Turkish Journal of Mathematics

Volume 39 | Number 5 Article 4

1-1-2015

Uniquely strongly clean triangular matrices

HUANYIN CHEN

ORHAN GÜRGÜN

HANDAN KOSE

Follow this and additional works at: https://journals.tubitak.gov.tr/math

Part of the Mathematics Commons

Recommended Citation

CHEN, HUANYIN; GÜRGÜN, ORHAN; and KOSE, HANDAN (2015) "Uniquely strongly clean triangular matrices," *Turkish Journal of Mathematics*: Vol. 39: No. 5, Article 4. https://doi.org/10.3906/mat-1408-13
Available at: https://journals.tubitak.gov.tr/math/vol39/iss5/4

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2015) 39: 645 – 649
© TÜBİTAK
doi:10.3906/mat-1408-13

Uniquely strongly clean triangular matrices

Huanyin CHEN[1], **Orhan GÜRGÜN**[2], **Handan KÖSE**[3,*]
[1]Department of Mathematics, Hangzhou Normal University, Hangzhou, P. R. China
[2]Department of Mathematics, Ankara University, Ankara, Turkey
[3]Department of Mathematics, Ahi Evran University, Kırşehir, Turkey

Received: 06.08.2014 • **Accepted/Published Online:** 31.03.2015 • **Printed:** 30.09.2015

Abstract: A ring R is uniquely (strongly) clean provided that for any $a \in R$ there exists a unique idempotent $e \in R$ $\big(e \in comm(a)\big)$ such that $a - e \in U(R)$. We prove, in this note, that a ring R is uniquely clean and uniquely bleached if and only if R is abelian, $\mathbb{T}_n(R)$ is uniquely strongly clean for all $n \geq 1$, i.e. every $n \times n$ triangular matrix over R is uniquely strongly clean, if and only if R is abelian, and $\mathbb{T}_n(R)$ is uniquely strongly clean for some $n \geq 1$. In the commutative case, more explicit results are obtained.

Key words: Uniquely strongly clean ring, uniquely bleached ring, triangular matrix ring

1. Introduction

Throughout this article, all rings are associative with unity. We write $U(R)$ for the set of all units in R. $\mathbb{T}_n(R)$ stands for the ring of all $n \times n$ triangular matrices over a ring R. Let $a, b \in R$. We denote the map from R to $R : x \mapsto ax - xb$ by $l_a - r_b$. We write $\mathbb{M}_n(R)$ for the ring of all $n \times n$ matrices over the ring R. The *commutant* of an element a in a ring R is defined by $comm(a) = \{x \in R \mid xa = ax\}$. $\mathbb{N}$ is the set of all natural numbers.

A ring R is *strongly clean* provided that for any $a \in R$ there exists an idempotent $e \in comm(a)$ such that $a - e \in U(R)$. Strongly clean triangular matrices are extensively studied by many authors, e.g., [1] and [3]. A ring R is called *uniquely clean* provided that for any $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in U(R)$. Many characterizations of such rings are studied in [2, 3, 4, 10] and [11]. Following Chen et al. [5], a ring R is called *uniquely strongly clean* provided that for any $a \in R$ there exists a unique idempotent $e \in comm(a)$ such that $a - e \in U(R)$. Uniquely strong cleanness behaves very differently from the properties of uniquely clean rings (cf. [5]). In general, matrix rings do not have such properties (see [13, Proposition 11.8]). Thus, it is attractive to investigate uniquely strong cleanness of triangular matrices over a ring. Chen et al. proved that if R is commutative, then R is uniquely clean if and only if $\mathbb{T}_n(R)$ is uniquely strongly clean for all $n \geq 1$ if and only if $\mathbb{T}_n(R)$ is uniquely strongly clean for some $n \geq 1$.

[5, Question 12] and [13, Question 11.13] asked if "commutative" in the preceding result can be replaced by "abelian". The motivation of this note is to explore this problem. Following [7], a ring R is *uniquely bleached* provided that for any $a \in J(R)$, $b \in U(R)$, $l_a - r_b$, and $l_b - r_a$ are isomorphism. We prove, in this note, that R is uniquely clean and uniquely bleached if and only if R is abelian, $\mathbb{T}_n(R)$ is uniquely strongly clean for all

*Correspondence: handankose@gmail.com
2010 *AMS Mathematics Subject Classification:* 16U99, 16S99.

