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Quenching behavior of a semilinear reaction-diffusion system with singular boundary condition

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Abstract: In this paper, we study the quenching behavior of the solution of a semilinear reaction-diffusion system with singular boundary condition. We first get a local exisence result. Then we prove that the solution quenches only on the right boundary in finite time and the time derivative blows up at the quenching time under certain conditions. Finally, we get lower bounds and upper bounds for quenching time.

Key words: Reaction-diffusion system, singular boundary condition, quenching, maximum principles, monotone iterations

1. Introduction

In this paper, we study the quenching behavior of the solution of the following semilinear reaction-diffusion system with singular boundary condition:

$$\begin{cases} u_t = u_{xx} + (1-v)^{-p_1}, \ 0 < x < 1, \ 0 < t < T, \\ v_t = v_{xx} + (1-u)^{-p_2}, \ 0 < x < 1, \ 0 < t < T, \\ u_x(0,t) = 0, \ u_x(1,t) = (1-v(1,t))^{-q_1}, \ 0 < t < T, \\ v_x(0,t) = 0, \ v_x(1,t) = (1-u(1,t))^{-q_2}, \ 0 < t < T, \\ u(x,0) = u_0(x) < 1, \ v(x,0) = v_0(x) < 1, \ 0 \le x \le 1, \end{cases} \tag{1.1}$$

where p_1, p_2, q_1, q_2 are positive constants and $u_0(x), v_0(x)$ are positive smooth functions satisfying the compatibility conditions:

$$u_0'(0) = v_0'(0) = 0, \ u_0'(1) = (1-v_0(1))^{-q_1}, \ v_0'(1) = (1-u_0(1))^{-q_2}.$$

Such systems arise in the study of simultaneous diffusion of several substances that decay spontaneously (see [14], page 189). Our main purpose is to examine the quenching behavior of the solution of problem (1.1).

Definition 1 *The solution of problem* (1.1) *is said to quench if there exists a finite time* T *such that*

$$\lim_{t \to T^-} \max\{u(x,t), v(x,t) : 0 \le x \le 1\} \to 1.$$

After this point, we denote the quenching time of problem (1.1) *with* T.

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The concept of quenching was first introduced by Kawarada [7]. Kawarada considered an initial-boundary value problem for the parabolic equation $u_t = u_{xx} + 1/(1-u)$. The quenching problems have since been studied extensively by several researchers (cf. the surveys by Chan [1, 2] and Kirk and Roberts [8], and [3, 4, 5, 10, 16]). There are many papers about the quenching behavior of the solutions of parabolic systems ([9, 17, 18, 19]).

For problem (1.1), if $p_1 = p_2$, $q_1 = q_2$ and $u_0 = v_0$, it is reduced to the following problem;

$$\begin{cases} u_t = u_{xx} + (1-u)^{-p_1}, 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) = 0, u_x(1,t) = (1-u(1,t))^{-q_1}, 0 < t < T, \\ u(x,0) = u_0(x) < 1, 0 \leq x \leq 1. \end{cases} \tag{1.2}$$

Recently, Ozalp and Selcuk [11] studied problem (1.2). They proved that the solution quenches only on the right boundary in finite time and the time derivative blows up at the quenching time under certain conditions. Finally, they obtained a lower bound and an upper bound for quenching time. Fu and Guo [6] investigated the blow-up behavior of the following semilinear reaction-diffusion system:

$$\begin{cases} u_t = u_{xx} + v^{p_1},\ 0 < x < 1,\ 0 < t < T, \\ v_t = v_{xx} + u^{p_2},\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) = 0,\ u_x(1,t) = v^{q_1}(1,t),\ 0 < t < T, \\ v_x(0,t) = 0,\ v_x(1,t) = u^{q_2}(1,t),\ 0 < t < T, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) > 0,\ 0 \leq x \leq 1, \end{cases} \tag{1.3}$$

where p_1, p_2, q_1, q_2 are positive constants and $u_0(x), v_0(x)$ are nonnegative smooth functions satisfying the compatibility conditions. They proved that the solution blows up only on the right boundary in finite time under certain conditions. Finally, they obtained the blow-up rate.

The equivalence between the blow-up problem and the quenching problem is well known; for example, see [10] and [15]. Motivated by problems (1.2) and (1.3), we investigate the quenching behavior of problem (1.1). In Section 2, we give a local existence result for problem (1.1). In Section 3, we prove that quenching occurs in finite time, the only quenching point is $x = 1$, and (u_t, v_t) blows up at quenching time under certain conditions. In Section 4, we obtain lower bounds and upper bounds for quenching time.

2. Local existence

It is well known that one of the most effective methods to obtain existence and uniqueness results of the solutions of a parabolic equation and system with initial conditions is the monotone iterative technique (for details see [5, 12, 13]). Interested readers may refer to [4] for the application of monotone iterative techniques to the quenching problem for a parabolic equation.

Let $C^m(Q), C^{0,\alpha}(Q)$ be the respective spaces of m-times differentiable and Hölder continuous functions in Q with exponent $\alpha \in (0,1)$, where Q is any domain. Assume that the set of functions that are twice continuously differentiable in x and continuously differentiable in t for $(x,t) \in [0,1] \times [0,T)$ are denoted by $C^{2,1}([0,1] \times [0,T))$. In addition, it assumed that initial function $u_0(x)$ is in $C^{2+\alpha}$.

