebra and $s,t\colon C\to C$ be endomorphisms such that $st=t$ and $ts=s$. Then the triple (C,s,t) is called a precat^1-algebra and the endomorphisms s,t are called unary operations. A morphism between two precat^1-algebras (C,s,t) and (C',s',t') is an algebra homomorphism $C\to C'$ compatible with the unary operations. The resulting category will be denoted here by $\mathbf{Precat^1-Comm}$.

Given a precrossed module $C_1\xrightarrow{d}C_0$. We have the corresponding precat^1-algebra $(C_1\rtimes C_0,s,t)$ where $s(c_1,c_0)=(0,c_0)$, $t(c_1,c_0)=(0,d(c_1)+c_0)$, for all $c_1\in C_1$, $c_0\in C_0$. Furthermore, for a given precat^1-algebra (C,s,t) we have the corresponding precrossed module $\mathrm{Ker}s\xrightarrow{t_{|kers}}\mathrm{Im}s$. This process gives rise to the natural equivalence of the categories of **PXComm** and $\mathbf{Precat^1-Comm}$ diagrammed as follows:

$$\mathbf{PXComm}\underset{PX}{\overset{PC}{\rightleftarrows}}\mathbf{Precat^1-Comm}$$

The same argument also gives rise to the natural equivalence of categories of crossed modules and that of cat^1-algebras.

2.2. Modified category of interest

Let $\mathbb{C}$ be a category of groups with a set of operations Ω and with a set of identities $\mathbb{E}$, such that $\mathbb{E}$ includes the group identities and the following conditions hold. If Ω_i is the set of i-ary operations in Ω, then:

(a) $\Omega=\Omega_0\cup\Omega_1\cup\Omega_2$;

(b) the group operations (written additively : $0,-,+$) are elements of Ω_0, Ω_1, and Ω_2, respectively. Let $\Omega_2'=\Omega_2\setminus\{+\}$, $\Omega_1'=\Omega_1\setminus\{-\}$. Assume that if $*\in\Omega_2$, then Ω_2' contains $*^\circ$ defined by $x*^\circ y=y*x$ and assume $\Omega_0=\{0\}$;

(c) for each $*\in\Omega_2'$, $\mathbb{E}$ includes the identity $x*(y+z)=x*y+x*z$;

(d) for each $\omega\in\Omega_1'$ and $*\in\Omega_2'$, $\mathbb{E}$ includes the identities $\omega(x+y)=\omega(x)+\omega(y)$ and $\omega(x*y)=\omega(x)*\omega(y)$.

Let C be an object of $\mathbb{C}$ and $x_1,x_2,x_3\in C$:

Axiom 1. $x_1+(x_2*x_3)=(x_2*x_3)+x_1$, for each $*\in\Omega_2'$.

Axiom 2. For each ordered pair $(*,\overline{*})\in\Omega_2'\times\Omega_2'$ there is a word W such that

$$(x_1*x_2)\overline{*}x_3=W(x_1(x_2x_3),x_1(x_3x_2),(x_2x_3)x_1,$$
$$(x_3x_2)x_1,x_2(x_1x_3),x_2(x_3x_1),(x_1x_3)x_2,(x_3x_1)x_2),$$

where each juxtaposition represents an operation in Ω_2'.

Definition 2.11 *A category of groups with operations* $\mathbb{C}$ *satisfying conditions* $(a)-(d)$, *Axiom 1 and Axiom 2, is called a modified category of interest.*

Let $\mathbb{E}_G$ be the subset of identities of $\mathbb{E}$ that includes the group identities and the identities (c) and (d). We denote by $\mathbb{C}_G$ the corresponding category of groups with operations. Thus we have $\mathbb{E}_G \hookrightarrow \mathbb{E}$, $\mathbb{C} = (\Omega, \mathbb{E})$, $\mathbb{C}_G = (\Omega, \mathbb{E}_G)$ and there is a full inclusion functor $\mathbb{C} \hookrightarrow \mathbb{C}_G$. $\mathbb{C}_G$ is called a general category of groups with operations of a modified category of interest $\mathbb{C}$.

Example 2.12 *The categories* $\mathbf{Cat^1}$**-Comm** *of* cat^1*-algebras and* $\mathbf{PreCat^1}$**-Comm** *of* $precat^1$*-algebras are modified categories of interest, which are not categories of interest. Further examples can be found in [8].*

Definition 2.13 *Let* $A,\ B \in \mathbb{C}$*. An extension of* B *by* A *is a sequence*

$$0 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B \longrightarrow 0\ , \tag{2.1}$$

in which p *is surjective and* i *is the kernel of* p*. We say that an extension is split if there is a morphism* $s : B \longrightarrow E$ *such that* $ps = 1_B$*.*

Definition 2.14 *For* $A, B \in \mathbb{C}$*, it is said that there is a set of actions of* B *on* A*, whenever there is a map* $f_* : A \times B \longrightarrow A$*, for each* $* \in \Omega_2$*.*
A split extension of B *by* A *induces an action of* B *on* A *corresponding to the operations in* $\mathbb{C}$*. For a given split extension* (2.1)*, we have*

$$b \cdot a = s(b) + a - s(b), \tag{2.2}$$

$$b * a = s(b) * a, \tag{2.3}$$

for all $b \in B$*,* $a \in A$ *and* $* \in {\Omega_2}'$*. Actions defined by* (2.2) *and* (2.3) *will be called derived actions of* B *on* A*. The notation* $b \dot{*} a$ *is used to denote both the dot and the star actions.*

