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On metallic Riemannian structures

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Abstract: The paper is devoted to the study of metallic Riemannian structures. An integrability condition and curvature properties for these structures by means of a Φ -operator applied to pure tensor fields are presented. Examples of these structures are also given.

Key words: Conformal metric, metallic structure, pure tensor, Riemannian manifold, tensor bundle, twin metric

1. Introduction

Let M be an n -dimensional manifold. We point out here and once that all geometric objects considered in this paper are supposed to be of class C^∞ .

The number $\eta = \frac{1+\sqrt{5}}{2} \approx 1,61803398874989\dots$, which is the positive root of the equation $x^2 - x - 1 = 0$, represents the golden mean. There are two most important generalizations of the golden mean. The first of them is the golden p -proportions being a positive root of the equation $x^{p+1} - x^p - 1 = 0$, ($p = 0, 1, 2, 3, \dots$) in [13]. The other, called the metallic means family or metallic proportions, was introduced by de Spinadel in [2, 3, 5, 4]. For two positive integers p and q , the positive solution of the equation $x^2 - px - q = 0$ is named members of the metallic means family. All the members of the metallic means family are positive quadratic irrational numbers $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$. These numbers $\sigma_{p,q}$ are also called (p, q) -metallic numbers. Inspired by the metallic means family, Hretcanu and Crasmareanu [8] constructed a new structure on a Riemannian manifold and named it a metallic structure. Indeed, a metallic structure is a polynomial structure with the structural polynomial $Q(J) = J^2 - pJ - qI$. Polynomial structures on a manifold were defined in [7]. A polynomial structure F of degree d on a connected manifold M means that a $(1, 1)$ -tensor field F satisfies the following algebraic polynomial equation:

$$Q(F) = F^d + a_1F^{d-1} + \dots + a_{d-1}F + a_dI = 0,$$

where a_1, a_2, \dots, a_d are real numbers and I is the identity tensor of type $(1, 1)$.

Given a Riemannian manifold (M, g) endowed with the metallic structure J , then the triple (M, J, g) is named a metallic Riemannian manifold if

$$g(JX, Y) = g(X, JY) \tag{1}$$

or equivalently

$$g(JX, JY) = g(J^2X, Y) = g((pJ + qI)X, Y) = pg(JX, Y) + qg(X, Y)$$

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for all vector fields X and Y on M [8]. The Riemannian metric (1) is referred to as J -compatible or pure metric [11, 14, 17].

In general, a $(0, s)$ -tensor field t is pure with respect to a $(1, 1)$ -tensor field ψ if and only if the following condition holds:

$$t(\psi Y_1, Y_2, \dots, Y_s) = t(Y_1, \psi Y_2, \dots, Y_s) = \dots = t(Y_1, Y_2, \dots, \psi Y_s)$$

for any vector fields Y_1, Y_2, \dots, Y_s on M . The Tachibana operator Φ_ψ applied to the $(0, s)$ -tensor field t is defined by

$$\begin{aligned} (\Phi_\psi t)(X, Y_1, \dots, Y_s) &= (\psi X)t(Y_1, \dots, Y_s) - Xt(\psi Y_1, \dots, Y_s) \\ &+ \sum_{\lambda=1}^s t(Y_1, \dots, (L_{Y_\lambda} \psi) X, \dots, Y_s), \end{aligned} \tag{2}$$

where L_Y denotes the Lie differentiation with respect to Y [11, 14, 17]. If the pure tensor t satisfies $\Phi_\psi t = 0$, then it is called a Φ -tensor. If ψ is a product structure, then a Φ -tensor is a decomposable tensor.

