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Jacobi-spectral method for integro-delay differential equations with weakly singular kernels

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Abstract: We present a numerical solution to the integro-delay differential equation with weakly singular kernels with the delay function $\theta(t)$ vanishing at the initial point of the given interval $[0,T]$ $(\theta(t)=qt, 0<q<1)$. In order to fully use the Jacobi orthogonal polynomial theory, we use some function and variable transformation to change the intergro-delay differential equation into a new equation defined on the standard interval $[-1,1]$. A Gauss–Jacobi quadrature formula is used to evaluate the integral term. The spectral rate of convergence is provided in infinity norm under the assumption that the solution of the given equation is sufficiently smooth. For validation of the theoretical exponential rate of convergence of our method, we provide some numerical examples.

Key words: Jacobi spectral methods, integro-delay differential, weakly singular kernels equations, Gauss–Jacobi quadrature formula, numerical examples

1. Introduction

Consider the integro-delay differential equation of the form

$$\begin{cases} y'(t) &= a(t)y(t)+b(t)y(qt)+\int_0^t K_0(t-s)y(s)ds+\int_0^{qt} K_1(t-s)y(s)ds+g(t), \quad t\in I:=[0,\ T\,] \\ y(0) &= y_0, \end{cases} \tag{1.1}$$

where $0<q<1$, $a(t)$ and $b(t)$ are smooth functions on $I:=[0,T]$, and $K_0(t-s)$, $K_1(t-s)$ are singular kernels of the form $(t-s)^{-\mu}$, $\mu\in(0,1)$. The special case corresponding to $K_0(t,s)\equiv 0$, $K_1(t,s)\equiv 0$, $g(t)\equiv zero$ yields the (variable coefficient) *pantograph equation*. Results on the existence, uniqueness, and regularity of solutions may be found in [4, 7]. For integro-delay differential equations with smooth kernels, it has been shown in [6] that the approximation of the solution of (1.1) by collocation using piecewise polynomials of degree $m\geq 1$ and uniform meshes does not lead to the classical $\mathcal{O}(h^{2m})$- superconvergence at the mesh points when collocation is at the Gauss points; for $m\geq 2$ the optimal order is only $m+2$. The numerical solution of (1.1) with smooth kernels was investigated by spectral methods, which leads to a higher (exponential) convergence order [3], where it was shown that the results on the exponential order of convergence of the spectral method for the pantograph DDE [1, 2] remain valid for pantograph-type integro-differential equation (1.1) with smooth kernels.

Several numerical methods have been proposed for (1.1) without a delay (see, e.g., [6-16, 24-28]), where they show that its numerical treatment is not simple, due to the fact that the solution of (1.1) usually has a

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weak singularity at $t=0$; even the inhomogeneous term $g(t)$ is regular. A Jacobi-collocation spectral method is developed for (1.1) in [16, 17] without a delay. To deal with the singular kernels we use the same technique presented by Chen and Tang in [16, 17].

The paper is organized as follows. In §2 we discuss the spectral approximation of (1.1) that yields spectral accuracy. This is followed in §3 by corresponding results on the attainable order of convergence of these spectral methods. §4 is used to illustrate the convergence results by numerical examples, followed by the concluding remarks in §5.

2. Jacobi spectral method

Let $\omega^{\alpha,\beta}(x)=(1-x)(1+x)$ be a weight function in the usual sense for $\alpha,\beta>-1$. It is well known that the set of Jacobi polynomials $\{J_n^{\alpha,\beta}n(x)\}_{n=0}^{\infty}$ forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$-orthogonal system, where $L^2_{\omega^{\alpha,\beta}}(-1,1)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1)=\{v : v \text{ is measurable and } \|v\|^2_{\omega}(-1,1)=\left(\int_{-1}^{1}|v(x)|^2\omega^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}}<\infty\}$$