Definition 2 $(\widetilde{u}, \widetilde{v})$ *is called an upper solution of problem* (1.1) *if* $\widetilde{u}, \widetilde{v} \in C([0,1] \times [0,T)) \cap C^{2,1}((0,1) \times (0,T))$

and $(\widetilde{u},\widetilde{v})$ *satisfies the following conditions:*

$$\begin{array}{l}
\widetilde{u}_t-\widetilde{u}_{xx}\geq(1-\widetilde{v})^{-p_1}\,,\ 0<x<1,\ 0<t<T,\\
\widetilde{v}_t-\widetilde{v}_{xx}\geq(1-\widetilde{u})^{-p_2}\,,\ 0<x<1,\ 0<t<T,\\
\widetilde{u}_x\left(0,t\right)=0,\ \widetilde{u}_x\left(1,t\right)\geq(1-\widetilde{v}(1,t))^{-q_1}\,,\ 0<t<T,\\
\widetilde{v}_x\left(0,t\right)=0,\ \widetilde{v}_x\left(1,t\right)\geq(1-\widetilde{u}(1,t))^{-q_2}\,,\ 0<t<T,\\
\widetilde{u}\left(x,0\right)\geq u_0\left(x\right),\ \widetilde{v}\left(x,0\right)\geq v_0\left(x\right),\ 0\leq x\leq 1.
\end{array}$$

Similarly, a lower solution $\widehat{u},\widehat{v}\in C([0,1]\times[0,T))\cap C^{2,1}((0,1)\times(0,T))$ *of problem* (1.1) *is defined by reversing the inequalities.*

Lemma 1 *Let* $(\widetilde{u},\widetilde{v})$ *and* $(\widehat{u},\widehat{v})$ *be a positive upper solution and a nonnegative lower solution of problem* (1.1) *in* $[0,1]\times[0,T)$*, respectively. Then we get the following results;*

(a) $\widetilde{u}\geq\widehat{u}$ *and* $\widetilde{v}\geq\widehat{v}$ *in* $[0,1]\times[0,T)$*,*

(b) *if* (u^*,v^*) *is a solution, then* $\widetilde{u}\geq u^*\geq\widehat{u}$ *and* $\widetilde{v}\geq v^*\geq\widehat{v}$ *in* $[0,1]\times[0,T)$*.*

Proof **(a)** The proof is given by utilizing Lemma 2.1 in [6]. Let $\Theta=\widetilde{u}-\widehat{u}$ and $\Psi=\widetilde{v}-\widehat{v}$. Then $\Theta(x,t)$ and $\Psi(x,t)$ satisfy

$$\begin{array}{l}
\Theta_t\geq\Theta_{xx}+a(x,t)\Psi,\Psi_t\geq\Psi_{xx}+b(x,t)\Theta,0<x<1,0<t<T,\\
\Theta_x\left(0,t\right)\leq 0,\Psi_x\left(0,t\right)\leq 0,0<t<T,\\
\Theta_x\left(1,t\right)\geq c(t)\Psi(1,t),\Psi_x\left(1,t\right)\geq d(t)\Theta(1,t),0<t<T,\\
\Theta\left(x,0\right)\geq 0,\Psi\left(x,0\right)\geq 0,0\leq x\leq 1,
\end{array}$$

where

$$\begin{aligned}
a(x,t) &= \frac{(1-\widetilde{v}(x,t))^{-p_1}-(1-\widehat{v}(x,t))^{-p_1}}{\widetilde{v}(x,t)-\widehat{v}(x,t)},\ \text{if } \widetilde{v}\neq\widehat{v}\text{ ; otherwise } a(x,t)=0,\\
b(x,t) &= \frac{(1-\widetilde{u}(x,t))^{-p_2}-(1-\widehat{u}(x,t))^{-p_2}}{\widetilde{u}(x,t)-\widehat{u}(x,t)},\ \text{if } \widetilde{u}\neq\widehat{u}\text{ ; otherwise } b(x,t)=0,\\
c(t) &= \frac{(1-\widetilde{v}(1,t))^{-q_1}-(1-\widehat{v}(1,t))^{-q_1}}{\widetilde{v}(1,t)-\widehat{v}(1,t)},\ \text{if } \widetilde{v}\neq\widehat{v}\text{ ; otherwise } c(t)=0,\\
d(t) &= \frac{(1-\widetilde{u}(1,t))^{-q_2}-(1-\widehat{u}(1,t))^{-q_2}}{\widetilde{u}(1,t)-\widehat{u}(1,t)},\ \text{if } \widetilde{u}\neq\widehat{u}\text{ ; otherwise } d(t)=0.
\end{aligned}$$

For any fixed $\tau\in(0,T)$, we will show that $\Psi\geq 0$ and $\Theta\geq 0$ for $0\leq x\leq 1$ and $0\leq t\leq\tau$. For contradiction, we assume that Θ has a negative minimum in $[0,1]\times[0,\tau]$ and $\min_{[0,1]\times[0,\tau]}\Theta\leq\min_{[0,1]\times[0,\tau]}\Psi$. Let $\bar{\Theta}(x,t)=e^{-Mt-Lx^2}\Theta(x,t)$ and $\bar{\Psi}(x,t)=e^{-Mt-Lx^2}\Psi(x,t)$, where

$$L=\max_{0\leq t\leq\tau}c(t)/2, M=2L+4L^2+\max_{[0,1]\times[0,\tau]}a(x,t)+\max_{[0,1]\times[0,\tau]}b(x,t).$$

Then $\bar{\Theta}$ and $\bar{\Psi}$ satisfy

$$\begin{array}{l}
\bar{\Theta}_t\geq\bar{\Theta}_{xx}+4Lx\bar{\Theta}_x+(2L+4L^2x^2-M)\bar{\Theta}+a(x,t)\bar{\Psi},\\
0<x<1,0<t<\tau,\\
\bar{\Psi}_t\geq\bar{\Psi}_{xx}+4Lx\bar{\Psi}_x+(2L+4L^2x^2-M)\bar{\Psi}+b(x,t)\bar{\Theta},\\
0<x<1,0<t<\tau.
\end{array}$$

