Turkish Journal of Mathematics

Volume 39 | Number 6 Article 12

1-1-2015

Some properties of a class of analytic functions defined bygeneralized Struve functions

MOHSAN RAZA

NİHAT YAĞMUR

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2015) 39: 931 – 944
© TÜBİTAK
doi:10.3906/mat-1501-48

Some properties of a class of analytic functions defined by generalized Struve functions

Mohsan RAZA[1], **Nihat YAĞMUR**[2,*]
[1]Department of Mathematics, Government College University Faisalabad, Faisalabad, Pakistan
[2]Department of Mathematics, Faculty of Arts and Sciences, Erzincan University, Erzincan, Turkey

Received: 20.01.2015 • **Accepted/Published Online:** 30.06.2015 • **Printed:** 30.11.2015

Abstract: The aim of this paper is to define a new operator by using the generalized Struve functions $\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(3/2)_n(k)_n}z^{n+1}$ with $k=p+(b+2)/2\neq 0,-1,-2,\dots$ and $b,c,k\in\mathbb{C}$. By using this operator we define a subclass of analytic functions. We discuss some properties of this class such as inclusion problems, radius problems, and some other interesting properties related to this operator.

Key words: Analytic functions, subordination, generalized Struve functions

1. Introduction

Let A be the class of functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $E=\{z:|z|<1\}$. A function f is said to be subordinate to a function g written as $f\prec g$, if there exists a Schwarz function w with $w(0)=0$ and $|w(z)|<1$ such that $f(z)=g(w(z))$. In particular, if g is univalent in E, then $f(0)=g(0)$ and $f(E)\subset g(E)$.

For any two analytic functions $f(z)$ and $g(z)$ with

$$f(z)=\sum_{n=0}^{\infty}b_nz^{n+1} \text{ and } g(z)=\sum_{n=0}^{\infty}c_nz^{n+1},\ z\in E,$$

the convolution (Hadamard product) is given by

$$(f*g)(z)=\sum_{n=0}^{\infty}b_nc_nz^{n+1},\ z\in E.$$

Consider the following second-order inhomogeneous differential equation and see [16] for more details:

$$z^2w''(z)+zw'(z)+\left(z^2-p^2\right)w(z)=\frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+1/2)}. \tag{1.2}$$

*Correspondence: nhtyagmur@gmail.com
2010 *AMS Mathematics Subject Classification:* 30C45, 30C80, 33C10.

The solution of the homogeneous part is a Bessel function of order p, where p is a real or complex number. The particular solution of the inhomogeneous equation defined in (1.2) is called the Struve function of order p. It is defined as

$$H_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+p+1}}{\Gamma(n+3/2)\Gamma(p+n+3/2)}. \tag{1.3}$$

Now we consider the differential equation

$$z^2 w''(z) + zw'(z) - \left(z^2 + p^2\right) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+1/2)}. \tag{1.4}$$

Equation (1.4) differs from equation (1.2) in the coefficients of $w(z)$. Its particular solution is called the modified Struve function of order p and is given as

$$L_p(z) = -ie^{-ip\pi/2} H_p(iz) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+p+1}}{\Gamma(n+3/2)\Gamma(p+n+3/2)}.$$

Again consider the second-order inhomogenous differential equation

$$z^2 w''(z) + bzw'(z) + \left[cz^2 - p^2 + (1-b)p\right] w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+b/2)}, \tag{1.5}$$

where $b, c, p \in \mathbb{C}$. Equation (1.5) generalizes equations (1.2) and (1.4). In particular for $b = 1$, $c = 1$, we obtain (1.2) and for $b = 1$, $c = -1$, we obtain (1.4). Its particular solution has the series form

$$M_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n (z/2)^{2n+p+1}}{\Gamma(n+3/2)\Gamma(p+n+(b+2)/2)} \tag{1.6}$$

and is called the generalized Struve function of order p. This series is convergent everywhere but not univalent in the open unit disk E. We take the transformation

$$N_{p,b,c}(z) = 2^p \sqrt{\pi}\Gamma(p+(b+2)/2)\, z^{(-p-1)/2} M_{p,b,c}\left(\sqrt{z}\right) = \sum_{n=0}^{\infty} \frac{(-c/4)^n z^n}{(3/2)_n (k)_n}, \tag{1.7}$$

where $k = p+(b+2)/2 \neq 0, -1, -2, \ldots$ and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \gamma(\gamma+1)\ldots(\gamma+n-1)$. This function is analytic in the whole complex plane and satisfies the differential equation

$$4z^2 w''(z) + 2(2p+b+3) zw'(z) + [cz + 2p + b] w(z) = 2p + b.$$

Some geometric properties such as univalency, starlikeness, convexity, and close-to-convexity of the function $N_{p,b,c}(z)$ were studied recently by Orhan and Yağmur [10] and Yağmur and Orhan [14, 15].