Let $\{t_k\}_{k=0}^{N}$ be the set of $(N+1)$ Gauss–Jacobi, or Gauss–Jacobi–Radau, or Gauss–Jacobi–Lobatto points. Integrating (1.1) from $[0,t_i]$ gives

$$\begin{aligned} y(t_i) &= y_0+\int_0^{t_i}a(s)y(s)ds+\int_0^{t_i}b(s)y(qs)ds+\int_0^{t_i}\left[\int_0^{s}K_0(t-v)y(v)dv\right]ds \\ &\quad+\int_0^{t_i}\left[\int_0^{qs}K_1(t-v)y(v)dv\right]ds+\int_0^{t_i}g(s)ds, \end{aligned} \tag{2.1}$$

exchanging the order of integration we get

$$\begin{aligned} y(t_i) &= y_0+\int_0^{t_i}a(s)y(s)ds+\int_0^{t_i}b(s)y(qs)ds+\int_0^{t_i}H_0(t-v)y(v)dv \\ &\quad+\int_0^{qt}H_1(t,v)y(v)dv+G(t_i), \end{aligned} \tag{2.2}$$

where

$$H_0(t-v)=\int_v^t K_0(s-v)ds, H_1(t,v)=\int_{\frac{v}{q}}^t K_1(s-v)ds, G(t_i)=\int_0^{t_i}g(s)ds.$$

Since K_0 and K_1 are singular kernels like $(s-v)^{-\alpha_0}$ and $(s-v)^{-\alpha_1}$. we can assume that H_0 and H_1 are also singular kernels of the form

$$H_0(t-v)=(t-v)^{-\mu_0}v^{-\upsilon_0}h_0(t-v), H_1(t,v)=(t-v)^{-\mu_1}v^{-\upsilon_1}h_1(t,v).$$

After using the linear transformation $s=\frac{t_i}{2}\theta+\frac{t_i}{2}$ and $(N+1)$-point Jacobi–Gauss, or Jacobi–Radau, or Jacobi–Lobatto quadrature formula relative to the Jacobi weight we get

$$\begin{aligned} y(t_i) &\approx y_0+\frac{t_i}{2}\sum_{k=0}^{N}a(\tau_{ik})y(\tau_{ik})\omega_k+\frac{t_i}{2}\sum_{k=0}^{N}b(\tau_{ik})y(q\tau_{ik})\omega_k+\left(\frac{t_i}{2}\right)^{1-\mu_0-\nu_0}\sum_{k=0}^{N}h_0(t_i-\hat{\tau}_{ik})y(\hat{\tau}_{ik})\hat{\omega}_k \\ &\quad+\left(\frac{qt_i}{2}\right)^{1-\nu_1}\sum_{k=0}^{N}(t_i-q\tilde{\tau}_{ik})^{-\mu_1}h_1(t_i-q\tilde{\tau}_{ik})y(q\tilde{\tau}_{ik})\tilde{\omega}_k+G(t_i), \end{aligned} \tag{2.3}$$

where

$\tau_{ik} = \frac{t_i}{2}(\theta_k + 1)$ θ_k, ω_k relative to the weight function $\omega^{0,0}(x)$

$\hat{\tau}_{ik} = \frac{t_i}{2}(\hat{\theta}_k + 1)$ $\hat{\theta}_k$, $\hat{\omega}_k$ relative to the weight function $\omega^{-\mu_0,-\nu_0}(x)$ $\tilde{\tau}_{ik} = \frac{t_i}{2}(\tilde{\theta}_k + 1)$ $\tilde{\theta}_k$, $\tilde{\omega}_k$ relative to the weight function $\omega^{0,-\nu_1}(x)$

Let $Y_j \approx y(t_j)$ and

$$Y(t) = \sum_{j=0}^{N} Y_j F_j(t), \tag{2.4}$$

where $F_j(t)$ is the standard Lagrange interpolation polynomial associated with $\{t_k\}_{k=0}^{N}$. It follows from (2.3) that the numerical scheme for solving (1.1) is given by