Since $\bar{\Theta} \geq -\delta$ and $\bar{\Psi} \geq -\delta$ on the boundary $([0,1] \times \{0\}) \cup (\{0,1\} \times (0,\tau])$, where $-\delta := \min_{[0,1]\times[0,\tau]} \bar{\Theta} < 0$, it follows from the strong maximum principle for weakly coupled parabolic systems (cf. Theorem 15 of chapter 3 in [14]) that $\bar{\Theta}$ cannot assume its negative minimum in the interior. Hence, $\bar{\Theta} > -\delta$ in $(0,1) \times (0,\tau]$. Let (x_0, t_0) be a minimum point on the boundary $\{0,1\} \times (0,\tau]$. Since $\bar{\Theta}_x(0,t) \leq 0, 0 < t \leq \tau$, then the strong maximum principle implies that $x_0 = 1$ and $\bar{\Theta}_x(x_0,t_0) < 0$. However,

$$\bar{\Theta}_x(1,t_0) = -(c(t_0) - 2L)\Theta \geq 0,$$

which is a contradiction. Thus, $\widetilde{u} \geq \widehat{u}$ and $\widetilde{v} \geq \widehat{v}$ in $[0,1] \times [0,T)$.

(b) It is clear from Definition 2 that every solution of problem (1.1) is an upper solution as well as a lower solution of the corresponding problem. If (u^*, v^*) is a solution, then we get

$$\begin{aligned} \widetilde{u} &\geq u^* \text{ and } \widetilde{v} \geq v^*, \\ u^* &\geq \widehat{u} \text{ and } v^* \geq \widehat{v}, \end{aligned}$$

and

$$\widetilde{u} \geq u^* \geq \widehat{u} \text{ and } \widetilde{v} \geq v^* \geq \widehat{v}$$

in $[0,1] \times [0,T)$ from Lemma 1(a). □

For a given pair of ordered upper and lower solutions $(\widetilde{u}, \widetilde{v})$ and $(\widehat{u}, \widehat{v})$, we set

$$\begin{aligned} S_1 &= \{u \in C([0,1] \times [0,T)) : \widehat{u} \leq u \leq \widetilde{u}\}, \\ S_2 &= \{v \in C([0,1] \times [0,T)) : \widehat{v} \leq v \leq \widetilde{v}\}, \\ S_1 \times S_2 &= \{(u,v) \in C([0,1] \times [0,T)) \times C([0,1] \times [0,T)) : (\widehat{u}, \widehat{v}) \leq (u,v) \leq (\widetilde{u}, \widetilde{v})\}. \end{aligned}$$

Let

$$\begin{aligned} f_1(x,t,v(x,t)) &= (1 - v(x,t))^{-p_1}, g_1(x,t,v(x,t)) = (1 - v(x,t))^{-q_1}, \\ f_2(x,t,u(x,t)) &= (1 - u(x,t))^{-p_2}, g_2(x,t,u(x,t)) = (1 - u(x,t))^{-q_2}. \end{aligned}$$

Throughout this section, we make the following hypothesis on the above functions in problem (1.1):

(H_1)-(i) The functions $f_1(x,t,.), f_2(x,t,.)$ are in $C^{\alpha,\alpha/2}([0,1] \times [0,T))$ and $g_1(x,t,.)$, $g_2(x,t,.)$ are in $C^{1+\alpha,(1+\alpha)/2}(\{1\} \times (0,T))$.

(H_1)-(ii) Let $f_1(.,v)$ and $g_1(.,v)$ be C^1-functions of v for $v \in S_2$, and $f_2(.,u)$ and $g_2(.,u)$ be C^1-functions of u for $u \in S_1$. Also,

$$\begin{cases} (f_1)_v (x,t,v) \geq 0 \text{ for } v \in S_2, (x,t) \in [0,1] \times [0,T), \\ (f_2)_u (x,t,u) \geq 0 \text{ for } u \in S_1, (x,t) \in [0,1] \times [0,T), \\ (g_1)_v (x,t,v) \geq 0 \text{ for } v \in S_2, (x,t) \in \{1\} \times (0,T), \\ (g_2)_u (x,t,u) \geq 0 \text{ for } u \in S_1, (x,t) \in \{1\} \times (0,T). \end{cases} \tag{2.1}$$

Condition (2.1) implies that $f_1(.,v), g_1(.,v)$ are nondecreasing in v and $f_2(.,u)$, $g_2(.,u)$ are nondecreasing in u, which is crucial for the construction of monotone sequences.

Next, we are going to construct monotone sequences of functions that give the estimation of the solution (u,v) of problem (1.1). Specifically, by starting from any initial iteration $\left(u^0,v^0\right)$ we can construct a sequence $\{u^{(k)},v^{(k)}\}$ from the linear iteration process

$$\begin{cases} u_t^{(k)}-u_{xx}^{(k)}=f_1(x,t,v^{(k-1)}),\ 0<x<1, 0<t<T, \\ v_t^{(k)}-v_{xx}^{(k)}=f_2(x,t,u^{(k-1)}),\ 0<x<1, 0<t<T, \\ u_x^{(k)}(0,t)=0,\ u_x^{(k)}(1,t)=g_1(1,t,v^{(k-1)}),\ 0<t<T, \\ v_x^{(k)}(0,t)=0,\ v_x^{(k)}(1,t)=g_2(1,t,u^{(k-1)}),\ 0<t<T, \\ u^{(k)}(x,0)=u_0(x), v^{(k)}(x,0)=v_0(x), 0\le x\le 1. \end{cases} \tag{2.2}$$

It is clear that the sequence governed by (2.2) is well defined and can be obtained by solving a linear initial boundary value problem. Starting from initial iteration $\left(u^0,v^0\right)=(\widetilde{u},\widetilde{v})$ and $\left(u^0,v^0\right)=(\widehat{u},\widehat{v})$, we define two sequences of the functions $\left\{\overline{u}^{(k)},\overline{v}^{(k)}\right\}$ and $\left\{\underline{u}^{(k)},\underline{v}^{(k)}\right\}$ for $k=1,2,\ldots$ and refer to them as maximal and minimal sequences, respectively, where those functions solve the above linear problem.