2. Locally decomposable metallic Riemannian structures

Let (M, g, F) be a locally decomposable Riemannian manifold. This means that the Riemannian manifold (M, g) is equipped with an almost product structure F , $F^2 = I$, such that

$$g(FX, Y) = g(X, FY)$$

and

$$\nabla F = 0$$

for all vector fields X and Y on M , where ∇ is the operator of the Riemannian covariant derivation. The theory of Riemannian almost product structures was initiated by Yano in [16]. The classification of Riemannian almost product structure with respect to their covariant derivatives was described by Naveira in [9]. In [10], it was shown that the condition $\nabla F = 0$ is equivalent to decomposability of the pure metric g , i.e. $\Phi_F g = 0$, where Φ_F is the Tachibana operator [11, 14, 17]:

$$(\Phi_F g)(X, Y, Z) = (FX)(g(Y, Z)) - X(g(FY, Z)) + g((L_Y F)X, Z) + g(Y, (L_Z F)X).$$

As is known, a polynomial structure F is integrable if and only if it is possible to introduce a torsion-free linear connection ∇ with respect to which the structure tensor F is covariantly constant [15]. By using the Tachibana operator, we can give another condition of integrability for a metallic Riemannian structure.

Theorem 2.1 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g . Then:*

- a) J is integrable if $\Phi_J g = 0$,
- b) the condition $\Phi_J g = 0$ is equivalent to $\nabla J = 0$, where ∇ is the Levi-Civita connection of g .

Proof The proof is similar to that of Theorem 2.1 in [6], so we omit it. □

Next, we are going to give relationships between the almost product structures and metallic structures on M .

Proposition 2.2 [8] *If J is a metallic structure on M , then*

$$F_{\pm} = \pm \left(\frac{2}{2\sigma_{p,q} - p} J - \frac{p}{2\sigma_{p,q} - p} I \right) \tag{3}$$

are two almost product structures on M . Conversely, every almost product structure F on M induces two metallic structures on M , given as follows:

$$J_{\pm} = \frac{p}{2} I \pm \left(\frac{2\sigma_{p,q} - p}{2} \right) F.$$

Due to (3), it follows that:

- i) A Riemannian metric g is pure with respect to a metallic structure J if and only if the Riemannian metric g is pure with respect to the almost product structures F_{\pm} associated with J .*
- ii) The dependence between $\Phi_{F_{\pm}}g$ and Φ_Jg is as follows:*

$$\Phi_{F_{\pm}}g = \pm \frac{2}{2\sigma_{p,q} - p} \Phi_Jg, \tag{4}$$

from which, in view of Theorem 2.1, we can say that the metallic Riemannian structure J is integrable if $\Phi_{F_{\pm}}g = 0$, i.e. the Riemannian metric g is decomposable. If (M, J, g) is a metallic Riemannian manifold with a decomposable pure metric, then we call it a locally decomposable metallic Riemannian manifold. Hence, we have the following.

Proposition 2.3 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g . The manifold M is a locally decomposable metallic Riemannian manifold if and only if $\Phi_{F_{\pm}}g = 0$, where F_{\pm} are the almost product structures associated with J .*

The twin metallic Riemannian metric is defined by

$$G(X, Y) = g(JX, Y)$$

for all vector fields X and Y on M . One can easily prove that G is pure with respect to J . If we apply the Φ_J -operator to the metric G , standard calculations give

$$(\Phi_JG)(X, Y, Z) = (\Phi_Jg)(X, JY, Z) + g(N_J(X, Y), Z). \tag{5}$$

Thus, (5) implies the following result.

Proposition 2.4 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g . Then $\Phi_Jg = 0$ is equivalent to $\Phi_JG = 0$ if $N_J = 0$, where N_J is Nijenhuis tensor constructed from J .*

We now turn our attention to the Riemannian curvature tensor field R of the locally decomposable metallic Riemannian manifold (M, J, g) .