$$\begin{aligned} Y_i \quad = \quad & y_0 + \frac{t_i}{2}\sum_{j=0}^{N} Y_j \left[\sum_{k=0}^{N} F_j(\tau_{ik})a(\tau_{ik})\omega_k\right] + \frac{t_i}{2}\sum_{j=0}^{N} Y_j \left[\sum_{k=0}^{N} F_j(q\tau_{ik})b(\tau_{ik})\omega_k\right] \\ & + \left(\frac{t_i}{2}\right)^{1-\mu_0-\nu_0} \sum_{j=0}^{N} Y_j \left[\sum_{k=0}^{N} h_0(t_i - \hat{\tau}_{ik})\Big)F_j\Big(\hat{\tau}_{ik}\Big)\hat{\omega}_k\right] \\ & + \left(\frac{qt_i}{2}\right)^{1-\nu_1} \sum_{j=0}^{N} Y_j \left[\sum_{k=0}^{N} (t_i - q\tilde{\tau}_{ik})^{-\mu_1} h_1(t_i - q\tilde{\tau}_{ik}) y(q\tilde{\tau}_{ik}) F_j(q\tilde{\tau}_{ik})\tilde{\omega}_k\right] + G(t_i). \end{aligned} \tag{2.5}$$

Denoting $Y = [Y_0, \cdots, Y_N]^T$ and $F_N = [y_0, \cdots, y_0]^T + [G(t_0), \cdots, G(t_N)]^T$, we can obtain a matrix form:

$$Y - (A_1 + A_2 + A_3 + A_4)Y = F_N \tag{2.6}$$

where the entries of the matrix A,s are given by:

$$A_1(i,j) = \frac{t_i}{2}\sum_{k=0}^{N} F_j(\tau_{ik})a(\tau_{ik})\omega_k,$$

$$A_2(i,j) = \frac{t_i}{2}\sum_{k=0}^{N} F_j(q\tau_{ik})b(\tau_{ik})\omega_k,$$

$$A_3(i,j) = \left(\frac{t_i}{2}\right)^{1-\mu_0-\nu_0} \sum_{k=0}^{N} h_0(t_i - \hat{\tau}_{ik})F_j(\hat{\tau}_{ik})\hat{\omega}_k,$$

$$A_4(i,j) = \left(\frac{qt_i}{2}\right)^{1-\nu_1} \sum_{k=0}^{N} (t_i - q\tilde{\tau}_{ik})^{-\mu_1} h_1(t_i - q\tilde{\tau}_{ik})F_j(q\tilde{\tau}_{ik})\tilde{\omega}_k.$$

The efficient way to compute $F_j(s)$ is to express it in terms of the Legendre functions [11, 24], that is:

$$F_j(s) = \sum_{p=0}^{N} f_{p,j} L_p(s),$$

where $f_{p,j}$ is called the discrete polynomial coefficients of F_j. The inverse relation is

$$f_{p,j} = \frac{1}{\gamma_p} \sum_{i=0}^{N} F_j(t_i) L_p(t_i) \omega_i = L_p(t_j)/\gamma_p,$$

where

$$\gamma_p = (p + 1/2)^{-1}, \qquad p < N$$

and $\gamma_N = (N + 1/2)^{-1}$ for the Gauss and Gauss–Radau formulas, and $\gamma_N = 2/N$ for the Gauss–Lobatto formula. Therefore, we can get the last formula of $F_j(s)$

$$F_j(s) = \sum_{p=0}^{N} L_p(t_j) L_p(s)/\gamma_p,$$

which together with the known recurrence formulas for $L_p(s)$ can be used to evaluate $F_j(s)$ in an efficient way.

3. Convergence analysis

To carry out the convergence analysis of our method, we first introduce some Hilbert spaces. For $m \geq 1$, define

$$H^m_{\omega^{\alpha,\beta}}(-1,1) := \{\upsilon : \partial_x^k \in L^2_{\omega^{\alpha,\beta}}(-1,1), 0 \leq k \leq m\}, \tag{3.1}$$

with the norm and seminorm as

$$\|\upsilon\|_{m,\omega^{\alpha,\beta}} = \left(\sum_{k=0}^{m} |\upsilon|^2_{k,\omega^{\alpha,\beta}} \right)^{1/2}, \tag{3.2}$$

$$|\upsilon|_{m,\omega^{\alpha,\beta}} = \|\partial_x^m \upsilon\|_{m,\omega^{\alpha,\beta}}. \tag{3.3}$$