Definition 2.15 *Given an action of* B *on* A*, a semidirect product* $A \rtimes B$ *is a universal algebra, whose underlying set is* $A \times B$ *and the operations are defined by*

$$\omega(a,b) = (\omega\,(a)\,, \omega\,(b)),$$
$$(a', b') + (a, b) = (a' + b' \cdot a, b' + b),$$
$$(a', b') * (a, b) = (a' * a + a' * b + b' * a, b' * b),$$

for all $a, a' \in A$*,* $b, b' \in B$*.*

Theorem 2.16 *[9] An action of* B *on* A *is a derived action if and only if* $A \rtimes B$ *is an object of* $\mathbb{C}$*.*

Now we will define the actions in the category $\mathbf{Precat^1}$**-Comm** of precat^1-algebras according to the definition of action in a modified category of interest.

Example 2.17 *Let* (C_0, s_0, t_0) *and* (C_1, s_1, t_1) *be* $precat^1$*-algebras with an action of* (C_0, s_0, t_0) *on* (C_1, s_1, t_1)*. According to Definition 2.15 we have*

$$c_0 \blacktriangleright (c_1 * c_1') = (c_0 \blacktriangleright c_1) * c_1'$$
$$(c_0 * c_0') \blacktriangleright c_1 = c_0 \blacktriangleright (c_0' * c_1)$$

and from the precat[1] *- structure we have*

$$
\begin{aligned}
(s_0(c_0)) \blacktriangleright (s_1(c_1)) &= s_1(c_0 \blacktriangleright c_1) \\
(t_0(c_0)) \blacktriangleright (t_1(c_1)) &= t_1(c_0 \blacktriangleright c_1) \\
(s_0(c_0)) \blacktriangleright (t_1(c_1)) &= s_1(c_0 \blacktriangleright t_1(c_1)) \\
(t_0(c_0)) \blacktriangleright (s_1(c_1)) &= t_1(c_0 \blacktriangleright s_1(c_1)) \\
s_0(c_0) \blacktriangleright (t_1(c_1)) &= t_1(s_0(c_0) \blacktriangleright c_1) \\
t_0(c_0) \blacktriangleright (s_1(c_1)) &= s_1(t_0(c_0) \blacktriangleright c_1)
\end{aligned}
$$

for any $c_0 \in (C_0, s_0, t_0)$*,* $c_1 \in (C_1, s_1, t_1)$.

The definition of a split extension classifier in modified categories of interest has the following form. Consider the category of all split extensions with fixed kernel A; thus the objects are

$$0 \to A \to C \overset{s}{\curvearrowleft}{\longrightarrow} C' \to 0$$

and the arrows are the triples of morphisms $(1_A, \gamma, \gamma')$) between the extensions, which commute with the section homomorphisms as well. By the definition, an object $[A]$ is a split extension classifier for A if there exists a derived action of $[A]$ on A, such that the corresponding extension

$$0 \to A \to A \rtimes [A] \overset{s}{\curvearrowleft}{\longrightarrow} [A] \to 0$$

is a terminal object in the above defined category.

Proposition 2.18 *[8] Let $\mathbb{C}$ be a modified category of interest and A be an object in $\mathbb{C}$. An object $B \in \mathbb{C}$ is a split extension classifier for A in the sense of [5] if and only if it satisfies the following condition: B has a derived action on A such that for all C in $\mathbb{C}$ and a derived action of C on A there is a unique morphism $\varphi : C \longrightarrow B$, with $c \cdot a = \varphi(c) \cdot a$, $c * a = \varphi(c) * a$, for all $* \in {\Omega_2}'$, $a \in A$ and $c \in C$.*

The object B in $\mathbb{C}$ satisfying the above stated condition is called an actor of A and denoted by Act(A). The corresponding universal acting object, which represents actions in the sense of [5, 6], in the categories equivalent to modified categories of interest is called a split extension classifier and denoted by $[A]$, as it is in semiabelian categories.

Remark 2.19 *As a consequence of this proposition, an actor of an object is unique up to an isomorphism.*

Definition 2.20 *[8] Let A, $B \in \mathbb{C}$. A set of actions of B on A is strict if for any two elements $b, b' \in B$, from the conditions $b \cdot a = b' \cdot a$, $\omega(b) \cdot a = \omega(b') \cdot a$, $b * a = b' * a$ and $\omega(b) * a = \omega(b') * a$, for all $a \in A$, $\omega \in {\Omega_1}'$ and $* \in {\Omega_2}'$; it follows that $b = b'$.*

Definition 2.21 *[8] A general actor $GA(A)$ of an object A in $\mathbb{C}$ is an object of $\mathbb{C}_G$, having a set of actions on A, which is a set of derived actions in $\mathbb{C}_G$, and for any object $C \in \mathbb{C}$ and a derived action of C on A in $\mathbb{C}$, there exists in $\mathbb{C}_G$ a unique morphism $\varphi : C \longrightarrow GA(A)$ such that $c \dot{*} a = \varphi(c) \dot{*} a$, for all $c \in C$, $a \in A$ and $* \in {\Omega_2}'$.*

Definition 2.22 *[8] If the action of a general actor $GA(A)$ on A is strict, then it is said that $GA(A)$ is a strict general actor of A and denoted by $\mathrm{SGA}(A)$.*

Condition 2.23 *[8] Let $A \in \mathbb{C}$ and $\{B_j\}_{j\in J}$ denote the set of all objects of $\mathbb{C}$ that have derived actions on A. Let $\varphi_j : B_j \longrightarrow GA(A)$, $j \in J$, denote the corresponding unique morphism such that $b_j \dot{*} a = \varphi_j(b_j) \dot{*} a$, for all $b_j \in B_j$, $a \in A$, $* \in {\Omega_2}'$. The elements of $GA(A)$ satisfy the following equality:*

$$(\varphi_i(b_i) * \varphi_j(b_j))\overline{*}a = W(\varphi_i(b), \varphi_j(b'); a; *, \overline{*})$$

for any $b_i \in B_i$, $b_j \in B_j$, $* \in {\Omega_2}'$ and $i, j \in J$.