Theorem 2.5 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g . The Riemannian curvature tensor field is a J -tensor field.*

Proof The Riemannian curvature tensor field R of the metallic Riemannian metric g is pure with respect to the metallic structure J , i.e.

$$R(JY_1, Y_2, Y_3, Y_4) = R(Y_1, JY_2, Y_3, Y_4) = R(Y_1, Y_2, JY_3, Y_4) = R(Y_1, Y_2, Y_3, JY_4).$$

From (2), the Tachibana operator Φ_J applied to the Riemannian curvature tensor field R of type $(0, 4)$ can be written as follows:

$$(\Phi_J R)(X, Y_1, Y_2, Y_3, Y_4) = (\nabla_{JX} R)(Y_1, Y_2, Y_3, Y_4) - (\nabla_X R)(JY_1, Y_2, Y_3, Y_4). \tag{6}$$

We can also say that in the locally decomposable metallic Riemannian manifold, the covariant derivative of the Riemannian curvature tensor R with respect to the Levi-Civita connection of g is pure. Using Bianchi's second identity and purity conditions, simple calculations give

$$(\Phi_J R)(X, Y_1, Y_2, Y_3, Y_4) = 0.$$

We omit standard calculations (see also [6]). □

By (3) and (6), we can find, in a similar way as for (4), the following:

$$\Phi_{F_{\pm}} R = \pm \frac{2}{2\sigma_{p,q} - p} \Phi_J R. \tag{7}$$

In view of Theorem 2.5 and (7), we have the result below.

Proposition 2.6 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g . The Riemannian curvature tensor field is a decomposable tensor field.*

3. Metallic structures with conformal metrics

Given a Riemannian metric g , we can easily define a new Riemannian metric \tilde{g} in terms of g by multiplying g by a smooth function f , or for vector fields X and Y on M ,

$$\tilde{g}(X, Y) = e^{2f} g(X, Y).$$

The metric \tilde{g} is called conformal to the Riemannian metric g .

Let us consider that (M, J, g) is a metallic Riemannian manifold. Immediately, we can say that (M, J, \tilde{g}) is also a metallic Riemannian manifold. If we apply the Φ_J -operator to the conformal metric \tilde{g} , we get

$$\begin{aligned} (\Phi_J \tilde{g})(X, Y, Z) &= (JX)(e^{2f} g(Y, Z)) - X(e^{2f} g(JY, Z)) + e^{2f} g((L_Y J)X, Z) \\ &\quad + e^{2f} g(Y, (L_Z J)X) \\ &= (JX)(e^{2f} g(Y, Z)) - X(e^{2f} g(JY, Z)) + e^{2f} (\Phi_J g)(X, Y, Z). \end{aligned}$$

By using (3) and (4), we have

$$\begin{aligned} (\Phi_J \tilde{g})(X, Y, Z) &= \pm \frac{2\sigma_{p,q} - p}{2} \{(F_{\pm} X)(e^{2f} g(Y, Z)) - X(e^{2f} g(F_{\pm} Y, Z)) \\ &\quad + e^{2f} (\Phi_{F_{\pm}} g)(X, Y, Z)\}, \end{aligned}$$

where F_{\pm} are the almost product structures associated with J . Therefore, we have following theorem.

Theorem 3.1 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g . Then $(M, J, \tilde{g} = e^{2f}g)$ is a locally decomposable metallic Riemannian manifold if and only if the function f is constant.*

4. Examples

Example 1. Let M be an n -dimensional Riemannian manifold with a Riemannian metric g and denote by $\pi : T_1^1(M) \rightarrow M$ its $(1,1)$ -tensor bundle with fibers the $(1,1)$ -tensor spaces to M . Then $T_1^1(M)$ is an $n + n^2$ -dimensional smooth manifold and some local charts induced naturally from local charts on M may be used. Namely, a system of local coordinates $(U; x^j)$ in M induces on $T_1^1(M)$ a system of local coordinates $(\pi^{-1}(U); x^j, x^{\bar{j}} = t_j^i)$ $j = 1, \dots, n, \bar{j} = n+1, \dots, n+n^2, J = 1, \dots, n+n^2$, where (t_j^i) are the Cartesian coordinates in each $(1,1)$ -tensor space $T_{1(P)}^1M$ at $P \in M$ with respect to the natural base.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $A = A_j^i \frac{\partial}{\partial x^i} \otimes dx^j$ be the local expressions in U of a vector field X and a $(1,1)$ -tensor field A on M , respectively. Then the vertical lift ${}^V A$ of A and the horizontal lift ${}^H X$ of X are given, with respect to the induced coordinates, by