Let $\omega(x) = \omega^{-\frac{1}{2},-\frac{1}{2}(x)}$ denote the Chebyshev weight function. For the error bound of Chebyshev polynomials, only some of the L^2- norms appearing on the right-hand side of the above norm enter play, and so for our convenience we introduce the seminorm

$$|\upsilon|_{H^{m;N}_{\omega(-1,1)}} = \left(\sum_{k=\min(m,N+1)}^{m} \|\partial_x^k \upsilon\|^2_{L^2_{\omega(-1,1)}} \right)^{1/2}, \tag{3.4}$$

To bound the approximation error for Jacobi polynomial, one needs the following nonuniformly weighted Sobolev spaces:

$$H^m_{\omega^{\alpha,\beta,\star}}(-1,1) := \{\upsilon : \partial_x^k \in L^2_{\omega^{\alpha+k,\beta+k}}(-1,1), 0 \leq k \leq m\}, \tag{3.5}$$

with the inner norm

$$(u,v)_{m,\omega^{\alpha,\beta,\star}} = \sum_{k=0}^{N} (\partial_x^k u, \partial_x^k v)_{\omega^{\alpha+k,\beta+k}}, \tag{3.6}$$

and

$$\|v\|_{m,\omega^{\alpha,\beta,\star}} = \sqrt{(u,v)_{m\omega^{\alpha,\beta,\star}}}. \tag{3.7}$$

We introduce the orthogonal projection $\pi_{N,\omega^{\alpha,\beta}} : L^2_{\omega^{\alpha,\beta}}(-1,1) \to \mathcal{P}_N$, which is the mapping such that any $v \in L^2_{\omega^{\alpha,\beta}}(-1,1)$

$$(v - \pi_{N,\omega^{\alpha,\beta}} v, \phi)_{\omega^{\alpha,\beta}} = 0, \forall \phi \in \mathcal{P}_N. \tag{3.8}$$

From [24] and [21], we have the following result.

For any function $v \in H^m_{\omega^{\alpha,\beta}}(-1,1)$ and for any $m \geq 1$ and $\alpha, \beta, > -1$, we have

$$\|\partial_x^k (v - \pi_{N,\omega^{\alpha,\beta}} v)\|_{\omega^{\alpha+k,\beta+k}} \leq C N^{k-m} \|\partial_x^k v\|_{\omega^{\alpha+m,\beta+m}}, 0 \leq k \leq m, \tag{3.9}$$

and

$$\|v - \pi_{N,\omega^{\alpha,\beta}} v\|_{\omega^{\alpha,\beta}} \leq C N^{-1} |v|_{1,\omega^{\alpha+1,\beta+1}}. \tag{3.10}$$

From [24] and [20] we have the following lemma.

Lemma 3.1 ***(Estimates for interpolation error)*** *Assume that $\partial_x v \in H^m(I)_{\omega^{\alpha,\beta,\star}}(-1,1)$ and denote by $I_N u$ the interpolation polynomial associated with the $(N+1)$ Jacobi–Gauss points, Jacobi–Gauss–Radau points, or Jacobi–Gauss–Lobatto points $\{t_k\}_{k=0}^N$. Then*

$$\|\partial_x^k (v - I_N^{\alpha,\beta} v)\|_{\omega^{\alpha+k,\beta+k}} \leq C N^{k-m} \|\partial_x^m v\|_{\omega^{\alpha+m,\beta+m}}, \tag{3.11}$$

$$\|v - I_N^{\alpha,\beta} v\|_{1,\omega^{\alpha,\beta}} \leq C(N(N+\alpha+\beta))^{1-m/2} \|\partial_x^m v\|_{\omega^{\alpha+m,\beta+m}}. \tag{3.12}$$

Lemma 3.2 ***(Integration error for Gauss quadrature)*** *Assume that a $(N+1)$-point Gauss–Jacobi, or Gauss–Radau, or Gauss–Lobatto quadrature formula relative to the Jacobi weights is used to integrate the product $\partial_x v \phi$, where $\partial_x v \in H^m(I)_{\omega^{\alpha,\beta,\star}}$ with $I := (-1,1)$ and some $m \geq 1$, and $\phi \in \mathcal{P}_N$. Then there exists a constant C not depending on N such that*

$$|(v,\phi)_{\omega^{\alpha,\beta}} - (v,\phi)_N| \leq \|v - I_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}} \|\phi\|_{\alpha,\beta} \leq C N^{-m} \|\partial_x^m v\|_{\omega^{\alpha+m,\beta+m}} \|\phi\|_{\alpha,\beta}. \tag{3.12}$$