Definition 2.24 *[8] A universal strict general actor of an object A, denoted by $\mathrm{USGA}(A)$, is a strict general actor with Condition 2.23, such that for any strict general actor $\mathrm{SGA}(A)$ with Condition 2.23 there exists a unique morphism $\eta : \mathrm{USGA}(A) \to \mathrm{SGA}(A)$ in the category $\mathbb{C}_G$, with $\psi_j\eta = \varphi_j$, for any $j \in J$, where $\varphi_j : B_j \to \mathrm{SGA}(A)$ and $\psi_j : B_j \to \mathrm{USGA}(A)$ denote the corresponding unique morphisms with the appropriate properties from the definition of a general actor.*

Proposition 2.25 *[8] Let $\mathbb{C}$ be a modified category of interest and $A \in \mathbb{C}$. If an actor $Act(A)$ exists, then the unique morphism $\eta : \mathrm{USGA}(A) \to \mathrm{Act}(A)$ is an isomorphism with $x \dot{*} a = \eta(x) \dot{*} (a)$, for all $x \in \mathrm{USGA}(A)$, $a \in A$.*

Theorem 2.26 *[8] Let $\mathbb{C}$ be a modified category of interest and $A \in \mathbb{C}$. A has an actor if and only if the semidirect product $A \rtimes \mathrm{USGA}(A)$ is an object of $\mathbb{C}$. If it is the case, then $\mathrm{Act}(A) \cong \mathrm{USGA}(A)$.*

Let (C, s^C, t^C) be a precat^1-algebra. Consider all split extensions of (C, s^C, t^C) in $\mathbf{Precat}^1\mathbf{-Comm}$

$$E_j : 0 \longrightarrow (C, s^C, t^C) \longrightarrow (K_j, s^{K_j}, t^{K_j}) \overset{\curvearrowleft}{\longrightarrow} (D_j, s^{D_j}, t^{D_j}) \longrightarrow 0 \quad j \in \mathbf{J}$$

where $(D_j, s^{D_j}, t^{D_j}) = (D_k, s^{Dk}, t^{Dk}) = (D, s^D, t^D)$, for $j \neq k$ in the case the corresponding extensions derive different actions of (D, s^D, t^D) on (C, s^C, t^C). Let $\{d_j\cdot, d_j \blacktriangleright\}$ be the set of functions defined by the action of $(D_j, s^{D_j}, t^{D_j}$ on (C, s^C, t^C). For any element $d_j \in D_j$ denotes $\mathbf{d}_j = \{d_j\cdot, d_j \blacktriangleright\}$. Let $\mathbb{D} = \{\mathbf{d}_j, d_j \in D_j, j \in \mathbb{J}\}$. Thus each element $\mathbf{d}_j \in \mathbb{D}$, $j \in \mathbb{J}$ is a special type of a function $\mathbf{d}_j = \{+, *, *^{op}\} \longrightarrow Maps,\ (C, s^C, t^C) \longrightarrow (C, s^C, t^C)$ defined by $\mathbf{d}_j(*) = d_j \blacktriangleright\_ : C \longrightarrow C$, $\mathbf{d}_j(*^{op}) = d_j \blacktriangleright\_ : C \longrightarrow C$, $\mathbf{d}_j(+) = (d_j +\_) : C \longrightarrow C$. The multiplication on $\mathbb{D}$ is defined by

$(\mathbf{d}_i * \mathbf{d}_k) \blacktriangleright c = (\mathbf{d}_k * \mathbf{d}_i) \blacktriangleright c = \mathbf{d}_i * (\mathbf{d}_k \blacktriangleright c)$,
$(\mathbf{d}_i * \mathbf{d}_k) \cdot (c) = c$.

Furthermore, we define

$(\mathbf{d}_i + \mathbf{d}_k) \cdot (c) = \mathbf{d}_i \cdot (\mathbf{d}_k \cdot c)$,
$(\mathbf{d}_i + \mathbf{d}_k) \blacktriangleright c = \mathbf{d}_i \blacktriangleright c + \mathbf{d}_k \blacktriangleright c$,
$s(\mathbf{d}_k) \cdot (c) = s^{D_k}(d_k) \cdot c$, $s(\mathbf{d}_k) \blacktriangleright c = s^{D_k}(d_k) \blacktriangleright c]$,
$t(\mathbf{d}_k) \cdot (c) = t^{D_k}(d_k) \cdot c$, $t(\mathbf{d}_k) \blacktriangleright c = t^{D_k}(d_k) \blacktriangleright c$,
$s(\mathbf{d}_i * \mathbf{d}_k) = s^{D_i}(d_i) * s^{D_k}(d_k)$, $t(\mathbf{d}_i * \mathbf{d}_k) = t^{D_i}(d_i) * t^{D_k}(d_k)$,

$s(d_1+d_2)=s^{D_1}(d_1)+s^{D_2}(d_2),\ t(d_1+d_2)=t^{D_1}(d_1)+t^{D_2}(d_2),$
$(-\mathbf{d}_i)\cdot(c)=(-d_i)\cdot c,\ (-d)\cdot c=c,$
$(-\mathbf{d}_i)\blacktriangleright c=-\left(d_i\blacktriangleright(c)\right),\ (-d)\blacktriangleright(c)=-\left(d\blacktriangleright(c)\right),$
$-(d_1+d_2)=-d_2-d_1$

where d,d_1,d_2 are certain combinations of the multiplication of the elements of $\mathbb{D}$.