$${}^V A = \begin{pmatrix} {}^V A^j \\ {}^V A^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_j^i \end{pmatrix} \tag{8}$$

and

$${}^H X = \begin{pmatrix} {}^H X^j \\ {}^H X^{\bar{j}} \end{pmatrix} = \begin{pmatrix} X^j \\ X^s (\Gamma_{sj}^m t_m^i - \Gamma_{sm}^i t_j^m) \end{pmatrix}, \tag{9}$$

where Γ_{ij}^h are the coefficients of the Levi-Civita connection ∇ of g . The vector fields γA and $\tilde{\gamma} A$ on $T_1^1(M)$ are respectively defined by

$$\begin{aligned} \gamma A &= \begin{pmatrix} 0 \\ t_j^m A_m^i \end{pmatrix}, \\ \tilde{\gamma} A &= \begin{pmatrix} 0 \\ (t_m^i A_j^m) \end{pmatrix}. \end{aligned}$$

From (8) we easily see that the vector fields γA and $\tilde{\gamma} A$ determine respectively the global vector fields on $T_1^1(M)$ [1].

The Sasaki type metric ${}^S g$ on $T_1^1(M)$ is defined by the following three equations:

$${}^S g ({}^V A, {}^V B) = g(A, B), \tag{10}$$

$${}^S g ({}^V A, {}^H Y) = 0, \tag{11}$$

$${}^S g ({}^H X, {}^H Y) = g(X, Y), \tag{12}$$

for any vector fields X and Y and $(1,1)$ -tensor fields A, B on M , where $g(A, B) = g_{it} g^{jl} A_j^i B_t^l$ (see [12]). From equations (10)–(12) we easily see that the horizontal distribution H , induced by ∇_g and determined by

the horizontal lifts, is orthogonal to the fibers of $T_1^1(M)$. Let now E be a nowhere zero vector field on M . For any vector field X and covector field $\tilde{E} = g \circ E$ on M , we define the vertical lift ${}^V(X \otimes \tilde{E})$ of X with respect to E . The map $X \rightarrow {}^V(X \otimes \tilde{E})$ is a monomorphism. Hence, an n -dimensional C^∞ vertical distribution V^E is defined on $T_1^1(M)$. Let V^\perp be the distribution on $T_1^1(M)$, which is orthogonal to H and V^E . Then H , V^E , and V^\perp are mutually orthogonal distributions with respect to the Sasaki type metric Sg . We define a $(1,1)$ -tensor field \tilde{J} on $T_1^1(M)$ by

$$\begin{cases} \tilde{J}^H X = \frac{p}{2} {}^H X + \left(\frac{2\sigma_{p,q}-p}{2}\right) {}^V(X \otimes \tilde{E}), \\ \tilde{J}^V(X \otimes \tilde{E}) = \frac{p}{2} {}^V(X \otimes \tilde{E}) + \left(\frac{2\sigma_{p,q}-p}{2}\right) {}^H X, \\ \tilde{J}({}^V A) = \sigma_{p,q} {}^V A, \end{cases} \tag{13}$$

for any vector field X and $(1,1)$ -tensor field A on M , where $\tilde{E} = g \circ E$ is a covector field on M . The restrictions of \tilde{J} to $H + V^E$ and V^\perp are endomorphisms, and hence \tilde{J} is a $(1,1)$ -tensor field on $T_1^1(M)$. It is easily see that $\tilde{J}^2 - p\tilde{J} - qI = 0$, i.e. \tilde{J} is a metallic structure on $T_1^1(M)$.