If we use the Jacobi–Gauss–Lobatto integration, then we have

$$|(v,\phi)_{\omega^{\alpha,\beta}} - (v,\phi)_N| \leq C(\|v - \pi_{N-1,\omega^{\alpha,\beta}} v + \|v - I_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}}) \|\phi\|_{\alpha,\beta} \tag{3.13}$$

$$\leq C N^{-m} \|\partial_x^m v\|_{\omega^{\alpha+m,\beta+m}} \|\phi\|_{\alpha,\beta}, \tag{3.14}$$

where the discrete inner product is given by

$$(u,\phi)_N = \sum_{k=0}^{N} \omega_k u(x_k) \phi(x_k). \tag{3.15}$$

From [22], we have the following result on the Lebesgue constant for Lagrange interpolation based on the zeros of the Jacobi polynomials.

Lemma 3.3 ***(Lebesgue constant for the Legendre series)*** *Assume that* $\{F_j(x)\}_{j=0}^{N}$ *are the Lagrange interpolation polynomials with respect to the Gauss–Jacobi, Gauss–Radau, or Gauss–Lobatto–Jacobi points* $\{x_j\}$. *Then*

$$||I_N^{-\mu,0}||_\infty := \max_{x\in(-1,1)} \sum_{j=0}^{N} |F_j(x)| = \mathcal{O}(\sqrt{N}),$$

where we use $\alpha = -\mu$ *and* $\beta = 0$.

Lemma 3.4 (Gronwall inequality) *Let* $T > 0$ *and* $C_1, C_2, C_3, C_4 \geq 0$. *If a nonnegative continuous function* $E(t)$ *satisfies*

$$\begin{aligned} E(t) &\leq C_1 \int_0^t E(s)ds + C_2 \int_0^t E(qs)ds + C_3 \int_0^t \int_v^t K_0(s-v)E(v)dvds \\ &+ C_4 \int_0^t \int_{\frac{v}{q}}^t K_1(s-v)E(v)dvds + H(t), \quad \forall t \in [0,T], \end{aligned} \tag{3.14}$$

where $0 < q < 1$ *is a constant and* $H(t)$ *is a continuous function, then*

$$\|E\|_{L^\infty(I)} \leq C\|H\|_{L^\infty(I)}. \tag{3.15}$$

By using the above lemmas and by following the same procedure of [17] one can easily state and prove the following main results, as our delay term $q \in (0,1)$, all the results of [17] are valid for the integro-delay differential equation with singular kernels. For this reason we leave the proof of the below theorem to the reader.

Theorem 3.1 *Consider the delay integro-differential equation (1.1) and its spectral approximation method (2.5). Let us denote the exact solution by* y *and the approximated solution by* Y. *If* μ *associated with the weakly singular kernel satisfies* $0 < \mu < \frac{1}{2}$ *and* $y \in H_\omega^m(-1,1) \cap H^m_{\omega^{-\mu,0,\star}}(-1,1)$, *then*

$$\begin{aligned} \|Y - y\|_{L^\infty} &\leq CN^{1/2-m}(D_1 + D_2 + B_0 + B_1)\|y\|_{L^\infty} + CN^{1-m} \\ &\times \Big(|y|_{H_\omega^{m;N}(-1,1)} + |K_0 y|_{H_\omega^{m;N}(-1,1)} + |K_1 y|_{H_\omega^{m;N}(-1,1)}\Big), \end{aligned} \tag{3.15}$$

for N *sufficiently large.* Y *is the polynomial of degree* N *associated with the spectral approximation (2.5), and* C, D_1, D_2, B_0 B_1 *are constants independent of* N.