Dziok and Srivastava [3, 4] defined the linear operator H by using the generalized hypergeometric functions and it is given as $H(\alpha_1, \ldots \alpha_s; \beta_1, \ldots \beta_q) : A \to A$ with $\alpha_i \in \mathbb{C}\,(i = 1, 2, \ldots, s)$ and $\beta_i \in \mathbb{C}\backslash\mathbb{Z}_0^-\ (i = 1, 2, \ldots, q)$ such that

$$H(\alpha_1, \ldots \alpha_s; \beta_1, \ldots \beta_q) f(z) = z\, {}_sF_q(\alpha_1, \ldots \alpha_s; \beta_1, \ldots \beta_q; z) * f(z),$$

where

$$_{s}F_{q}\left(\alpha_{1},\ldots\alpha_{s};\beta_{1},\ldots\beta_{q};z\right)=\sum_{n=0}^{\infty}\frac{\left(\alpha_{1}\right)_{n}\ldots\left(\alpha_{s}\right)_{n}z^{n}}{\left(\beta_{1}\right)_{n}\ldots\left(\beta_{q}\right)_{n}n!},\quad s\leq q+1;s,q\in\mathbb{N}_{0}=\mathbb{N}\cup\{0\}$$

is the generalized hypergeometric function. Baricz et al. [2] used a similar argument to define a convolution operator $B_k^c : A \to A$ by using generalized Bessel functions and it is given as

$$B_{k}^{c}f\left(z\right)=\varphi_{k,c}\left(z\right)*f\left(z\right)=z+\sum_{n=1}^{\infty}\frac{\left(-c/4\right)^{n}a_{n+1}z^{n+1}}{\left(k\right)_{n}n!},\quad\left(k=p+\frac{b+1}{2}\notin\mathbb{Z}_{0}^{-},c\in\mathbb{C}\right),$$

where

$$\varphi_{k,c}\left(z\right)=z+\sum_{n=1}^{\infty}\frac{\left(-c/4\right)^{n}z^{n+1}}{\left(k\right)_{n}n!}.$$

For some refrences for convolution operators see [11, 12, 13].

Now using (1.7), we define the following convolution operator. Let

$$\varphi_{p,b,c}\left(z\right)=2^{p}\sqrt{\pi}\Gamma\left(p+\left(b+2\right)/2\right)z^{\left(-p+1\right)/2}M_{p,b,c}\left(\sqrt{z}\right)=z+\sum_{n=1}^{\infty}\frac{\left(-c/4\right)^{n}z^{n+1}}{\left(3/2\right)_{n}\left(k\right)_{n}}.$$

Then

$$S_{k}^{c}f\left(z\right)=\varphi_{p,b,c}\left(z\right)*f\left(z\right)=z+\sum_{n=1}^{\infty}\frac{\left(-c/4\right)^{n}a_{n+1}z^{n+1}}{\left(3/2\right)_{n}\left(k\right)_{n}}\quad\left(k=p+\frac{b+2}{2}\notin\mathbb{Z}_{0}^{-},b,c,p\in\mathbb{C}\right).\tag{1.8}$$

It can easily be seen that

$$z\left(S_{k+1}^{c}f\left(z\right)\right)'=kS_{k}^{c}f\left(z\right)-\left(k-1\right)S_{k+1}^{c}f\left(z\right).\tag{1.9}$$

Special cases

(i) For $b=1,\ c=1,$ we have the operator $\mathcal{S}_p : A \to A$ related with the Struve function of order p. It is given as

$$\begin{aligned}\mathcal{S}_{p}f\left(z\right)&=\varphi_{p,1,1}\left(z\right)*f\left(z\right)=\left[2^{p}\sqrt{\pi}\Gamma\left(p+3/2\right)z^{\left(-p+1\right)/2}M_{p,1,1}\left(\sqrt{z}\right)\right]*f\left(z\right)\\&=z+\sum_{n=1}^{\infty}\frac{\left(-1/4\right)^{n}a_{n+1}z^{n+1}}{\left(3/2\right)_{n}\left(p+3/2\right)_{n}}\end{aligned}$$

and the recursive relation

$$z\left[\mathcal{S}_{p+1}f\left(z\right)\right]'=\left(p+3/2\right)\mathcal{S}_{p}f\left(z\right)-\left(p+1/2\right)\mathcal{S}_{p+1}f\left(z\right)$$

holds.

(ii) For $b=1,\ c=-1,$ we obtain the operator $\mathfrak{S}_p : A \to A$ related with the modified Struve function of order p. It is given as

$$\begin{aligned}\mathfrak{S}_{p}f\left(z\right)&=\varphi_{p,1,-1}\left(z\right)*f\left(z\right)=\left[2^{p}\sqrt{\pi}\Gamma\left(p+3/2\right)z^{\left(-p+1\right)/2}M_{p,1,-1}\left(\sqrt{z}\right)\right]*f\left(z\right)\\&=z+\sum_{n=1}^{\infty}\frac{\left(1/4\right)^{n}a_{n+1}z^{n+1}}{\left(3/2\right)_{n}\left(p+3/2\right)_{n}}\end{aligned}$$

and the recursive relation

$$z\left[\mathfrak{S}_{p+1}f\left(z\right)\right]' = \left(p+3/2\right)\mathfrak{S}_{p}f\left(z\right) - \left(p+1/2\right)\mathfrak{S}_{p+1}f\left(z\right)$$

holds.