Theorem 4.1 *Let (M, g) be a Riemannian manifold and $T_1^1(M)$ be its tensor bundle equipped with the Sasaki type metric Sg and the metallic structure \tilde{J} defined by (13). The triple $(T_1^1(M), \tilde{J}, {}^Sg)$ is a metallic Riemannian manifold if and only if $g(E, E) = 1$.*

Proof We calculate

$$A(\tilde{X}, \tilde{Y}) = {}^Sg(\tilde{J}\tilde{X}, \tilde{Y}) - {}^Sg(\tilde{X}, \tilde{J}\tilde{Y})$$

for any vector fields \tilde{X} and \tilde{Y} on $T_1^1(M)$. From (10)–(12) and (13), we obtain

$$\begin{aligned} A({}^H X, {}^H Y) &= {}^Sg(\tilde{J}^H X, {}^H Y) - {}^Sg({}^H X, \tilde{J}^H Y) \\ &= {}^Sg\left(\frac{p}{2} {}^H X + \left(\frac{2\sigma_{p,q}-p}{2}\right) {}^V(X \otimes \tilde{E}), {}^H Y\right) \\ &\quad - {}^Sg\left({}^H X, \frac{p}{2} {}^H Y + \left(\frac{2\sigma_{p,q}-p}{2}\right) {}^V(Y \otimes \tilde{E})\right) \\ &= 0, \\ A({}^V(X \otimes \tilde{E}), {}^V(Y \otimes \tilde{E})) &= {}^Sg(\tilde{J}^V(X \otimes \tilde{E}), {}^V(Y \otimes \tilde{E})) \\ &\quad - {}^Sg({}^V(X \otimes \tilde{E}), \tilde{J}^V(Y \otimes \tilde{E})) \\ &= {}^Sg\left(\frac{p}{2} {}^V(X \otimes \tilde{E}) + \left(\frac{2\sigma_{p,q}-p}{2}\right) {}^H X, {}^V(Y \otimes \tilde{E})\right) \\ &\quad - {}^Sg\left({}^V(X \otimes \tilde{E}), \frac{p}{2} {}^V(Y \otimes \tilde{E}) + \left(\frac{2\sigma_{p,q}-p}{2}\right) {}^H Y\right) \\ &= 0, \end{aligned}$$

$$\begin{aligned}
 A\left({}^V(X \otimes \tilde{E}), {}^HY\right) &= {}^Sg\left(\tilde{J}^V(X \otimes \tilde{E}), {}^HY\right) - {}^Sg\left({}^V(X \otimes \tilde{E}), \tilde{J}^HY\right) \\
 &= {}^Sg\left(\frac{p}{2}{}^V(X \otimes \tilde{E}) + \left(\frac{2\sigma_{p,q}-p}{2}\right) {}^HX, {}^HY\right) \\
 &\quad - {}^Sg\left({}^V(X \otimes \tilde{E}), \frac{p}{2}{}^HY + \left(\frac{2\sigma_{p,q}-p}{2}\right) {}^V(Y \otimes \tilde{E})\right) \\
 &= \left(\frac{2\sigma_{p,q}-p}{2}\right)\{ {}^Sg({}^HX, {}^HY) - {}^Sg({}^V(X \otimes \tilde{E}), {}^V(Y \otimes \tilde{E}))\} \\
 &= \left(\frac{2\sigma_{p,q}-p}{2}\right)\{g(X, Y) - g(X, Y)g(E, E)\} \\
 A\left({}^VA, {}^VB\right) &= {}^Sg\left(\tilde{J}^VA, {}^VB\right) - {}^Sg\left({}^VA, \tilde{J}^VB\right) \\
 &= \sigma_{p,q}\{ {}^Sg({}^VA, {}^VB) - {}^Sg({}^VA, {}^VB)\} \\
 &= 0, \\
 A\left({}^VA, {}^V(Y \otimes \tilde{E})\right) &= {}^Sg\left(\tilde{J}^VA, {}^V(Y \otimes \tilde{E})\right) - {}^Sg\left({}^VA, \tilde{J}^V(Y \otimes \tilde{E})\right) \\
 &= \sigma_{p,q} {}^Sg\left({}^VA, {}^V(Y \otimes \tilde{E})\right) \\
 &\quad - {}^Sg\left({}^VA, \frac{p}{2}{}^V(Y \otimes \tilde{E}) + \left(\frac{2\sigma_{p,q}-p}{2}\right) {}^HY\right) \\
 &= 0, \\
 A\left({}^VA, {}^HY\right) &= {}^Sg\left(\tilde{J}^VA, {}^HY\right) - {}^Sg\left({}^VA, \tilde{J}^HY\right) \\
 &= \sigma_{p,q} {}^Sg\left({}^VA, {}^HY\right) \\
 &\quad - {}^Sg\left({}^VA, \frac{p}{2}{}^HX + \left(\frac{2\sigma_{p,q}-p}{2}\right) {}^V(X \otimes \tilde{E})\right) \\
 &= 0.
 \end{aligned}$$