Lemma 2. The sequences $\left\{\overline{u}^{(k)},\overline{v}^{(k)}\right\},\left\{\underline{u}^{(k)},\underline{v}^{(k)}\right\}$ possess the monotone property

$$(\widehat{u},\widehat{v})\le\left(\underline{u}^{(k)},\underline{v}^{(k)}\right)\le\left(\underline{u}^{(k+1)},\underline{v}^{(k+1)}\right)\le\left(\overline{u}^{(k+1)},\overline{v}^{(k+1)}\right)\le\left(\overline{u}^{(k)},\overline{v}^{(k)}\right)\le(\widetilde{u},\widetilde{v})$$

for $(x,t)\in[0,1]\times[0,T)$ and every $k=1,2,\ldots$.

Proof Let $\mu=\widetilde{u}-\overline{u}^{(1)}$ and $\lambda=\widetilde{v}-\overline{v}^{(1)}$. From (2.2) and Definition 2, we get

$$\begin{array}{l} \mu_t-\mu_{xx}=\widetilde{u}_t-\widetilde{u}_{xx}-f_1(x,t,\widetilde{v})\ge 0,\ 0<x<1,\ 0<t<T, \\ \lambda_t-\lambda_{xx}=\widetilde{v}_t-\widetilde{v}_{xx}-f_2(x,t,\widetilde{u})\ge 0,\ 0<x<1,\ 0<t<T, \\ \mu_x\left(0,t\right)=0,\ \mu_x\left(1,t\right)=\widetilde{u}_x(1,t)-g_1(1,t,\widetilde{v})\ge 0,\ 0<t<T, \\ \lambda_x\left(0,t\right)=0,\ \lambda_x\left(1,t\right)=\widetilde{v}_x(1,t)-g_2(1,t,\widetilde{u})\ge 0,\ 0<t<T, \\ \mu\left(x,0\right)=\widetilde{u}(x,0)-u_0\left(x\right)\ge 0,\ \lambda\left(x,0\right)=\widetilde{v}(x,0)-v_0\left(x\right)\ge 0,\ 0\le x\le 1. \end{array}$$

From the Maximum Principle and Hopf's lemma for parabolic equations, we get $\mu,\lambda\ge 0$ for $(x,t)\in[0,1]\times[0,T)$, i.e. $\overline{u}^{(1)}\le\widetilde{u}$ and $\overline{v}^{(1)}\le\widetilde{v}$. Similarly, using the property of a lower solution, we obtain $\underline{u}^{(1)}\ge\widehat{u}$ and $\underline{v}^{(1)}\ge\widehat{v}$.

Let $\mu^{(1)}=\overline{u}^{(1)}-\underline{u}^{(1)}$ and $\lambda^{(1)}=\overline{v}^{(1)}-\underline{v}^{(1)}$. From (2.1) and (2.2), we get

$$\begin{array}{l} \mu_t^{(1)}-\mu_{xx}^{(1)}=f_1(x,t,\widetilde{v})-f_1(x,t,\widehat{v})\ge 0,\ 0<x<1,\ 0<t<T, \\ \lambda_t^{(1)}-\lambda_{xx}^{(1)}=f_2(x,t,\widetilde{u})-f_2(x,t,\widehat{u})\ge 0,\ 0<x<1,\ 0<t<T, \\ \mu_x^{(1)}\left(0,t\right)=0,\ \mu_x^{(1)}\left(1,t\right)=g_1(1,t,\widetilde{v})-g_1(1,t,\widehat{v})\ge 0,\ 0<t<T, \\ \lambda_x^{(1)}\left(0,t\right)=0,\ \lambda_x^{(1)}\left(1,t\right)=g_2(1,t,\widetilde{u})-g_2(1,t,\widehat{u})\ge 0,\ 0<t<T, \\ \mu^{(1)}\left(x,0\right)=u_0\left(x\right)-u_0\left(x\right)=0,\ \lambda^{(1)}\left(x,0\right)=v_0\left(x\right)-v_0\left(x\right)=0,\ 0\le x\le 1. \end{array}$$

From the Maximum Principle and Hopf's lemma for parabolic equations, we get $\mu^{(1)},\lambda^{(1)}\ge 0$ for $(x,t)\in[0,1]\times[0,T)$, i.e. $\underline{u}^{(1)}\le\overline{u}^{(1)}$ and $\underline{v}^{(1)}\le\overline{v}^{(1)}$. Therefore,

$$(\widehat{u},\widehat{v})\le\left(\underline{u}^{(1)},\underline{v}^{(1)}\right)\le\left(\overline{u}^{(1)},\overline{v}^{(1)}\right)\le(\widetilde{u},\widetilde{v})$$

for $(x,t)\in[0,1]\times[0,T)$.