We define the following class of analytic functions by using the operator $S_k^c f(z)$.

Definition 1.1 *Let $f \in A$. Then $f \in N_{k,c}^{\alpha}(\lambda,\mu,\phi)$ for $0<\mu<1, \lambda \in \mathbb{C}, k = p+(b+2)/2 \neq 0,-1,-2,\ldots, b,c,p \in \mathbb{C}$, and $|\alpha| < \frac{\pi}{2}$, if and only if*

$$e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda\frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu}\right\} \prec \cos\alpha\phi(z) + i\sin\alpha, \tag{1.10}$$

where $\phi(z)$ is a convex univalent function with $\phi(0)=1$.

(i) For $\phi(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, we have the class $N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+Az}{1+Bz}\right)$, which consists of functions f such that

$$J(\alpha,c,k,f(z)) \prec \frac{1+Az}{1+Bz},$$

where

$$J(\alpha,c,k,f(z)) = \frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda\frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu}\right\} - i\sin\alpha\right].$$

(ii) For $\phi(z) = \frac{1+z}{1-z}$, we have the class $N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+z}{1-z}\right)$. That is, $f \in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+z}{1-z}\right)$ if

$$J(\alpha,c,k,f(z)) \prec \frac{1+z}{1-z}.$$

Since it is well known that for a function $p(z) \prec \frac{1+z}{1-z}$, then $\mathrm{Re}p(z) > 0$. This implies that $f \in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+z}{1-z}\right)$ if

$$\mathrm{Re}J(\alpha,c,k,f(z)) > 0.$$

Lemma 1.2 *[8] Let F be analytic and convex in E. If $f,\ g \in A$ and $f,\ g \prec F$, then*

$$\sigma f + (1-\sigma)g \prec F, \quad 0 \leq \sigma \leq 1.$$

Lemma 1.3 *[6] Let h be convex in E with $h(0)=a$ and $\beta \in \mathbb{C}$ such that $Re\beta \geq 0$. If $p \in H[a,n]$ and*

$$p(z) + \frac{zp'(z)}{\beta} \prec h(z),$$

then $p(z) \prec q(z) \prec h(z)$, where

$$q(z) = \frac{\beta}{nz^{\beta/n}}\int_0^z h(t)\,t^{\beta/n-1}dt$$

and $q(z)$ is the best dominant.

Lemma 1.4 *[1]. Let a, b, and $c \neq 0, -1, -2\ldots$ be complex numbers. Then, for $Re c > Re b > 0$,*

$$\begin{aligned}
(i)\quad {}_2F_1(a,b,c;z) &= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt,\\
(ii)\quad {}_2F_1(a,b,c;z) &= {}_2F_1(b,a,c;z),\\
(iii)\quad {}_2F_1(a,b,c;z) &= (1-z)^{-a}\ {}_2F_1\left(a,c-b,c;\frac{z}{z-1}\right).
\end{aligned}$$

Lemma 1.5 *[7] Let $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. Then*

$$\frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z}.$$

Lemma 1.6 *[9] Let the function $g(z)$ be analytic and univalent in E and let the functions $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(E)$, with $\theta(w) \neq 0$ $(w \in g(E))$. Set $Q(z) = zg'(z)\varphi(g(z))$ and $h(z) = \theta(g(z)) + Q(z)$ and suppose that*

(i) $Q(z)$ is univalently starlike in E;

(ii) $Re\frac{zh'(z)}{Q(z)} = Re\left\{\frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0$ $(z \in E)$. If $q(z)$ is analytic in E with $q(0) = g(0), q(E) \subset D$ and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z)\ (z \in E),$$

then $q(z) \prec g(z)$ $(z \in E)$ and $g(z)$ is the best dominant.

2. Main results

Theorem 2.1 *Let $f \in N_{k,c}^{\alpha}(\lambda,\mu,\phi)$. Then for $Re\frac{\mu k}{\lambda} \geq 0$,*

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \prec \frac{\mu k}{\lambda}\cos\alpha z^{-\frac{\mu k}{\lambda}}\int_0^z \phi(t)\,t^{\frac{\mu k}{\lambda}-1}dt + i\sin\alpha \prec (\cos\alpha)\,\phi(z) + i\sin\alpha.$$

This result is the best possible.