Denote by $\mathfrak{L}'(M)$ the set of all functions $(\Omega_2\to Maps(C,s^C,t^C)\longrightarrow(C,s^C,t^C))$ obtained by performing all kind of operations defined above on the elements of $\mathbb{L}$ and on the new obtained elements as the results of operations. Note that it may happen that $\mathbf{d}\blacktriangleright c=\mathbf{d}'\blacktriangleright c$, for any $c\in C$, but we do not have the equalities $\sigma(\mathbf{d})\blacktriangleright c=\sigma(\mathbf{d}')\blacktriangleright c$, for any $c\in C$, where σ is a finite combination of s and t. Define a relation on $\mathfrak{L}'(C)$ by " $d\sim d'$ if and only if $d\blacktriangleright c=d'\blacktriangleright c,\ s(d)\blacktriangleright c=s(d')\blacktriangleright c,\ t(d)\blacktriangleright c=t(d')\blacktriangleright c$ " for any $d,d'\in\mathfrak{L}'(C)$, $c\in C$. This is a congruence relation on $\mathfrak{L}'(C)$. Denote $\mathfrak{L}'(C)/\sim$ by $\mathfrak{L}(C)$. The operations defined on $\mathfrak{L}'(C)$ define the corresponding operations on $\mathfrak{L}(C)$.

Theorem 2.27 *[8]Let $A\in\mathbb{C}$. Then we have $\mathfrak{B}(A)\cong\mathrm{USGA}(A)$.*

Corollary 2.28 *Let $(C,s^c,t^c)\in$ **Precat**^{1}**–Comm**. Then $(\mathfrak{L}(C),s^{\mathfrak{L}(C)},t^{\mathfrak{L}(C)})$ is a universal strict general actor of (C,s^C,t^C).*

Proof Follows from Theorem 2.27. □

3. Actor of an object in Precat^1-Comm

In this section, we will construct an object $\left(\mathfrak{A}(C),\overline{s},\overline{t}\right)$ according to a given precat^1-algebra (C,s,t) and then we show that it is an actor of (C,s,t) in **Precat**^{1}**-Comm** under certain conditions. The construction is deduced from the interpretation of $\mathfrak{L}(C)$ given in Section 2.

Let (C,s,t) be a precat^1-algebra. Consider the triples $(\theta,\theta^0,\theta^1)$ of multipliers of C such that
C1) $\theta^0 s=s\theta$,
C2) $\theta^1 t=t\theta$,
C3) $\theta^j s=s\theta^i$, for $i=0,1$,
C4) $\theta^j t=t\theta^i$, for $i=0,1$.
The set of all these kinds of triples will be denoted by $\mathfrak{A}(C)$ and it is an algebra with componentwise addition, scalar multiplication, and the multiplication defined as the componentwise composition such as

$$(\theta,\theta^0,\theta^1)*(\psi,\psi^0,\psi^1):=(\theta\psi,\theta^0\psi^0,\theta^1\psi^1),$$

for all $(\theta,\theta^0,\theta^1),(\psi,\psi^0,\psi^1)\in\mathfrak{A}(C)$. The zero element is the triple $(0,0,0)$ of zero maps. Now we introduce a precat^1 structure on $\mathfrak{A}(C)$. Define $\overline{s}:\mathfrak{A}(C)\longrightarrow\mathfrak{A}(C),\ \overline{t}:\mathfrak{A}(C)\longrightarrow\mathfrak{A}(C)$ by $\overline{s}(\theta,\theta^0,\theta^1)=(\theta^0,\theta^0,\theta^0)$, $\overline{t}(\theta,\varphi^0,\theta^1)=(\theta^1,\theta^1,\theta^1)$, respectively. $(\mathfrak{A}(C),\overline{s},\overline{t})$ is a precat^1-algebra with the defined unary operations $\overline{s}$, $\overline{t}$.

There is an action of $\left(\mathfrak{A}(C),\overline{s},\overline{t}\right)$ on (C,s,t) defined by the map

$$\begin{array}{rcl}\left(\mathfrak{A}(C),\overline{s},\overline{t}\right)\times(C,s,t) & \longrightarrow & (C,s,t)\\ ((\theta,\theta^0,\theta^1),c) & \longmapsto & \theta(c),\end{array}$$

for all $c \in C$ and $(\theta, \theta^0, \theta^1) \in \mathfrak{A}(C)$.

Condition A For an algebra C, $Ann(C) = 0$ or $C^2 = C$.