$n \geq 1$ if and only if R is abelian, and $\mathbb{T}_n(R)$ is uniquely strongly clean for some $n \geq 1$. In the commutative case, more explicit results are obtained. These also generalize the main theorems in [5] and [6], and provide many new classes of such rings.

2. The main results

It is well known that every uniquely clean ring is a uniquely strongly clean ring, but the converse is not true. For instance, $\mathbb{T}_2(\mathbb{Z}_{(2)})$ is uniquely strongly clean, while it is not uniquely clean. We are concerned with uniquely strongly clean triangular matrix rings over a uniquely clean base ring. We begin with

Lemma 1 *Let R be a ring. If $\mathbb{T}_2(R)$ is uniquely strongly clean, then R is uniquely bleached.*

Proof In view of [5, Example 5], R is uniquely strongly clean. Let $a \in J(R)$ and $b \in U(R)$, and let $r \in R$. Choose $A = \begin{bmatrix} a & -r \\ & b \end{bmatrix} \in \mathbb{T}_2(R)$. Then there exists a unique idempotent $E = [e_{ij}] \in \mathbb{T}_2(R)$ such that $A - E \in U(\mathbb{T}_2(R))$ and $EA = AE$. It can be easily seen that e_{11} and $e_{22} \in R$ are idempotents. Further, $a - e_{11} \in U(R)$ and $b - e_{22} \in U(R)$. As $a - 0 \in U(R)$ and $b - 1 \in U(R)$, by the uniquely strong cleanness of R, we get $e_{11} = 0$ and $e_{22} = 1$. Thus, $E = \begin{bmatrix} 0 & x \\ & 1 \end{bmatrix}$ for some $x \in R$. It follows from $EA = AE$ that $ax - xb = r$. Assume that $ay - yb = r$. Then we have an idempotent $F = \begin{bmatrix} 0 & y \\ & 1 \end{bmatrix}$ such that $A - F \in U(\mathbb{T}_2(R))$ and $AF = FA$. By the uniqueness of E, we get $x = y$. Therefore, $l_a - r_b : R \to R$ is an isomorphism. Likewise, $l_b - r_a : R \to R$ is an isomorphism. Accordingly, R is uniquely bleached, as asserted. $\square$

The following theorem is a generalization of Theorem 1 in [6].

Theorem 2 *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely clean and uniquely bleached.*

(2) *R is abelian, and $\mathbb{T}_n(R)$ is uniquely strongly clean for all $n \in \mathbb{N}$.*

(3) *R is abelian, $\mathbb{T}_n(R)$ is uniquely strongly clean for some $n \in \mathbb{N}$.*

Proof (1) $\Rightarrow$ (2) In view of [10, Theorem 20], R is abelian. Clearly, the result holds for $n = 1$. Assume that the result holds for $n (n \geq 1)$. Let $A = \begin{bmatrix} a_{11} & \alpha \\ & A_1 \end{bmatrix} \in \mathbb{T}_{n+1}(R)$ where $a_{11} \in R$, $\alpha \in \mathbb{M}_{1\times n}(R)$, and $A_1 \in \mathbb{T}_n(R)$. Since R is uniquely clean, we can find a unique idempotent $e_{11} \in R$ such that $u_{11} := a_{11} - e_{11} \in U(R)$ and $a_{11}e_{11} = e_{11}a_{11}$. Furthermore, we have a unique idempotent $E_1 \in \mathbb{T}_n(R)$ such that $U_1 := A_1 - E_1 \in U(\mathbb{T}_n(R))$ and $A_1E_1 = E_1A_1$; hence, $U_1E_1 = E_1U_1$. Let $E = \begin{bmatrix} e_{11} & x \\ & E_1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & \alpha - x \\ & U_1 \end{bmatrix}$, where $x \in \mathbb{M}_{1\times n}(R)$. Observing that