Proof Consider

$$p(z) = \frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - i\sin\alpha\right\}. \tag{2.1}$$

Then p is analytic in E with $p(0) = 1$. Therefore, we have

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} = (\cos\alpha)\,p(z) + i\sin\alpha.$$

Differentiating both sides and using (1.9) and simplifying, we obtain

$$\frac{\lambda(\cos\alpha)\,zp'(z)}{\mu k} = \lambda e^{i\alpha}\left\{\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu}\right\}.$$

It follows from the above equation and (2.1) that

$$\begin{aligned} & p(z)+\frac{\lambda}{\mu k}zp'(z) \\ = & \frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}-\lambda\frac{S_{k}^{c}f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\}-i\sin\alpha\right]. \end{aligned}$$

Since $f\in N_{k,c}^{\alpha}(\lambda,\mu,\phi)$, therefore

$$p(z)+\frac{\lambda}{\mu k}zp'(z)\prec\phi(z).$$

Now using Lemma 1.3 for $\beta=\frac{\mu k}{\lambda}$ with $Re\frac{\mu k}{\lambda}\geq 0$, we obtain the required result. □

Corollary 2.2 *Let* $f\in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+Az}{1+Bz}\right)$*. Then for* $k,\lambda\in\mathbb{R}$ *and* $\frac{\mu k}{\lambda}\geq 0$,

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\prec h(z)\cos\alpha+i\sin\alpha,$$

where

$$h(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1,1,\frac{\mu k}{\lambda}+1;\frac{Bz}{1+Bz}\right), & B\neq 0,\\ 1+\frac{\mu k}{\mu k+\lambda}Az, & B=0.\end{cases}$$

Furthermore,

$$Re\left[e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right]>(\cos\alpha)\,h(-1).$$

Proof Since $f\in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+Az}{1+Bz}\right)$, therefore from Theorem 2.1, we have

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\prec\frac{\mu k}{\lambda}(\cos\alpha)\,z^{-\frac{\mu k}{\lambda}}\int_{0}^{z}\frac{1+At}{1+Bt}t^{\frac{\mu k}{\lambda}-1}dt+i\sin\alpha. \tag{2.2}$$

Putting $t=zu$ and after simple calculations, one can get

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\prec\left\{\frac{A}{B}+\frac{\mu k}{\lambda}\left(1-\frac{A}{B}\right)\int_{0}^{1}(1+Buz)^{-1}u^{\frac{\mu k}{\lambda}-1}dt\right\}\cos\alpha+i\sin\alpha.$$

Now using Lemma 1.4 for $a=1$, $b=\frac{\mu k}{\lambda}$, $c=b+1$, and $B\neq 0$, we obtain

$$\begin{aligned} & e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \\ \prec & \left(\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1,1,\frac{\mu k}{\lambda}+1;\frac{Bz}{1+Bz}\right)\right)\cos\alpha+i\sin\alpha. \end{aligned}$$

For the case of $B=0$, it can easily be followed from (2.2) that

$$\begin{aligned} e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} &\prec \left(\frac{\mu k}{\lambda}\int_{0}^{1}(1+Atz)\,t^{\frac{\mu k}{\lambda}-1}dt\right)\cos\alpha+i\sin\alpha. \\ &= \frac{\mu k}{\lambda}\left\{\left(\int_{0}^{1}t^{\frac{\mu k}{\lambda}-1}dt\right)+\int_{0}^{1}Azt^{\frac{\mu k}{\lambda}}dt\right\}\cos\alpha+i\sin\alpha. \\ &= \left\{1+\frac{\mu k}{\mu k+\lambda}Az\right\}\cos\alpha+i\sin\alpha. \end{aligned}$$

Now we have to prove that $\mathrm{Re}\left[e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right]>(\cos\alpha)\,h(-1)$. From (2.2), we can have this relation by using subordination

$$\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}-i\sin\alpha\right\}=h(w(z)),$$

where $h(z)=\frac{\mu k}{\lambda}z^{-\frac{\mu k}{\lambda}}\int_{0}^{z}\frac{1+At}{1+Bt}t^{\frac{\mu k}{\lambda}-1}dt$. Therefore,

$$\begin{aligned} \mathrm{Re}\left[\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\}\right] &= \mathrm{Re}\frac{\mu k}{\lambda}\int_{0}^{1}\frac{1+Atw(z)}{1+Btw(z)}t^{\frac{\mu k}{\lambda}-1}dt \\ &> \frac{\mu k}{\lambda}\int_{0}^{1}\frac{1-At}{1-Bt}t^{\frac{\mu k}{\lambda}-1}dt \\ &= h(-1). \end{aligned}$$

To show that this result is sharp, we have to prove that $\inf\limits_{|z|<1}\{\mathrm{Re}h(z)\}=h(-1)$. Now

$$\mathrm{Re}h(z)\geq\frac{\mu k}{\lambda}\int_{0}^{1}t^{\frac{\mu k}{\lambda}-1}\frac{1-Atr}{1-Btr}dt=h(-r).$$

Therefore, $h(-r)\to h(-1)$ as $r\to1^{-}$. □

Theorem 2.3 *Let* $e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\prec\phi(z)\cos\alpha+i\sin\alpha$ *with* $\phi(z)=\frac{1+z}{1-z}$*. Then* $f\in N_{k,c}^{\alpha}(\lambda,\mu,\phi(z))$ *for* $|z|=r<-c+\sqrt{c^{2}+1}$*, where* $c=\left|\frac{\lambda}{\mu k}\right|$.