4. Numerical examples

In the following, we use two examples to illustrate the accuracy and efficiency of the spectral methods (2.5). For our numerical computation we use the Gauss–Jacobi quadrature rule nodes and weights (see the detailed algorithm in [5]). The Gauss–Jacobi quadrature formula is used to numerically calculate the integral

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)dx, f(x) \in [-1,1], \alpha, \beta > -1,$$

by using the formula

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)dx = \sum_{i=0}^{N} \omega_i f(x_i).$$

With the help of a change in the variables (which changes both weights w_i and nodes x_i), we can get onto the arbitrary interval $[a, b]$.

Example 4.1 *Let* $a(t) = t, b(t) = -(\frac{1}{q^5})$ *and* $K_0(s-v) = K_1(s-v) = (s-v)^{-0.5}$ *with*

$$g(t) = -\frac{5t^2}{q^6} - \frac{2t^2(t-tq)^{1/2}}{693q^4}(256 + 128q + 96q^2 + 80q^3 + 70q^4 - \frac{256t^{1/2}}{(t-tq)^{-1/2}}).$$

The exact solution of the problem is

$$y(t) = \left(\frac{t}{q}\right)^5.$$

The point-wise error between different norms of numerical solution and exact solution for $q = .2$ and $q = .99$ and for different value of N is shown in Table 1 and Figure 1, respectively, while Figure 2 shows the comparison between exact and approximate solutions.

Table 1. The point-wise error for $q = .2$ using (2.5).

N	L_∞ error $(q = .2)$	N	L_1 error$(q = .2)$	N	L_2 error$(q = .2)$
2	1.617e+003	2	4.794e+002	2	8.536e+002
4	1.776e+003	4	3.430e+002	4	6.625e+002
6	2.254e-008	6	3.006e-009	6	2.955e-008
8	1.017e-007	8	1.421e-008	8	4.676e-009
10	1.620e-008	10	2.298e-009	10	1.554e-009
12	1.604e-007	12	2.764e-008	12	5.053e-008

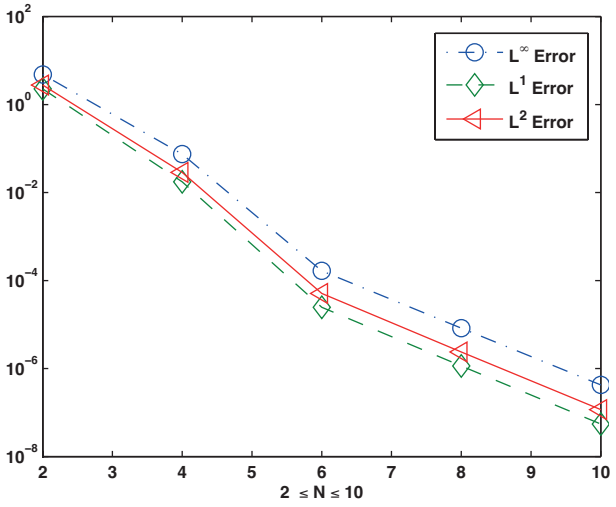

Figure 1. L^∞, L^2, and L^1 error for $q = .99$.

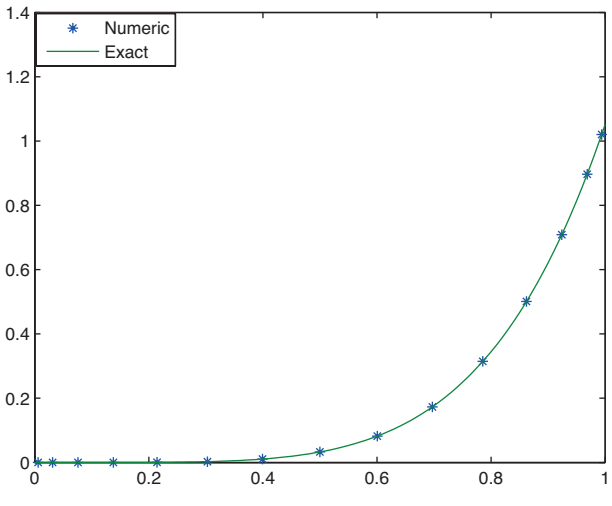

Figure 2. Exact vs. approximate solution.