$$\begin{array}{lll} E^2 = E & \Leftrightarrow & e_{11}x + xE_1 = x; \qquad (i) \\ UE = EU & \Leftrightarrow & u_{11}x + (\alpha - x)E_1 = e_{11}(\alpha - x) + xU_1, \qquad (ii) \end{array}$$

and then combining (i) with (ii) yields that

$$(u_{11} + 2e_{11} - 1)x - xU_1 = e_{11}\alpha - \alpha E_1. \qquad (*)$$

It is enough to show that there exists a unique $x \in \mathbb{M}_{1\times n}(R)$ such that $(*)$ holds. In view of [10, Theorem 20], $R/J(R)$ is Boolean, and so $2 \in J(R)$. Furthermore, $u_{11} \in 1 + J(R)$. This shows that $u_{11} + 2e_{11} - 1 \in J(R)$.

Write $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$, $e_{11}\alpha - \alpha E_1 = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$, and $U_1 = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ & c_{22} & \cdots & c_{2n} \\ & & \ddots & \\ & & & c_{nn} \end{bmatrix}$ where each $c_{ii} \in U(R)$. The equation $(*)$ is equivalent to the n equations:

$$\begin{aligned} (u_{11} + 2e_{11} - 1)x_1 - x_1c_{11} &= v_1; \\ (u_{11} + 2e_{11} - 1)x_2 - x_2c_{22} &= v_1 + x_1c_{12}; \\ &\vdots \\ (u_{11} + 2e_{11} - 1)x_n - x_nc_{nn} &= v_n + x_1c_{1n} + \cdots + x_{n-1}c_{(n-1)n}. \end{aligned}$$

As R is uniquely bleached, we have a unique $x_i \in R$ $(i = 1, \ldots, n)$, and so there exists a unique x such that $(*)$ holds. Further, we see that

$$\begin{aligned} & (u_{11} + 2e_{11} - 1)x(e_{11}I_n + E_1) - x(e_{11}I_n + E_1)U_1 \\ = \;& \alpha(e_{11}I_n - E_1)(e_{11}I_n + E_1) \\ = \;& \alpha(e_{11}I_n - E_1) \\ = \;& (u_{11} + 2e_{11} - 1)x - xU_1 \end{aligned}$$

because $E_1U_1 = U_1E_1$. Set $y = x(e_{11}I_n + E_1)$. This implies that $(u_{11} + 2e_{11} - 1)(y - x) - (y - x)U_1 = 0$. Write $y - x = \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix}$. Then

$$\begin{aligned} (u_{11} + 2e_{11} - 1)z_1 - z_1c_{11} &= 0; \\ (u_{11} + 2e_{11} - 1)z_2 - z_2c_{22} &= z_1c_{12}; \\ &\vdots \\ (u_{11} + 2e_{11} - 1)z_n - z_nc_{nn} &= z_1c_{1n} + \cdots + z_{n-1}c_{(n-1)n}. \end{aligned}$$

Since R is uniquely bleached, we get each $z_i = 0$, and so $y = x$. This gives that $e_{11}x + xE_1 = x$. Furthermore, $u_{11}x - (\alpha - x)E_1 = e_{11}(\alpha - x) + xU_1$. Therefore, we have a uniquely strongly clean expression

$$A = \begin{bmatrix} e_{11} & x \\ & E_1 \end{bmatrix} + \begin{bmatrix} a_{11} - e_{11} & \alpha - x \\ & A_1 - E_1 \end{bmatrix}.$$

By induction, $\mathbb{T}_n(R)$ is uniquely strongly clean for all $n \in \mathbb{N}$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) In light of [5, Example 5], both R and $\mathbb{T}_2(R)$ are uniquely strongly clean, hence the result by [10, Theorem 20] and Lemma 1. □

Remark 3 Examples of uniquely bleached rings include the ring with nil Jacobson radical, the ring for which some power of each element in $J(R)$ is central, and commutative rings.