Assume that

$$\left(\underline{u}^{(k-1)},\underline{v}^{(k-1)}\right)\leq\left(\underline{u}^{(k)},\underline{v}^{(k)}\right)\leq\left(\overline{u}^{(k)},\overline{v}^{(k)}\right)\leq\left(\overline{u}^{(k-1)},\overline{v}^{(k-1)}\right)$$

for $(x,t)\in[0,1]\times[0,T)$ and for some integer $k>1$. Let $\mu^{(k)}=\overline{u}^{(k)}-\overline{u}^{(k+1)}$ and $\lambda^{(k)}=\overline{v}^{(k)}-\overline{v}^{(k+1)}$. From (2.1) and (2.2), we get

$$\begin{array}{l}\mu_t^{(k)}-\mu_{xx}^{(k)}=f_1(x,t,\overline{v}^{(k-1)})-f_1(x,t,\overline{v}^{(k)})\geq 0,\ 0<x<1,\ 0<t<T,\\ \lambda_t^{(k)}-\lambda_{xx}^{(k)}=f_2(x,t,\overline{u}^{(k-1)})-f_2(x,t,\overline{u}^{(k)})\geq 0,\ 0<x<1,\ 0<t<T,\\ \mu_x^{(k)}(0,t)=0,\ \mu_x^{(k)}(1,t)=g_1(1,t,\overline{v}^{(k-1)})-g_1(1,t,\overline{v}^{(k)})\geq 0,\ 0<t<T,\\ \lambda_x^{(k)}(0,t)=0,\ \lambda_x^{(k)}(1,t)=g_2(1,t,\overline{u}^{(k-1)})-g_2(1,t,\overline{u}^{(k)})\geq 0,\ 0<t<T,\\ \mu^{(k)}(x,0)=0,\ \lambda^{(k)}(x,0)=0,\ 0\leq x\leq 1.\end{array}$$

From the Maximum Principle and Hopf's lemma for parabolic equations, we get $\mu^{(k)},\lambda^{(k)}\geq 0$ for $(x,t)\in[0,1]\times[0,T)$, i.e. $\overline{u}^{(k+1)}\leq\overline{u}^{(k)}$ and $\overline{v}^{(k+1)}\leq\overline{v}^{(k)}$. A similar argument gives $\underline{u}^{(k+1)}\geq\underline{u}^{(k)}$, $\underline{v}^{(k+1)}\geq\underline{v}^{(k)}$, $\overline{u}^{(k+1)}\geq\underline{u}^{(k+1)}$ and $\overline{v}^{(k+1)}\geq\underline{v}^{(k+1)}$. Therefore, from the mathematical induction, the result follows. □

Lemma 2 *For each positive integer* k, $\left(\overline{u}^{(k)},\overline{v}^{(k)}\right)$ *is an upper solution,* $\left(\underline{u}^{(k)},\underline{v}^{(k)}\right)$ *is a lower solution, and* $\underline{u}^{(k)}\leq\overline{u}^{(k)}$ *and* $\underline{v}^{(k)}\leq\overline{v}^{(k)}$ *for* $(x,t)\in[0,1]\times[0,T)$.

Proof From $(2.1),(2.2)$, and Lemma 2, $\left(\overline{u}^{(k)},\overline{v}^{(k)}\right)$ satisfies

$$\begin{array}{l}\overline{u}_t^{(k)}-\overline{u}_{xx}^{(k)}=f_1(x,t,\overline{v}^{(k-1)})=f_1(x,t,\overline{v}^{(k-1)})-f_1(x,t,\overline{v}^{(k)})+f_1(x,t,\overline{v}^{(k)})\geq f_1(x,t,\overline{v}^{(k)}),\\ \overline{v}_t^{(k)}-\overline{v}_{xx}^{(k)}=f_2(x,t,\overline{u}^{(k-1)})=f_2(x,t,\overline{u}^{(k-1)})-f_2(x,t,\overline{u}^{(k)})+f_2(x,t,\overline{u}^{(k)})\geq f_2(x,t,\overline{u}^{(k)}),\\ \overline{u}_x^{(k)}(0,t)=0,\overline{v}_x^{(k)}(0,t)=0,\\ \overline{u}_x^{(k)}(1,t)=g_1(1,t,\overline{v}^{(k-1)})=g_1(1,t,\overline{v}^{(k-1)})-g_1(1,t,\overline{v}^{(k)})+g_1(1,t,\overline{v}^{(k)})\geq g_1(1,t,\overline{v}^{(k)})\\ \overline{v}_x^{(k)}(1,t)=g_2(1,t,\overline{u}^{(k-1)})=g_2(1,t,\overline{u}^{(k-1)})-g_2(1,t,\overline{u}^{(k)})+g_2(1,t,\overline{u}^{(k)})\geq g_2(1,t,\overline{u}^{(k)})\\ \overline{u}^{(k)}(x,0)=u_0(x),\overline{v}^{(k)}(x,0)=v_0(x),0\leq x\leq 1,\end{array}$$

and $\left(\underline{u}^{(k)},\underline{v}^{(k)}\right)$ satisfies

$$\begin{array}{l}\underline{u}_t^{(k)}-\underline{u}_{xx}^{(k)}=f_1(x,t,\underline{v}^{(k-1)})=f_1(x,t,\underline{v}^{(k-1)})-f_1(x,t,\underline{v}^{(k)})+f_1(x,t,\underline{v}^{(k)})\leq f_1(x,t,\underline{v}^{(k)}),\\ \underline{v}_t^{(k)}-\underline{v}_{xx}^{(k)}=f_2(x,t,\underline{u}^{(k-1)})=f_2(x,t,\underline{u}^{(k-1)})-f_2(x,t,\underline{u}^{(k)})+f_2(x,t,\underline{u}^{(k)})\leq f_2(x,t,\underline{u}^{(k)}),\\ \underline{u}_x^{(k)}(0,t)=0,\underline{v}_x^{(k)}(0,t)=0,\\ \underline{u}_x^{(k)}(1,t)=g_1(1,t,\underline{v}^{(k-1)})=g_1(1,t,\underline{v}^{(k-1)})-g_1(1,t,\underline{v}^{(k)})+g_1(1,t,\underline{v}^{(k)})\leq g_1(1,t,\underline{v}^{(k)})\\ \underline{v}_x^{(k)}(1,t)=g_2(1,t,\underline{u}^{(k-1)})=g_2(1,t,\underline{u}^{(k-1)})-g_2(1,t,\underline{u}^{(k)})+g_2(1,t,\underline{u}^{(k)})\leq g_2(1,t,\underline{u}^{(k)})\\ \underline{u}^{(k)}(x,0)=u_0(x),\underline{v}^{(k)}(x,0)=v_0(x),0\leq x\leq 1.\end{array}$$