Proposition 3.1 *If C satisfies Condition* **A***, then $(\mathfrak{A}(C), \overline{s}, \overline{t})$ and $(\mathfrak{L}(C),\ s^{\mathfrak{L}(C)}, t^{\mathfrak{L}(C)})$ are isomorphic.*

Proof The action of $\left(\mathfrak{A}(C), \overline{s}, \overline{t}\right)$ on (C, s, t) defined above is a derived action in $\mathbf{Precat^1}$**-Comm** under Condition **A** . Therefore, from Definition 2.24, we have the unique morphism $\eta : \mathfrak{A}(C) \longrightarrow \mathfrak{L}(C)$ defined as $\eta((\theta, \theta^0, \theta^1) \blacktriangleright c) = (\theta, \theta^0, \theta^1) \blacktriangleright c$, for all $c \in C$ and $(\theta, \theta^0, \theta^1) \in \mathfrak{A}(C)$. By the constructions of $(\mathfrak{A}(C), \overline{s}, \overline{t})$ and $(\mathfrak{L}(C), s^{\mathfrak{L}(C)}, t^{\mathfrak{L}(C)})$, we find that η is an isomorphism. □

Corollary 3.2 *If C satisfies Condition* **A***, then $(\mathfrak{A}(C), \overline{s}, \overline{t})$ is an actor of (C, s, t).*

Proof Since $(\mathfrak{A}(C), \overline{s}, \overline{t}) \in$ $\mathbf{Precat^1}$**-Comm** and its action on C is a derived action, then the result follows from Theorem 2.26, Corollary 2.28, and Proposition 3.1. □

3.1. Actor of a precat^1-algebra corresponding to a given precrossed module

Let C_1, C_0 be algebras with an action of C_0 on C_1. Let θ be a multiplier of the algebra $C_1 \rtimes C_0$. Then $\theta : C_1 \rtimes C_0 \longrightarrow C_1 \rtimes C_0$ can be represented by four k-linear maps

$$\alpha : C_1 \longrightarrow C_1, \gamma : C_1 \longrightarrow C_0, \beta : C_0 \longrightarrow C_0 \text{ and } \partial : C_0 \longrightarrow C_1$$

such that

$$\theta(c_1, c_0) = (\alpha(c_1) + \partial(c_0), \beta(c_0) + \gamma(c_1)),$$

for all $c_1 \in C_1$, $c_0 \in c_0$.

Let $\mathcal{C} : C_1 \xrightarrow{d} C_0$ be a precrossed module and $(C_1 \rtimes C_0, s, t)$ be the corresponding precat^1-algebra. Suppose θ satisfies the Condition C3 or C4. Then $\gamma = 0$. On the other hand, any multiplier θ of the algebra $C_1 \rtimes C_0$ can be represented by the triple $(\alpha, \partial, \beta)$. Moreover, for any $c := (c_1, c_0)$, $c' := (c_1', c_0') \in C_1 \times C_0$, we obtain

$$\theta((c_1, c_0) * (c_1', c_0')) = (c_1, c_0) * \theta((c_1', c_0')) = (c_1', c_0') * \theta((c_1, c_0))$$

By direct calculations we get

$$\begin{aligned} \alpha(c_1 * c_1') &= c_1 * \alpha(c_1'), \\ \partial(c_0 * c_0') &= c_0 \blacktriangleright \partial(c_0'), \\ \beta(c_0 * c_0') &= c_0 * \beta(c_0'), \\ \alpha(c_0 \blacktriangleright c_1) &= c_0 \blacktriangleright \alpha(c_1) = c_1 * \partial(c_0) + \beta(c_0) \blacktriangleright c_1, \end{aligned}$$

for all c_1, $c_1' \in C_1$, $c_0, c_0' \in C_0$.

Proposition 3.3 *Let C: $C_0 \xrightarrow{d} C_1$ be a precrossed module and $(C_1 \rtimes C_0, s, t)$ be the corresponding precat^1-algebra. Let θ, θ^0, θ^1 be multipliers of the algebra $C_1 \rtimes C_0$ and denote θ, θ^0, θ^1 by the triples $(\alpha, \partial, \beta)$, $(\alpha^0, \partial^0, \beta^0)$, $(\alpha^1, \partial^1, \beta^1)$, respectively. Then $(\theta, \theta^0, \theta^1) \in \mathfrak{A}(C_1 \rtimes C_0)$ if and only if $(\theta, \theta^0, \theta^1)$ satisfies the following identities:*

1.) $\beta(c_0) = \beta^0(c_0)$,
2.) $\partial^i(c_0) = 0,\ for\ i = 0,1$,
3.) $\beta^1(c_0) = \beta(c_0) + d\partial(c_0)$,
4.) $\beta^1 d(c_1) = d\alpha(c_1)$,
5.) $\beta^i d(c_1) = d\alpha^i(c_1), for\ i = 0,1$,
for all $c_0 \in C_0,\ c_1 \in C_1$.

Proof Follows from definition of unary operations $s,\ t$ and the structure of $\mathfrak{A}(C_1 \rtimes C_0)$. □

Let $\mathcal{C} : C_1 \xrightarrow{d} C_0$ be a precrossed module, $C_0,\ C_1$ satisfy Condition **A** and $(\mathfrak{A}(C_1 \rtimes C_0), \overline{s}, \overline{t})$ be the actor of $(C_1 \rtimes C_0, s, t)$. Then from the definition of $\overline{s}$ and Proposition 3.3 we have $\ker \overline{s} = \{(\theta, 0, \theta^1) \in \mathfrak{A}(C_1 \rtimes C_0)\}$.