From the equations above, we say that Sg is pure with respect to \tilde{J} if and only if $g(E, E) = 1$. This completes the proof. \square

Now we consider the covariant derivative of \tilde{J} with respect to the Levi-Civita connection of Sg . For this, first we state the following proposition.

Proposition 4.2 [12] *Let (M, g) be a Riemannian manifold and $T_1^1(M)$ be its tensor bundle equipped with the Sasaki type metric Sg . Then the corresponding Levi-Civita connection satisfies the following relations:*

- i) ${}^S\nabla_{HX} {}^HY = {}^H(\nabla_X Y) + \frac{1}{2}(\tilde{\gamma} - \gamma)R(X, Y),$
- ii) ${}^S\nabla_{VA} {}^HY = \frac{1}{2}{}^H\left(g^{bl}R(t_b, A_l)Y + g_{at}(t^a(g^{-1} \circ R(\quad, Y)\tilde{A}^t))\right),$
- iii) ${}^S\nabla_{HX} {}^VB = {}^V(\nabla_X B) + \frac{1}{2}{}^H\left(g^{bj}R(t_b, B_j)X + g_{ai}(t^a(g^{-1} \circ R(\quad, X)\tilde{B}^i))\right),$
- iv) ${}^S\nabla_{VA} {}^VB = 0,$

for all vector fields X, Y and $(1,1)$ -tensor fields A, B on M , where $A_l = (A_l^i), \tilde{A}^t = (g^{bl}A_l \quad t) = (A^bt),$

$t_l = (t_l^a)$, $t^a = (t_b^a)$, $R(\cdot, X)Y$ is a $(1, 1)$ -tensor field and $g^{-1} \circ R(\cdot, X)Y$ is a vector field.

By using Proposition 4.2 we calculate

$$({}^S\nabla_{\tilde{X}}\tilde{J})\tilde{Y} = {}^S\nabla_{\tilde{X}}(\tilde{J}\tilde{Y}) - \tilde{J}({}^S\nabla_{\tilde{X}}\tilde{Y})$$

for all vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ on $T_1^1(M)$. Then we get