Example 4.2 *Consider the following nonlinear delay-integro differential equation*

$$y'(t) = \int_0^t (t-s)^{-1/3} y^3(s) ds + \int_0^{qt} (t-s)^{-1/3} y^3(s) ds + g(t)$$

with

$$g(t) = \frac{1}{3}(t+q)^{-2/3} - \frac{9}{10}t^{2/3} + \sqrt{(t-qt)}(\frac{4t}{3} + \frac{2qt}{3} - 2) - \frac{4t^{2/3}}{3} - 2q^{1/2}.$$

The exact solution of the problem is

$$y(t) = (t+q)^{1/3}.$$

The point-wise error between different norms of numerical solution and exact solution for $q = .05$ and $q = .99$ for different value of N is shown in Table 2 and Figure 3, respectively. Figure 4 illustrates the exact solution vs. numerical solution.

Table 2. The point-wise error for $q = .05$.

N	L_∞ error $(q = .05)$	N	L_1 error$(q = .05)$	N	L_2 error$(q = .05)$
4	9.168e-003	4	1.795e-003	4	3.380e-003
6	1.424e-004	6	2.373e-005	6	4.485e-005
8	4.256e-006	8	6.656e-007	8	1.252e-006
10	9.175e-008	10	2.040e-008	10	3.446e-008
12	7.520e-009	12	1.191e-009	12	2.218e-009
14	4.331e-010	14	7.874e-011	14	1.414e-010

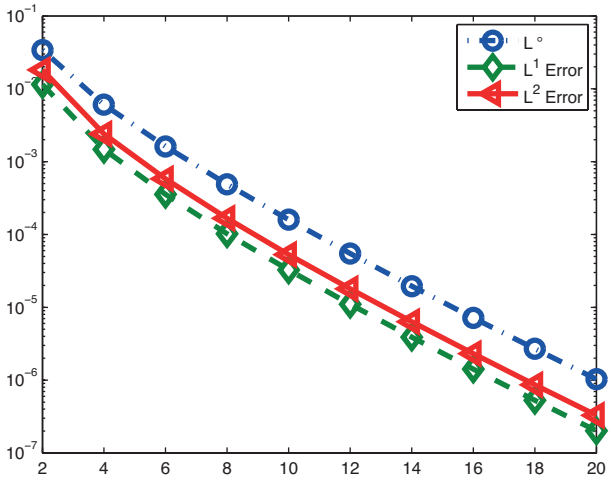

Figure 3. L^∞, L^2, and L^1 for $q = .99$.

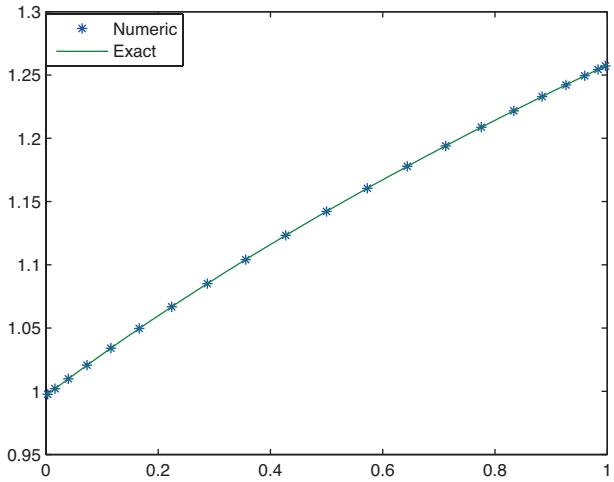

Figure 4. Exact vs. approximate solution.

5. Conclusion

A Jacobi spectral-collocation method was introduced for the numerical solution of integro-delay differential equations with weakly singular kernels of the form $(t-s)^{-\alpha}$, where $\alpha \in (0,1)$. Near the origin, the derivative of the solution behaves like $t^{-\alpha}$, which causes the global convergence order. To avoid this problem, we use some transformation to change the given equation into a new equation, which possesses better regularity properties. Our method has spectral accuracy, which means that a very accurate solution can be obtained using relatively few data points.

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