From Lemma 2 and the above inequalities, the functions $\left(\overline{u}^{(k)},\overline{v}^{(k)}\right)$ and $\left(\underline{u}^{(k)},\underline{v}^{(k)}\right)$ are ordered upper and lower solutions of problem (2.2). □

We have the following existence theorem for problem (1.1) via Lemma 2 and Lemma 3.

Theorem 1 *Let* $(\widetilde{u},\widetilde{v}),(\widehat{u},\widehat{v})$ *be a pair of ordered upper and lower solutions of problem* (1.1), *and let hypothesis* (H_1) *hold. Then the sequences* $\left\{\overline{u}^{(k)},\overline{v}^{(k)}\right\}$ *and* $\left\{\underline{u}^{(k)},\underline{v}^{(k)}\right\}$ *are given by problem* (2.2) *with* $\left(u^0,v^0\right)=(\widetilde{u},\widetilde{v})$

and $\left(u^0, v^0\right) = (\widehat{u}, \widehat{v})$ *converge monotonically to a maximal solution* $(\overline{u}, \overline{v})$ *and a minimal solution* $(\underline{u}, \underline{v})$ *of problem* (1.1)*, respectively. Furthermore,*

$$(\widehat{u}, \widehat{v}) \leq \left(\underline{u}^{(k)}, \underline{v}^{(k)}\right) \leq \left(\underline{u}^{(k+1)}, \underline{v}^{(k+1)}\right) \leq (\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v}) \leq \left(\overline{u}^{(k+1)}, \overline{v}^{(k+1)}\right) \leq \left(\overline{u}^{(k)}, \overline{v}^{(k)}\right) \leq (\widetilde{u}, \widetilde{v}) \tag{2.3}$$

for $(x,t) \in [0,1] \times [0,T)$ *and each positive integer* k*. Furthermore,* $(\underline{u}, \underline{v}) = (\overline{u}, \overline{v})$ $(\equiv (u^*, v^*))$*, and then* (u^*, v^*) *is the unique solution of problem* (1.1) *in* $S_1 \times S_2$*.*

Proof The pointwise limits

$$\lim_{k\to\infty} \left(\overline{u}^{(k)}(x,t), \overline{v}^{(k)}(x,t)\right) = (\overline{u}(x,t), \overline{v}(x,t)), \lim_{k\to\infty} \left(\underline{u}^{(k)}(x,t), \underline{v}^{(k)}(x,t)\right) = (\underline{u}(x,t), \underline{v}(x,t))$$

exist and satisfy relation (2.3). Indeed, the sequence $\left\{\overline{u}^{(k)}, \overline{v}^{(k)}\right\}$ is monotone nonincreasing, which is bounded from below, while the sequence $\left\{\underline{u}^{(k)}, \underline{v}^{(k)}\right\}$ is monotone nondecreasing and is bounded from Lemma 2.

Let $\Theta = \underline{u}(x,t) - \overline{u}(x,t)$ and $\Psi = \underline{v}(x,t) - \overline{v}(x,t)$. From (2.3), we have $\underline{u}(x,t) \leq \overline{u}(x,t)$ and $\underline{v}(x,t) \leq \overline{v}(x,t)$ for $(x,t) \in [0,1] \times [0,T)$. Then $\Theta(x,t)$ and $\Psi(x,t)$ satisfy

$$\begin{array}{l}
\Theta_t - \Theta_{xx} = f_1(x,t,\underline{v}) - f_1(x,t,\overline{v}), 0 < x < 1, 0 < t < T, \\
\Psi_t - \Psi_{xx} = f_2(x,t,\underline{u}) - f_2(x,t,\overline{u}), 0 < x < 1, 0 < t < T, \\
\Theta_x(0,t) = 0, \Psi_x(0,t) = 0, 0 < t < T, \\
\Theta_x(1,t) = g_1(1,t,\underline{v}) - g_1(1,t,\overline{v}), 0 < t < T, \\
\Psi_x(1,t) = g_2(1,t,\underline{u}) - g_2(1,t,\overline{u}), 0 < t < T, \\
\Theta(x,0) = 0, \Psi(x,0) = 0, 0 \leq x \leq 1.
\end{array}$$

By using Lemma 1(a) and Lemma 4(a), $\Theta \geq 0$ and $\Psi \geq 0$ for $(x,t) \in [0,1] \times [0,T)$, i.e. $\underline{u}(x,t) \geq \overline{u}(x,t)$ and $\underline{v}(x,t) \geq \overline{v}(x,t)$. Then we get $\underline{u}(x,t) = \overline{u}(x,t)$ and $\underline{v}(x,t) = \overline{v}(x,t)$.