$$\begin{aligned} ({}^S\nabla_{H_X}\tilde{J})^HY &= \frac{2\sigma_{p,q}-p}{2} V(Y \otimes (g \circ \nabla_X E)) + \frac{p-2\sigma_{p,q}}{4} (\tilde{\gamma} - \gamma)R(X, Y) \\ &\quad + \frac{2\sigma_{p,q}-p}{4} H\{g^{bj} R(t_b, (Y \otimes \tilde{E})_j)X + g_{ai} (t^a(g^{-1} \circ R(\cdot, X)(Y \otimes \tilde{E})^i))\}, \\ ({}^S\nabla_{H_X}\tilde{J})^VB &= \frac{2\sigma_{p,q}-p}{4} H\{g^{bj} R(t_b, B_j)X + g_{ai} (t^a(g^{-1} \circ R(\cdot, X)\tilde{B}^i))\} \\ &\quad + \frac{p-2\sigma_{p,q}}{4} V\{[g^{bj} R(t_b, B_j)X + g_{ai} (t^a(g^{-1} \circ R(\cdot, X)\tilde{B}^i))] \otimes \tilde{E}\}, \\ ({}^S\nabla_{H_X}\tilde{J})^V(Y \otimes \tilde{E}) &= \frac{p}{2} V(Y \otimes (g \circ \nabla_X E)) + \frac{2\sigma_{p,q}-p}{4} (\tilde{\gamma} - \gamma)R(X, Y) \\ &\quad + \frac{p-2\sigma_{p,q}}{4} V\{[g^{bj} R(t_b, (Y \otimes \tilde{E})_j)X + g_{ai} (t^a(g^{-1} \circ R(\cdot, X)(Y \otimes \tilde{E})^i))] \otimes \tilde{E}\}, \\ ({}^S\nabla_{V_A}\tilde{J})^HY &= \frac{p-2\sigma_{p,q}}{4} V\{[g^{bj} R(t_b, A_j)Y + g_{ai} (t^a(g^{-1} \circ R(\cdot, Y)\tilde{A}^i))] \otimes \tilde{E}\}, \\ ({}^S\nabla_{V_A}\tilde{J})^V(Y \otimes \tilde{E}) &= \frac{2\sigma_{p,q}-p}{4} H\{g^{bj} R(t_b, A_j)Y + g_{ai} (t^a(g^{-1} \circ R(\cdot, Y)\tilde{A}^i))\}, \\ ({}^S\nabla_{V_{(X \otimes \tilde{E})}}\tilde{J})^V(Y \otimes \tilde{E}) &= \frac{2\sigma_{p,q}-p}{4} H\{g^{bj} R(t_b, (X \otimes \tilde{E})_j)Y + g_{ai} (t^a(g^{-1} \circ R(\cdot, Y)(X \otimes \tilde{E})^i))\}, \\ ({}^S\nabla_{V_{(X \otimes \tilde{E})}}\tilde{J})^HY &= \frac{p-2\sigma_{p,q}}{4} V\{[g^{bj} R(t_b, (X \otimes \tilde{E})_j)Y + g_{ai} (t^a(g^{-1} \circ R(\cdot, Y)(X \otimes \tilde{E})^i))] \otimes \tilde{E}\}, \\ ({}^S\nabla_{V_A}\tilde{J})^VB = 0, ({}^S\nabla_{V_{(X \otimes \tilde{E})}}\tilde{J})^V(Y \otimes \tilde{E}) &= 0. \end{aligned}$$

Therefore, from the last equations, we have the following result.

Theorem 4.3 *Let (M, g) be a Riemannian manifold and $T_1^1(M)$ be its tensor bundle equipped with the Sasaki type metric Sg and the metallic structure \tilde{J} defined by (13). The triple $(T_1^1(M), \tilde{J}, {}^Sg)$ is a locally decomposable*

metallic Riemannian manifold if and only if M is locally flat and $g(E, E) = 1, \nabla E = 0$, where ∇ is the Levi-Civita connection of g .

Example 2. Let us consider the $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ endowed with the Euclidean metric g , i.e.

$$g = \begin{pmatrix} \delta_j^i & 0 \\ 0 & \delta_{\bar{j}}^{\bar{i}} \end{pmatrix}, \quad i, j = 1, \dots, k, \quad \bar{i}, \bar{j} = k + 1, \dots, n.$$

The canonical product structure on \mathbb{R}^n is given by

$$F = \begin{pmatrix} 0 & \delta_j^i \\ \delta_{\bar{j}}^{\bar{i}} & 0 \end{pmatrix}, \quad i, j = 1, \dots, k, \quad \bar{i}, \bar{j} = k + 1, \dots, n.$$

The triple (\mathbb{R}^n, F, g) is a locally decomposable Euclidean space. Metallic structures J_{\pm} on \mathbb{R}^n obtained from F are as follows:

$$J_{\pm} = \begin{pmatrix} \frac{p}{2}\delta_j^i & \pm(\frac{2\sigma_{p,q}-p}{2})\delta_j^i \\ \pm(\frac{2\sigma_{p,q}-p}{2})\delta_{\bar{j}}^{\bar{i}} & \frac{p}{2}\delta_{\bar{j}}^{\bar{i}} \end{pmatrix}.$$

The triple $(\mathbb{R}^n, J_{\pm}, g)$ is a locally decomposable metallic Euclidean space.

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