Proof Let

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}=p(z)\cos\alpha+i\sin\alpha,$$

where $p(z) \prec \frac{1+z}{1-z}$. Then from Theorem 2.1, we have

$$\begin{aligned} & p(z) + \frac{\lambda}{\mu k} z p'(z) \\ = & \frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} - \lambda \frac{S_{k}^{c} f(z)}{S_{k+1}^{c} f(z)}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu}\right\} - i\sin\alpha\right]. \end{aligned}$$

Since $p(z) \prec \frac{1+z}{1-z}$, then it is well known (see [5]) that:

$$\frac{1-r}{1+r} \leq \mathrm{Re}p(z) \leq |p(z)| \leq \frac{1+r}{1-r} \text{ and } |zp'(z)| \leq \frac{2r\mathrm{Re}p(z)}{1-r^2}. \tag{2.3}$$

Thus, we have

$$\begin{aligned} & \mathrm{Re}\frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} - \lambda \frac{S_{k}^{c} f(z)}{S_{k+1}^{c} f(z)}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu}\right\} - i\sin\alpha\right] \\ \geq & \mathrm{Re}p(z) - \left|\frac{\lambda}{\mu k}\right| |zp'(z)|. \end{aligned}$$

Using (2.3), we obtain

$$\begin{aligned} & \mathrm{Re}\frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} - \lambda \frac{S_{k}^{c} f(z)}{S_{k+1}^{c} f(z)}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu}\right\} - i\sin\alpha\right] \\ \geq & \mathrm{Re}p(z) - \frac{2cr\mathrm{Re}p(z)}{1-r^2} \\ = & \mathrm{Re}p(z)\frac{1-r^2-2cr}{1-r^2}. \end{aligned}$$

Since $p(z) \prec \frac{1+z}{1-z}$, therefore $\mathrm{Re}p(z) > 0$. This implies that $f \in N_{k,c}^{\alpha}(\lambda, \mu, \phi(z))$ for $r < -c + \sqrt{c^2+1}$. This result is sharp for the function $p(z) = \frac{1+z}{1-z}$. □

Theorem 2.4 *Let* $0 < \mu < 1$, $k = p + (b+2)/2 \neq 0, -1, -2, \ldots, b, c, p \in \mathbb{C}$. *Then*

$$N_{k,c}^{0}(\lambda_2, \mu, \phi) \subset N_{k,c}^{0}(\lambda_1, \mu, \phi), \ 0 \leq \lambda_1 < \lambda_2.$$

Proof Since $f \in N_{k,c}^{\alpha}(\lambda_2, \mu, \phi)$, therefore we have

$$h_1(z) = (1+\lambda_2)\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} - \lambda_2 \frac{S_{k}^{c} f(z)}{S_{k+1}^{c} f(z)}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} \prec \phi(z).$$

From Theorem 2.1 for $\alpha = 0$, we write

$$h_2(z) = \left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} \prec \phi(z), \ z \in E.$$

Now for $\lambda_1 \geq 0$, we obtain

$$\begin{aligned}
& (1+\lambda_1)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda_1 \frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \\
= & (1-\frac{\lambda_1}{\lambda_2})\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} + \\
& \frac{\lambda_1}{\lambda_2}\left\{(1+\lambda_2)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu}\right\} \\
= & \frac{\lambda_1}{\lambda_2} h_1(z) + (1-\frac{\lambda_1}{\lambda_2}) h_2(z).
\end{aligned}$$

Using the convexity of the class of the functions $\phi(z)$ and Lemma 1.2, we write

$$\frac{\lambda_1}{\lambda_2} h_1(z) + (1-\frac{\lambda_1}{\lambda_2}) h_2(z) \prec \phi(z), \;\; z \in E,$$

and this implies that $f \in N_{k,c}^0(\lambda_1, \mu, \phi)$. Hence, the proof of the theorem is complete. □

Corollary 2.5 *Let* $0 < \mu < 1,\; k = p + (b+2)/2 \neq 0, -1, -2, \ldots, b, c, p \in \mathbb{C}$. *Then for* $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$,

$$N_{k,c}^0\left(\lambda_2, \mu, \frac{1+A_2 z}{1+B_2 z}\right) \subset N_{k,c}^0\left(\lambda_1, \mu, \frac{1+A_1 z}{1+B1 z}\right), \;\; 0 \leq \lambda_1 < \lambda_2, \; z \in E.$$

Proof Let $f \in N_{k,c}^0\left(\lambda_2, \mu, \frac{1+A_2 z}{1+B_2 z}\right)$. Then

$$h_1(z) = (1+\lambda_2)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \prec \frac{1+A_2 z}{1+B_2 z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, therefore by Lemma 1.5, we have

$$h_1(z) = (1+\lambda_2)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \prec \frac{1+A_1 z}{1+B_1 z}.$$