The double commutant of an element a in a ring R is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$. Clearly, $comm^2(a) \subseteq comm(a)$. This concept is closely related to quasipolar, perfectly clean, and pseudopolar elements (for details see [4, 8, 9, 12]). We end this note by a more explicit result than [5, Theorem 10].

Theorem 4 *Let R be a commutative ring, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *R is uniquely clean.*

(2) *For any $A \in \mathbb{T}_n(R)$, there exists a unique idempotent $E \in comm^2(A)$ such that $A - E \in U\big(\mathbb{T}_n(R)\big)$.*

Proof. (1) ⇒ (2) For any $A \in \mathbb{T}_n(R)$, we claim that there exists an idempotent $E \in comm^2(A)$ such that $A - E \in U\big(\mathbb{T}_n(R)\big)$. Suppose that the result holds for $n-1 (n \geq 2)$. Let $A = \begin{bmatrix} a_{11} & \alpha \\ & A_1 \end{bmatrix} \in \mathbb{T}_n(R)$, where $a_{11} \in R, \alpha \in \mathbb{M}_{1\times(n-1)}(R)$ and $A_1 \in \mathbb{T}_{n-1}(R)$. Since R is a commutative uniquely clean ring, there exists a unique idempotent $E = \begin{bmatrix} e_{11} & x \\ & E_1 \end{bmatrix}$ such that $A - E \in U\big(\mathbb{T}_n(R)\big)$ and $E \in comm(A)$ by Theorem 2. Write $A - E = \begin{bmatrix} u_{11} & \alpha - x \\ & U_1 \end{bmatrix}$. According to the proof of Theorem 2, we know that $\alpha(E_1 - e_{11}I_{n-1}) = x\big(U_1 - (u_{11} + 2e_{11} - 1)I_{n-1}\big)$ by $(*)$ and A_1 is uniquely strongly clean with E_1. This implies that $E_1 \in comm^2(A_1)$ by induction. For any $X = \begin{bmatrix} x_{11} & \beta \\ & X_1 \end{bmatrix} \in comm(A)$, we have $x_{11}\alpha + \beta A_1 = a_{11}\beta + \alpha X_1$, and so $\alpha(X_1 - x_{11}I_{n-1}) = \beta(A_1 - a_{11}I_{n-1})$. We check that

$$\begin{aligned} & \beta(A_1 - a_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) \\ = \; & \alpha(X_1 - x_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) \\ = \; & \alpha(E_1 - e_{11}I_{n-1})(X_1 - x_{11}I_{n-1}) \\ = \; & x\big(U_1 - (u_{11} + 2e_{11} - 1)I_{n-1}\big)(X_1 - x_{11}I_{n-1}) \\ = \; & x(X_1 - x_{11}I_{n-1})\big(U_1 - (u_{11} + 2e_{11} - 1)I_{n-1}\big) \end{aligned}$$

because $E_1 \in comm^2(A_1)$ and $X_1 \in comm(A_1)$. Moreover,

$$\begin{aligned} & \beta(A_1 - a_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) \\ = \; & \beta(E_1 - e_{11}I_{n-1})\big(E_1 + U_1 - (e_{11} + u_{11})I_{n-1}\big) \\ = \; & \beta(E_1 - e_{11}I_{n-1})\big(E_1 + e_{11}I_{n-1} + (U_1 - 2e_{11} - u_{11})I_{n-1}\big) \\ = \; & \beta\big(E_1 - e_{11}I_{n-1} + (E_1 - e_{11}I_{n-1})(U_1 - 2e_{11} - u_{11})I_{n-1}\big) \\ = \; & \beta(E_1 - e_{11}I_{n-1})\big(U_1 + (1 - 2e_{11} - u_{11})I_{n-1}\big) \\ = \; & \beta(E_1 - e_{11}I_{n-1})\big(U_1 - (u_{11} + 2e_{11} - 1)I_{n-1}\big). \end{aligned}$$

This shows that $\beta(E_1 - e_{11}I_{n-1}) = x(X_1 - x_{11}I_{n-1})$ since $U_1 - (u_{11} + 2e_{11} - 1)I_{n-1} \in U\big(\mathbb{T}_{n-1}(R)\big)$. Thus, we get $e_{11}\beta + xX_1 = x_{11}x + \beta E_1$; hence, $EX = XE$. That is, $E \in comm^2(A)$, as claimed.