Proposition 3.4 $\ker \overline{s} \cong \mathcal{GMUL}(\mathcal{C})$

Proof Let $(\theta, 0, \theta^1) \in \ker \overline{s}$. It follows from Propositions 3.3 that $(\theta, 0, \theta^1) = ((\alpha, \partial, 0), (0,0,0), (\alpha^1, 0, \beta^1))$ and the resulting triple $(\alpha, \partial, \alpha^1)$ is a generalized crossed multiplier. Conversely, for any $(\gamma, \lambda, \gamma^1) \in \mathcal{GMUL}(\mathcal{C})$ we have the triples $\varphi = (\gamma, \lambda, 0),\ \varphi^1 = \left(\gamma^1, 0, d\lambda\right)$ such that $(\varphi, 0, \varphi^1) \in \ker \overline{s}$, which completes the proof. □

By a similar calculation we have $\mathrm{Im}(\overline{s}) \cong \mathcal{MUL}(\mathcal{C})$.

4. Split extension classifier of a precrossed module

As indicated in [12], the category of precrossed modules of commutative algebras is a semiabelian category. For defining actions in **PXComm** in the sense of [6], we will define an action in **PXComm** in an analogous way as it is defined in a modified category of interest.

In this section we will construct a precrossed module Δ for a given precrossed module $\mathcal{C} : C_1 \xrightarrow{d} C_0$ and prove that if C_0 and C_1 satisfy Condition **A**, then Δ is isomorphic to $PX(\mathrm{Act}(PC(\mathcal{C})))$. Consequently, Δ is the split extension classifier of $\mathcal{C}$.

Consider the precrossed module $\mathcal{C} : C_1 \xrightarrow{d} C_0$ where C_0 and C_1 satisfy Condition **A**. We have the corresponding precat^1-algebra $(C_1 \rtimes C_0, s, t)$ and its actor $(\mathfrak{A}(C_1 \rtimes C_0), \overline{s}, \overline{t})$ in **Precat^1-Comm**.

Proposition 4.1 *The k-bilinear map $\mathcal{MUL}(\mathcal{C}) \times \mathcal{GMUL}(\mathcal{C}) \longrightarrow \mathcal{GMUL}(\mathcal{C})$, $((\mu_1, \mu_0, (\alpha, \partial, \alpha^1)) \longmapsto (\mu_1\alpha, \mu_1\partial, \mu_1\alpha^1)$ defines an action of $\mathcal{MUL}(\mathcal{C})$ on $\mathcal{GMUL}(\mathcal{C})$ where $\mu_1\alpha,\ \mu_1 d$, and $\mu_1\alpha^1$ are compositions. In addition, the map $\Delta : \mathcal{GMUL}(\mathcal{C}) \longrightarrow \mathcal{MUL}(\mathcal{C})$ defined by $(\alpha, \partial, \alpha^1) \longmapsto (\alpha, \beta)$ is a precrossed module with this action where $\beta = d\partial$.*

Proof Since

$$\begin{aligned}
(\mu_1, \mu_0) \blacktriangleright ((\alpha, \partial, \alpha^1)(\delta, \partial', \delta^1)) &= (\mu_1, \mu_0)(\alpha\delta, \alpha\partial', \alpha^1\delta^1) \\
&= (\mu_1\alpha\delta, \mu_1\alpha\partial', \mu_1\alpha^1\delta^1) \\
&= ((\mu_1\alpha)\delta, (\mu_1\alpha)\partial', (\mu_1\alpha^1)\delta^1) \\
&= (\mu_1\alpha, \mu_1\partial, \mu_1\alpha^1)(\delta, \partial', \delta^1) \\
&= ((\mu_1, \mu_0) \blacktriangleright (\alpha, \partial, \alpha^1))(\delta, \partial', \delta^1)
\end{aligned}$$

and

$$\begin{aligned}
((\mu_1, \mu_0)(\mu_1', \mu_0')) \blacktriangleright (\alpha, d, \alpha^1) &= (\mu_1\mu_1', \mu_0\mu_0') \blacktriangleright (\alpha, \partial, \alpha^1) \\
&= (\mu_1\mu_1'\alpha, \mu_1\mu_1'\partial, \mu_1\mu_1'\alpha^1) \\
&= (\mu_1(\mu_1'\alpha), \mu_1(\mu_1'\partial), \mu_1(\mu_1'\alpha^1)) \\
&= (\mu_1, \mu_0) \blacktriangleright (\mu_1'\alpha, \mu_1'\partial, \mu_1'\alpha^1) \\
&= (\mu_1, \mu_0) \blacktriangleright ((\mu_1', \mu_0') \blacktriangleright (\alpha, \partial, \alpha^1)),
\end{aligned}$$

for all $(\mu_1,\mu_0),(\mu_1',\mu_0')\in \mathcal{MUL}(\mathcal{C})$, $(\alpha,\partial,\alpha^1),(\delta,\partial',\delta^1)\in \mathcal{GMUL}(\mathcal{C})$ we have a good definition of the action. On the other hand, we have

$$\begin{aligned}\Delta((\mu_1,\mu_0)\blacktriangleright(\delta,\partial,\delta_1)) &= \Delta(\mu_1\delta,\mu_1\partial,\mu_1\delta^1)\\ &= (\mu_1\delta, d\mu_1\partial)\\ &= (\mu_1\delta,\mu_0 d\partial)\\ &= (\mu_1,\mu_0)(\delta,d\partial)\\ &= (\mu_1,\mu_0)\left(\Delta(\delta,\partial,\delta^1)\right),\end{aligned}$$

for all $(\mu_1,\mu_0)\in \mathcal{MUL}(\mathcal{C})$, $(\delta,\partial,\delta^1)\in \mathcal{GMUL}(\mathcal{C})$, as required. □

Proposition 4.2 $\Delta \cong PX(\mathfrak{A}(CX(\mathcal{C})))$.