Theorem 2.1 implies for $\phi(z) = \frac{1+A_1 z}{1+B_1 z}$ that

$$h_2(z) = \left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \prec \frac{1+A_1 z}{1+B_1 z}.$$

Now for $\lambda_2 > \lambda_1 \geq 0,$

$$\begin{aligned} & (1+\lambda_1)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda_1 \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \\ = & (1-\frac{\lambda_1}{\lambda_2})\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} + \\ & \frac{\lambda_1}{\lambda_2}\left\{(1+\lambda_2)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu}\right\} \\ = & \frac{\lambda_1}{\lambda_2} h_1(z) + (1-\frac{\lambda_1}{\lambda_2}) h_2(z). \end{aligned}$$

Using the convexity of the function $\frac{1+A_1 z}{1+B_1 z}$ with Lemma 1.2, we write

$$\frac{\lambda_1}{\lambda_2} h_1(z) + (1-\frac{\lambda_1}{\lambda_2}) h_2(z) \prec \frac{1+A_1 z}{1+B_1 z}, \quad z \in E,$$

and this implies that $f \in N_{k,c}^0\left(\lambda_1, \mu, \frac{1+A_1 z}{1+B_1 z}\right)$. □

Theorem 2.6 *Let* $f \in N_{k,c}^0(\lambda, \mu, \phi)$, $0 < \mu < 1$, $k = p + (b+2)/2 \neq 0, -1, -2, \ldots, b, c, p \in \mathbb{C}$ *and* $\lambda \leq -1$. *Then*

$$\frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \prec \phi(z).$$

Proof Since $f \in N_{k,c}^0(\lambda, \mu, \phi)$, therefore we have

$$(1+\lambda)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \prec \phi(z).$$

Now consider

$$\begin{aligned} \lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} = & (1+\lambda)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} + \lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \\ & -(1+\lambda)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu}. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} = & \left(1+\frac{1}{\lambda}\right)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \\ & -\frac{1}{\lambda}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} + \lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu}\right\}. \end{aligned}$$

Using Theorem 2.1, Lemma 1.2, and the convexity of $\phi(z)$ with $\lambda \leq -1$, we have the required result. □

Theorem 2.7 *Let* $f \in N^{\alpha}_{k,c}(\lambda, \mu, h), h(z) = \frac{1+Az}{1+Bz} + \frac{\lambda\mu}{k}\frac{(A-B)z}{(1+Bz)^2}$. *Then for* $Re\frac{\lambda}{\mu k} > 0$,

$$e^{i\alpha}\left(\frac{z}{S^c_{k+1}f(z)}\right)^{\mu} \prec (\cos\alpha)\,\phi(z) + i\sin\alpha,$$

where $\phi(z) = \frac{1+Az}{1+Bz}$. *This result is the best possible.*

Proof Consider

$$p(z) = \frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S^c_{k+1}f(z)}\right)^{\mu} - i\sin\alpha\right\}.$$

Then p is analytic in E with $p(0) = 1$. Therefore, we have

$$e^{i\alpha}\left(\frac{z}{S^c_{k+1}f(z)}\right)^{\mu} = (\cos\alpha)\,p(z) + i\sin\alpha.$$

Differentiating both sides, using (1.9), and simplifying, we obtain

$$\frac{\lambda(\cos\alpha)\,zp'(z)}{\mu k} = \lambda e^{i\alpha}\left\{\left(\frac{z}{S^c_{k+1}f(z)}\right)^{\mu} - \frac{S^c_k f(z)}{S^c_{k+1}f(z)}\left(\frac{z}{S^c_{k+1}f(z)}\right)^{\mu}\right\}.$$

It follows from the above equation and (2.1) that

$$\begin{aligned} & p(z) + \frac{\lambda}{\mu k}zp'(z) \\ = \; & \frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S^c_{k+1}f(z)}\right)^{\mu} - \lambda\frac{S^c_k f(z)}{S^c_{k+1}f(z)}\left(\frac{z}{S^c_{k+1}f(z)}\right)^{\mu}\right\} - i\sin\alpha\right]. \end{aligned}$$

Since $f \in N^{\alpha}_{k,c}(\lambda, \mu, h)$, therefore

$$p(z) + \frac{\lambda}{\mu k}zp'(z) \prec h(z).$$

Now we choose $g(z) = \frac{1+Az}{1+Bz}$, and then $\theta(w) = w$ and $\varphi(w) = \frac{\mu k}{\lambda}$. It is clear that $g(z)$ is analytic in E with $g(0) = 1$. Also, $\theta(w)$ and $\varphi(w)$ are analytic with $\theta(w) \neq 0$.