(2) ⇒ (1) Let $a \in R$. Then $A = diag(a, a, \ldots, a) \in \mathbb{T}_n(R)$. Hence, we can find a unique idempotent $E = [e_{ij}] \in comm^2(A)$ such that $A - E \in U\big(\mathbb{T}_n(R)\big)$. This implies that $e_{11} \in R$ is an idempotent and $a - e_{11} \in U(R)$. Suppose that $a - e \in U(R)$ with an idempotent $e \in R$. Then $F = diag(e, e, \ldots, e) \in \mathbb{T}_n(R)$ is an idempotent. Further, $F \in comm^2(A)$, and that $A - F \in U\big(\mathbb{T}_n(R)\big)$. By the uniqueness, we get $E = F$, and then $e = e_{11}$. Therefore R is uniquely clean, as asserted. □

The next result showed that if R is commutative uniquely clean, then both A and $-A$ are uniquely strongly clean for any $A \in \mathbb{T}_n(R)$.

Corollary 5 *Let R be a commutative uniquely clean ring. Then for any $A \in \mathbb{T}_n(R)$, there exists a unique idempotent $E \in comm(A)$ such that $A - E, A + E \in U\big(\mathbb{T}_n(R)\big)$.*

Proof In view of Theorem 4, we have a unique idempotent $E \in comm^2\big(A^2\big)$ such that $A^2 - E$ is invertible. Obviously, $EA = AE$. Then $A - E$ and $A + E$ are invertible. On the other hand, if $A - F$ and $A + F$ are invertible for some idempotent F that commutes with A, then $A^2 - F$ is invertible. Then $A - E, A - F \in U\big(\mathbb{T}_n(R)\big)$ and $EA = AE, FA = AF$. By Theorem 2, we have $E = F$, as desired. □

Acknowledgments

The authors are grateful to the referee for his/her helpful suggestions, which led to the new version being clearer. This research was supported by the Natural Science Foundation of Zhejiang Province (LY13A010019) and the Scientific and Technological Research Council of Turkey (TÜBİTAK-2221 Visiting Scientists Fellowship Programme).

References

[1] Borooah G, Diesl AJ, Dorsey TJ. Strongly clean triangular matrix rings over local rings. J Algebra 2007; 312: 773–797.

[2] Chen H. On uniquely clean rings. Comm Algebra 2011; 39: 189–198.

[3] Chen H. Rings Related to Stable Range Conditions. Series in Algebra 11, World Scientific, Hackensack, NJ, 2011.

[4] Chen H, Halicioglu S, Kose H. On perfectly clean rings. preprint, arXiv:1307.6087.

[5] Chen J, Wang Z, Zhou Y. Rings in which elements are uniquely the sum of an idempotent and a unit that commute. J Pure Appl Algebra 2009; 213: 215–223.

[6] Cui J, Chen J. Uniquely strongly clean triangular matrix rings. J Southeast Univ (English Edition) 2011; 27: 463–465.

[7] Diesl AJ. Classes of strongly clean rings. PhD, University of California, Berkeley. 2006.

[8] Harte RE. On quasinilpotents in rings. Panamer Math J 1991; 1: 10–16.

[9] Koliha JJ, Patricio P. Elements of rings with equal spectral idempotents. J Austral Math Soc 2002; 72: 137–152.

[10] Nicholson WK, Zhou Y. Rings in which elements are uniquely the sum of an idempotent and a unit. Glasgow Math J 2004; 46: 227–236.

[11] Wang XL. Uniquely strongly clean group rings. Commun Math Research 2012; 28: 17–25.

[12] Wang Z., Chen J. Pseudo Drazin inverses in associative rings and Banach algebras. Linear Algebra Appl 2012; 437: 1332–1345.

[13] Yang X. A survey of strongly clean rings. Acta Appl Math 2009; 108: 157–173.