If (u^*, v^*) is any other solution in $S_1 \times S_2$, then we get from Lemma 3,

$$\begin{aligned}
\overline{u} &\geq u^* \text{ and } \overline{v} \geq v^*, \\
u^* &\geq \underline{u} \text{ and } v^* \geq \underline{v},
\end{aligned}$$

and

$$\overline{u} \geq u^* \geq \underline{u} \text{ and } \overline{v} \geq v^* \geq \underline{v}$$

in $[0,1] \times [0,T)$. This implies that

$$\overline{u} = u^* = \underline{u} \text{ and } \overline{v} = v^* = \underline{v}$$

and hence (u^*, v^*) is the unique solution of the problem (1.1). □

3. Finite time quenching

Throughout this section and the next section, we also assume that the initial function (u_0, v_0) satisfies the following inequalities:

$$\begin{array}{l}
u_0''(x) + (1 - v_0(x))^{-p_1} \geq 0 \ (\neq 0), \\
v_0''(x) + (1 - u_0(x))^{-p_2} \geq 0 \ (\neq 0),
\end{array} \tag{3.1}$$

$$u_0'(x) \geq 0, v_0'(x) \geq 0. \tag{3.2}$$

Remark 1. We assume that conditions (3.1) and (3.2) are proper. Namely, we can easily construct such initial functions $u_0(x) = u(x,0), v_0(x) = v(x,0)$ satisfying $(3.1) - (3.2)$ and compatibility conditions. For example, for $p_1 \neq p_2$ and $q_1 = 1, q_2 = 2$, $u_0(x) = \frac{1}{2}x^4, v_0(x) = \frac{1}{2}x^8$ satisfies compatibility conditions and $(3.1) - (3.2)$.

Remark 2. If (u_0, v_0) satisfies (3.2), then we get $u_x(x,t), v_x(x,t) > 0$ in $(0,1] \times (0,T)$ by the maximum principle. Thus, we get $u(1,t) = \max\limits_{0 \leq x \leq 1} u(x,t)$ and $v(1,t) = \max\limits_{0 \leq x \leq 1} v(x,t)$.

Lemma 4. We assume that (u_0, v_0) satisfies (3.1). Then we get:

(a) $u_t(x,t), v_t(x,t) \geq 0$ in $[0,1] \times [0,T)$,

(b) $u_t(x,t), v_t(x,t) > 0$ in $(0,1) \times [0,T)$.

Proof **(a)** Since $u_0''(x) + (1 - v_0(x))^{-p_1} \geq 0$ and $v_0''(x) + (1 - u_0(x))^{-p_2} \geq 0$ in $(0,1)$, $u_0'(0) = 0$, $u_0'(1) = (1 - v_0(1))^{-q_1}$, $v_0'(0) = 0$, $v_0'(1) = (1 - u_0(1))^{-q_2}$, it follows that $(u_0(x), v_0(x))$ is a lower solution of problem (1.1). The strong maximum principle implies that

$$u(x,t) \geq u_0(x), v(x,t) \geq v_0(x) \text{ in } (0,1) \times (0,T).$$

The proof is given by utilizing Lemma 2.1 in [6]. Let $\Theta = u(x,t+h) - u(x,t)$ and $\Psi = v(x,t+h) - v(x,t)$ for $(x,t) \in [0,1] \times [0,T-h)$. Then $\Theta(x,t)$ satisfies

$$\begin{aligned} &\Theta_t = \Theta_{xx} + p_1 (1 - \beta_1)^{-p_1 - 1} \Psi, \ 0 < x < 1, \ 0 < t < T - h, \\ &\Theta_x(0,t) = 0, \ \Theta_x(1,t) = q_1 (1 - \beta_2(1,t))^{-q_1 - 1} \Psi(1,t), \ 0 < t < T - h, \\ &\Theta(x,0) = u(x,h) - u(x,0) \geq 0, \ 0 \leq x \leq 1, \end{aligned}$$

where $\beta_1(x,t)$ and $\beta_2(1,t)$ lie, respectively, between $v(x,t+h)$ and $v(x,t)$, and between $v(1,t+h)$ and $v(1,t)$. Similarly, $\Psi(x,t)$ satisfies

$$\begin{aligned} &\Psi_t = \Psi_{xx} + p_2 (1 - \xi_1)^{-p_2 - 1} \Theta, \ 0 < x < 1, \ 0 < t < T - h, \\ &\Psi_x(0,t) = 0, \ \Psi_x(1,t) = q_2 (1 - \xi_2(1,t))^{-q_2 - 1} \Theta(1,t), \ 0 < t < T - h, \\ &\Psi(x,0) = v(x,h) - v(x,0) \geq 0, \ 0 \leq x \leq 1, \end{aligned}$$

where $\xi_1(x,t)$ and $\xi_2(1,t)$ lie, respectively, between $u(x,t+h)$ and $u(x,t)$, and between $u(1,t+h)$ and $u(1,t)$.

For any fixed $\tau \in (0, T-h)$, we will show that $\Psi \geq 0$ and $\Theta \geq 0$ for $0 \leq x \leq 1$ and $0 \leq t \leq \tau$. For contradiction, we assume that Θ has a negative minimum in $[0,1] \times [0,\tau]$ and $\min_{[0,1]\times[0,\tau]} \Theta \leq \min_{[0,1]\times[0,\tau]} \Psi$. Let $\bar{\Theta}(x,t) = e^{-Mt - Lx^2} \Theta(x,t)$ and $\bar{\Psi}(x,t) = e^{-Mt - Lx^2} \Psi(x,t)$, where

$$\begin{aligned} L &= \max_{0 \leq t \leq \tau} \left(\frac{1}{2} q_1 (1 - \beta_2(1,t))^{-q_1 - 1} \right), \\ M &= 2L + 4L^2 + \max_{[0,1]\times[0,\tau]} \left(p_1 (1 - \beta_1(x,t))^{-p_1 - 1} \right) + \max_{[0,1]\times[0,\tau]} \left(p_2 (1 - \xi_1(x,t))^{-p_2 - 1} \right). \end{aligned}$$