We see that

$$Q(z) = zg'(z)\varphi(g(z)) = \frac{\mu k}{\lambda}\frac{(A-B)z}{(1+Bz)^2}. \tag{2.4}$$

We have to prove that $Q(z)$ is starlike. In other words, we show that $\mathrm{Re}\frac{zQ'(z)}{Q(z)} > 0$. From (2.4), we have

$$\begin{aligned} \mathrm{Re}\frac{zQ'(z)}{Q(z)} &= \mathrm{Re}\left\{1 - \frac{2Bz}{1+Bz}\right\} \\ &= 1 - 2B\mathrm{Re}\frac{re^{i\psi}}{1+Bre^{i\psi}} \qquad (z = re^{i\psi}) \\ &= \frac{1-B^2r^2}{(1+Br\cos\psi)^2 + B^2r^2\sin^2\psi}. \end{aligned}$$

Since $-1 \leq B < 1,\ r < 1$. This implies that $\mathrm{Re}\frac{zQ'(z)}{Q(z)} > 0$. Consider

$$\begin{aligned}\mathrm{Re}\frac{zh'(z)}{Q(z)} &= \mathrm{Re}\left\{\frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)}\right\} \\ &= \mathrm{Re}\frac{\lambda}{\mu k} + \mathrm{Re}\frac{zQ'(z)}{Q(z)} > 0.\end{aligned}$$

Using Lemma 1.6, we have $e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \prec (\cos\alpha)\,\phi(z) + i\sin\alpha$. The function $\phi(z) = \frac{1+Az}{1+Bz}$ is the best possible. □

Theorem 2.8 *Let $f \in N_{k,c}^{\alpha}\left(\lambda, \mu, \frac{1+Az}{1+Bz}\right)$. Then for $k, \lambda \in \mathbb{R}$ and $\frac{\mu k}{\lambda} \geq 0$,*

$$\begin{aligned}&\left.\begin{matrix}\frac{A}{B} + \left(1 - \frac{A}{B}\right)\ {}_2F_1\left(1, \frac{\mu k}{\lambda}, \frac{\mu k}{\lambda} + 1; B\right), & B \neq 0, \\ 1 - \frac{\mu k}{\mu k+\lambda}A, & B = 0.\end{matrix}\right\} \\ &< \frac{1}{\cos\alpha} Re\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} \\ &< \left\{\begin{matrix}\frac{A}{B} + \left(1 - \frac{A}{B}\right)\ {}_2F_1\left(1, \frac{\mu k}{\lambda}, \frac{\mu k}{\lambda} + 1; -B\right), & B \neq 0, \\ 1 + \frac{\mu k}{\mu k+\lambda}A, & B = 0.\end{matrix}\right.\end{aligned}$$

Proof Since $f \in N_{k,c}^{\alpha}\left(\lambda, \mu, \frac{1+Az}{1+Bz}\right)$, therefore, by using (2.2), we have

$$\frac{1}{\cos\alpha}\mathrm{Re}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} \prec \mathrm{Re}\frac{\mu k}{\lambda}\int_0^1 \frac{1+Atz}{1+Btz}t^{\frac{\mu k}{\lambda}-1}dt.$$

It follows from the definition of subordination that

$$\begin{aligned}\frac{1}{\cos\alpha}\mathrm{Re}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} &< \sup_{|z|<1}\mathrm{Re}\left\{\frac{\mu k}{\lambda}\int_0^1 \frac{1+Atz}{1+Btz}t^{\frac{\mu k}{\lambda}-1}dt\right\} \\ &\leq \left\{\frac{\mu k}{\lambda}\int_0^1 \sup_{|z|<1}\mathrm{Re}\left\{\frac{1+Atz}{1+Btz}\right\}t^{\frac{\mu k}{\lambda}-1}dt\right\} \\ &< \frac{\mu k}{\lambda}\int_0^1 \frac{1+At}{1+Bt}t^{\frac{\mu k}{\lambda}-1}dt \\ &= \frac{\mu k}{\lambda}\int_0^1 \left\{A/B + \left(\frac{1-A/B}{1+Bt}\right)\right\}t^{\frac{\mu k}{\lambda}-1}dt.\end{aligned}$$

Now using Lemma 1.4 for the case $B \neq 0$, we have

$$\frac{1}{\cos\alpha}\mathrm{Re}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} < \frac{A}{B} + \left(1 - \frac{A}{B}\right)\ {}_2F_1\left(1, \frac{\mu k}{\lambda}, \frac{\mu k}{\lambda} + 1; -B\right).$$

When $B=0$, it can be easily seen that

$$\begin{aligned}\frac{1}{\cos\alpha}\mathrm{Re}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} &< \frac{\mu k}{\lambda}\int_{0}^{1}(1+At)\,t^{\frac{\mu k}{\lambda}-1}dt \\ &= 1+\frac{\mu k}{\mu k+\lambda}A.\end{aligned}$$

We also have

$$\begin{aligned}\frac{1}{\cos\alpha}\mathrm{Re}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} &> \inf_{|z|<1}\mathrm{Re}\left\{\frac{\mu k}{\lambda}\int_{0}^{1}\frac{1+Atz}{1+Btz}t^{\frac{\mu k}{\lambda}-1}dt\right\} \\ &\geq \left\{\frac{\mu k}{\lambda}\int_{0}^{1}\inf_{|z|<1}\mathrm{Re}\left\{\frac{1+Atz}{1+Btz}\right\}t^{\frac{\mu k}{\lambda}-1}dt\right\} \\ &> \frac{\mu k}{\lambda}\int_{0}^{1}\frac{1-At}{1-Bt}t^{\frac{\mu k}{\lambda}-1}dt \\ &= \frac{\mu k}{\lambda}\int_{0}^{1}\left\{A/B+\left(\frac{1-A/B}{1-Bt}\right)\right\}t^{\frac{\mu k}{\lambda}-1}dt.\end{aligned}$$