Proof Follows from Propositions 3.4 and 4.1, since Δ is isomorphic to the restriction of $\bar{t}$. □

Theorem 4.3 *If C_1 and C_0 satisfy Condition* **A***, then the precrossed module $\Delta:\mathcal{GMUL}(\mathcal{C})\longrightarrow \mathcal{MUL}(\mathcal{C})$ defined in Proposition 4.1 is the split extension classifier of $\mathcal{C}$.*

Proof The semidirect product $C_1\rtimes C_0$ also satisfies Condition **A**. Therefore, the result is a direct consequence of Corollary 3.2, Proposition 4.2, and the fact that PX and CX define an equivalence between the categories **PXComm** and **Precat**^{1}**-Comm**. □

The split extension classifier of a precrossed module $\mathcal{C}:C_1\xrightarrow{d}C_0$ will be denoted here by $[\mathcal{C}]_{\mathbf{PXComm}}$.

Remark 4.4 *Let $\mathcal{C}:C_1\xrightarrow{d}C_0$ be a crossed module in the category of* **PXComm***. Define the set*

$$M^*(\mathcal{C}):=\{(\alpha,\partial,\alpha^1)\in M(C_0,C_1):\alpha=\alpha^1=\partial d\}$$

Evidently, M^* is an ideal of $\mathcal{GMUL}(\mathcal{C})$ and

$$M^*(\mathcal{C})\xrightarrow{\Delta|}\mathcal{MUL}(\mathcal{C})$$

is a precrossed ideal of the precrossed module $[\mathcal{C}]_{\mathbf{PXComm}}$. By direct checking we have that

$$M^*(\mathcal{C})\xrightarrow{\Delta|}\mathcal{MUL}(\mathcal{C})$$

is isomorphic to the split extension classifier of $\mathcal{C}$ in the category **Xcomm** *of crossed modules defined in [3] where the split extension classifier is named as an actor of $\mathcal{C}$.*

Examples 4.5 *i) Let A be an algebra. Consider the precrossed module $\mathcal{C}:A\xrightarrow{id}A$. Then the split extension classifier of $\mathcal{C}$ is $[\mathcal{C}]_{\mathbf{PXComm}}:\mathcal{M}(A)\xrightarrow{id}\mathcal{M}(A)$, which coincides with the split extension classifier of $\mathcal{C}$ in the category of crossed modules.*

ii) Consider the precrossed modules $\mathcal{C}:A\xrightarrow{0}A$ and $\mathcal{C}':A\xrightarrow{0}0$ where A is a nonsingular algebra. Then $[\mathcal{C}]_{\mathbf{PXComm}}:A_1\xrightarrow{\Delta}A_0$ where A_1 is the set of all triples $(\delta,\partial,0)\in\mathcal{GMUL}(\mathcal{C})$ with $\delta=\partial$, $A_0\cong\mathcal{M}(A)$ and Δ is defined by $\Delta(\delta,\partial,0)=(0,0)$.

$[\mathcal{C}']_{\mathbf{PXComm}}$ *is the precrossed module* $A'_1 \xrightarrow{\Delta'} A'_0$, *where* A'_1 *is the set of all triples* $(\delta, 0, \delta^1) \in \mathcal{GMUL}(\mathcal{C})$, $A'_0 \cong \mathcal{M}(A)$ *and* Δ' *is defined by* $\Delta'(\delta, 0, \delta^1) = (\delta^1, 0)$.

iii) Consider the precrossed module $\mathcal{C} : A \times A \xrightarrow{\pi_1} A$ *defined in Examples 2.4. Then* $[\mathcal{C}]_{\mathbf{PXComm}}$ *is the precrssed module* $A_1 \xrightarrow{\Delta} A_0$ *where* A_1 *is the set of all triples* $\big((\alpha, \delta), (\partial, 0), (\alpha^1, \delta^1)\big)$ *where* $\alpha, \delta, \partial, \alpha^1, \delta^1$ *are multipliers of* A *such that* $\alpha = \partial = \alpha^1$, $A_0 \cong \mathcal{M}(A) \times \mathcal{M}(A)$ *and* $\Delta\big((\alpha, \delta), (\partial, 0), (\alpha^1, \delta^1)\big) = \big((\alpha, \delta^1), \alpha\big)$.

Now we are going to define the canonical map $(\xi, \eta) : \mathcal{C} \longrightarrow [\mathcal{C}]_{\mathbf{PXComm}}$ of a given precrossed module $\mathcal{C}$.

Proposition 4.6 *Let* $\mathcal{C} : C_1 \xrightarrow{d} C_0$ *be precrossed module with its split extension classifier* $[\mathcal{C}]_{\mathbf{PXComm}}$.

a) *Define* $\xi : C_1 \longrightarrow \mathcal{GMUL}(\mathcal{C})$, $\xi(c_1) = (\delta_{c_1}, \partial_{c_1}, \delta^1_{c_1})$ *where* $\delta_{c_1}(c'_1) = c_1 * c'_1$, $\delta^1_{c_1}(c'_1) = d(c_1) \blacktriangleright c'_1$ *and* $\partial_{c_1}(c_0) = c_0 * c_1$, *for all* $c_1, c'_1 \in C_1$, $c_0 \in C_0$. *Then* $\xi : C_1 \longrightarrow \mathcal{GMUL}(\mathcal{C})$ *is a precrossed module with action defined by* $(\delta, \partial, \delta^1) \blacktriangleright c_1 = \delta_1(c_1)$.

b) *Define* $\eta : C_0 \longrightarrow \mathcal{MUL}(\mathcal{C})$, $c_0 \longmapsto (\delta_{c_0}, \gamma_{c_0})$ *where* $\delta_{c_0}(c_1) = c_0 \blacktriangleright c_1$, $\gamma_{c_0}(c'_0) = c_0 * c'_0$, *for all* $c_1 \in C_1$, $c_0, c'_0 \in C_0$. η *is a precrossed module with the action defined by* $(\mu_1, \mu_0) \blacktriangleright c_0 = \mu_0(c_0)$.

Proof **a)** Since

$\delta_{c_1 * c'_1}(c''_1) = (c_1 * c'_1) * c''_1 = c_1 * (c'_1 * c''_1) = \delta_{c_1}\delta_{c'_1}(c''_1)$,

$\delta^1_{c_1 * c'_1}(c''_1) = d(c_1 * c'_1) \blacktriangleright c''_1 = d(c_1) \blacktriangleright (d(c'_1) \blacktriangleright c''_1) = \delta^1_{c_1}\delta^1_{c'_1}(c''_1)$,

$\partial_{c_1 * c'_1}(c_0) = c_0 \blacktriangleright (c_1 * c'_1) = (c_0 \blacktriangleright c'_1) * c_1$,

$\partial_{c_1}\partial_{c'_1}(c_0) = \delta_{c_1}\partial_{c'_1}(c_0) = \delta_{c_1}(c_0 \blacktriangleright c'_1) = (c_0 \blacktriangleright c'_1)c_1$

for all $c_1, c'_1 \in C_1$, $c_0 \in C_0$ we have

$$\begin{aligned} \xi(c_1 * c'_1) &= (\delta_{c_1}, \partial_{c_1}, \delta^1_{c_1}) * (\delta_{c'_1}, \partial_{c'_1}, \delta^1_{c'_1}) \\ &= \xi(c_1) * \xi(c'_1) \end{aligned}$$

for all $c_1, c'_1 \in C_1$, which makes ξ a homomorphism. Other conditions can be easily checked.

b) It can be checked by similar calculations. □

Proposition 4.7 *Let* $\mathcal{C} : C_1 \xrightarrow{d} C_0$ *be a precrossed module. Then* $(\xi, \eta) : \mathcal{C} \longrightarrow [\mathcal{C}]_{\mathbf{PXComm}}$ *is a homomorphism of precrossed modules.*

Proof Direct checking □

Remark 4.8 *Im*(ξ, η) *is an precrossed ideal of* $[\mathcal{C}]_{\mathbf{PXComm}}$ *and the* $\ker(\xi, \eta) : Z_1 \xrightarrow{d} Z_0$ *is also a precrossed ideal of* $\mathcal{C}$ *where* $Z_1 = \ker(\xi) = \{c_1 \in C_1 : c_1 * c'_1 = 0, c_0 \blacktriangleright c_1 = 0, d(c_1) \blacktriangleright c'_1 = 0,$ *for all* $c'_1 \in C_1, c_0 \in C_0\}$ *and* $Z_2 = \ker(\eta) = \{c_0 \in C_0 : c_0 * c'_0 = 0, c_0 \blacktriangleright c_1 = 0,$ *for all* $c'_0 \in C_0, c_1 \in C_1\}$.

Proof Direct checking. □

Definition 4.9 *Let* $\mathcal{C} : C_1 \xrightarrow{d} C_0$ *be precrossed module. Then the precrossed module* $Z_1 \xrightarrow{d} Z_0$ *is called the center of the precrossed module* $\mathcal{C}$.

Remark 4.10 *This definition recovers the Huq's definition in [11]. Also* $PC(Z_1 \xrightarrow{\Delta} Z_0)$ *is the center of* $C_1 \rtimes C_0$*, and it is compatible with the definition of center of an object in an modified interest category, given in [8].*

Now we define an action in the category **PXComm** in the sense of [6].

Definition 4.11 *Let* $\mathcal{C} : C_1 \xrightarrow{d} C_0$*, and* $\mathcal{C}' : C_1' \xrightarrow{d'} C_0'$ *be precrossed modules. We say that* $\mathcal{C}'$ *has an action on* $\mathcal{C}$ *if there exists a split extension*

$$0 \to \mathcal{C} \to \mathcal{E} \overset{s}{\curvearrowleft}\!\!\longrightarrow \mathcal{C}' \to 0$$

in **PXComm**. *Equivalently, an action of* $\mathcal{C}'$ *on* $\mathcal{C}$ *is defined by an homomorphism* $\mathcal{C}' \to [\mathcal{C}]_{\mathbf{PXComm}}$.

Example 4.12 *The homomorphism* $(\xi, \eta) : \mathcal{C} \longrightarrow [\mathcal{C}]_{\mathbf{PXComm}}$ *given in Proposition 4.7 defines an action of* $\mathcal{C}$ *on itself.*

5. Conclusion

The definition of action allows one to generalize some notions and properties from module theory such as morphisms preserving actions and the semidirect products to precrossed modules. It also gives rise to definition of obstructions of precrossed modules.

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