Using again Lemma 1.4, we have the required result. □

Theorem 2.9 *Let* $f\in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+Az}{1+Bz}\right)$. *Then for* $k,\lambda\in\mathbb{R}$ *and* $\frac{\mu k}{\lambda}\geq 0$,

$$\begin{aligned}&\left.\begin{array}{ll}\frac{A}{B}+\left(1-\frac{A}{B}\right)\ {}_2F_1\left(1,\frac{\mu k}{\lambda},\frac{\mu k}{\lambda}+1;Br\right), & B\neq 0,\\ 1-\frac{\mu k}{\mu k+\lambda}A, & B=0.\end{array}\right\} \\ &\leq \left|\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\}-i\sin\alpha\right| \\ &\leq \left\{\begin{array}{ll}\frac{A}{B}+\left(1-\frac{A}{B}\right)\ {}_2F_1\left(1,\frac{\mu k}{\lambda},\frac{\mu k}{\lambda}+1;-Br\right), & B\neq 0,\\ 1+\frac{\mu k}{\mu k+\lambda}A, & B=0.\end{array}\right.\end{aligned}$$

Proof Since $f\in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+Az}{1+Bz}\right)$, therefore, by using (2.2), we have

$$\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}-i\sin\alpha\right\}\prec\frac{\mu k}{\lambda}\int_{0}^{1}\frac{1+Atz}{1+Btz}t^{\frac{\mu k}{\lambda}-1}dt.$$

It follows from the definition of subordination that

$$\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}-i\sin\alpha\right\}=\left\{\frac{\mu k}{\lambda}\int_{0}^{1}\frac{1+Atw(z)}{1+Btw(z)}t^{\frac{\mu k}{\lambda}-1}dt\right\},$$

where $w(z) = c_1 z + c_2 z^2 + \ldots$ is analytic and $|w(z)| \leq |z|$. Therefore,

$$\left|\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} - i\sin\alpha\right\}\right| \leq \left\{\frac{\mu k}{\lambda}\int_0^1 \frac{1+Atr}{1+Btr} t^{\frac{\mu k}{\lambda}-1} dt\right\}.$$

Now using the same process as in the theorem above, we get the required result. □

References

[1] Abramowitz M, Stegun IA. Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. New York, NY, USA: Dover Publications, 1971.

[2] Baricz A, Deniz E, Çağlar M, Orhan H. Differential subordinations involving generalized Bessel functions. Bull Malays Math Sci Soc 2015; 38: 1255–1280.

[3] Dziok J, Srivastava HM. Certain subclasses of analytic functions associated with the generalized hypergeometric function. Integral Transforms Spec Funct 2003; 14: 7–18.

[4] Dziok J, Srivastava HM. Classes of analytic functions associated with the generalized hypergeometric function. Appl Math Comput 1999; 103: 1–13.

[5] Goodman AW. Univalent Functions, Washington, NJ, USA: Polygonal Publishing House, 1983.

[6] Hallenbeck DJ, Ruscheweyh S. Subordination by convex functions. P Am Math Soc 1975; 52: 191–195.

[7] Liu MS. On a subclass of p-valent close-to-convex functions of order β and type α. J Math Study 1997; 30: 102–104.

[8] Liu MS. On certain subclass of analytic functions. J South China Normal Univ 2002; 4: 15–20.

[9] Miller SS, Mocanu PT. Differential Subordinations: Theory and Applications. Series in Pure and Applied Mathematics, No. 225. NewYork, NY, USA: Marcel Dekker, 2000.

[10] Orhan H, Yağmur N. Geometric properties of generalized Struve functions. Scientific Annals of "Al I Cuza" University of Iasi (in press).

[11] Raina RK, Sharma P. Harmonic univalent functions associated with Wright's generalized hypergeometric functions. Integral Transforms Spec Funct 2011; 22: 561–572.

[12] Shareef Z, Hussain S, Darus M. Convolution operators in the geometric function theory. J Inequal Appl 2012; 2012: 213.

[13] Srivastava HM, Attiya AA. An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination. Integral Transforms Spec Funct 2007; 18: 207–216.

[14] Yağmur N, Orhan H. Hardy space of generalized Struve functions. Complex Var Elliptic Equ 2014; 59: 929–936.

[15] Yağmur N, Orhan H. Starlikeness and convexity of generalized Struve functions. Abstr Appl Anal 2013; 2013: 954513.

[16] Zhang S, Jin J. Computation of Special Functions. New York, NY, USA: Wiley Interscience Publication, 1996.