Then $\bar{\Theta}$ and $\bar{\Psi}$ satisfy

$$\begin{aligned} &\bar{\Theta}_t = \bar{\Theta}_{xx} + 4Lx\bar{\Theta}_x + (2L + 4L^2x^2 - M)\bar{\Theta} + p_1 (1 - \beta_1(x,t))^{-p_1 - 1} \bar{\Psi}, \\ &0 < x < 1, 0 < t < \tau, \\ &\bar{\Psi}_t = \bar{\Psi}_{xx} + 4Lx\bar{\Psi}_x + (2L + 4L^2x^2 - M)\bar{\Psi} + p_2 (1 - \xi_1(x,t))^{-p_2 - 1} \bar{\Theta}, \\ &0 < x < 1, 0 < t < \tau. \end{aligned}$$

Since $\bar{\Theta} \geq -\delta$ and $\bar{\Psi} \geq -\delta$ on the boundary $([0,1] \times \{0\}) \cup (\{0,1\} \times (0,\tau])$, where $-\delta := \min_{[0,1]\times[0,\tau]} \bar{\Theta} < 0$, it follows from the strong maximum principle for weakly coupled parabolic systems (cf. Theorem 15 of chapter 3 in [14]) that $\bar{\Theta}$ cannot assume its negative minimum in the interior. Hence, $\bar{\Theta} > -\delta$ in $(0,1) \times (0,\tau]$. Let (x_0, t_0) be a minimum point on the boundary $\{0,1\} \times (0,\tau]$. Since $\bar{\Theta}_x(0,t) \leq 0, 0 < t \leq \tau$, the same strong maximum principle implies that $x_0 = 1$ and $\bar{\Theta}_x(x_0,t_0) < 0$. However,

$$\bar{\Theta}_x(1,t_0) = (q_1(1-\beta_2(1,t_0))^{-q_1-1} - 2L)\Theta = -(q_1(1-\beta_2(1,t_0))^{-q_1-1} - 2L)\delta \geq 0,$$

which is a contradiction. Then, we obtain that $\bar{\Theta} \geq 0$ and $\bar{\Psi} \geq 0$ in $[0,1] \times [0,\tau]$. As $h \to 0$, $u_t(x,t) \geq 0$ and $v_t(x,t) \geq 0$ in $[0,1] \times [0,T)$.

(b) For any $(\xi,\eta) \in (0,1) \times (0,T)$, there exists a subset $[x_1,x_2] \times [t_1,t_2]$ of $(0,1) \times (0,T)$ such that $(\xi,\eta) \in [x_1,x_2] \times [t_1,t_2]$. Define $H = u_t, K = v_t$ in $[x_1,x_2] \times [t_1,t_2]$. We can obtain

$$\begin{aligned} H_t - H_{xx} &= p_1(1-v)^{-p_1-1}K \text{ in } (x_1,x_2) \times (t_1,t_2), \\ K_t - K_{xx} &= p_2(1-u)^{-p_2-1}H \text{ in } (x_1,x_2) \times (t_1,t_2), \\ H, K &\geq 0 \text{ on } [x_1,x_2] \times [t_1,t_2]. \end{aligned}$$

The strong maximum principle implies that either $H, K > 0$ or $H, K \equiv 0$ in $(x_1,x_2) \times (t_1,t_2)$. Since $H, K \equiv 0$ contradicts the fact that $u(x,t)$ and $v(x,t)$ is strictly increasing t, therefore, $u_t, v_t > 0$. Because (ξ,η) is arbitrary in $(0,1) \times (0,T)$, we have $u_t, v_t > 0$ in $(0,1) \times (0,T)$. □

Theorem 2 *If (u_0,v_0) satisfies (3.1), then there exists a finite time T, such that the solution (u,v) of problem (1.1) quenches at time T.*

Proof Assume that (u_0,v_0) satisfies (3.1). Then there exist

$$\begin{aligned} w_1 &= (1-v(1,0))^{-q_1} + \int_0^1 (1-v(x,0))^{-p_1}\,dx > 0, \\ w_2 &= (1-u(1,0))^{-q_2} + \int_0^1 (1-u(x,0))^{-p_2}\,dx > 0. \end{aligned}$$

Define $m_1(t) = \int_0^1 (1-u(x,t))\,dx$ and $m_2(t) = \int_0^1 (1-v(x,t))\,dx, 0 < t < T$. Then

$$\begin{aligned} m_1'(t) &= -(1-v(1,t))^{-q_1} - \int_0^1 (1-v(x,t))^{-p_1}\,dx \leq -w_1, \\ m_2'(t) &= -(1-u(1,t))^{-q_2} - \int_0^1 (1-u(x,t))^{-p_2}\,dx \leq -w_2, \end{aligned}$$

by Lemma 4(a). Thus, $m_1(t) \leq m_1(0) - w_1 t$ and $m_2(t) \leq m_2(0) - w_2 t$, which means that $m_1(T_0) = 0$ or $m_2(T_0) = 0$ for some $T_0 = \min(\frac{m_1(0)}{w_1}, \frac{m_2(0)}{w_2}), (0 < T \leq T_0)$. Then (u,v) quenches in finite time. □

Theorem 3 *If (u_0,v_0) satisfies (3.2), then $x = 1$ is the only quenching point.*

Proof Define

$$J(x,t) = u_x - \varepsilon\,(x-(1-\eta)) \text{ in } [1-\eta,1]\times[\tau,T),$$

where $\eta \in (0,1), \tau \in (0,T)$ and ε is a positive constant to be specified later. Then $J(x,t)$ satisfies

$$J_t - J_{xx} = p_1(1-v)^{-p_1-1}v_x > 0 \text{ in } (1-\eta,1)\times(\tau,T),$$

since $v_x(x,t) > 0$ in $(0,1]\times(0,T)$. Thus, $J(x,t)$ cannot attain a negative interior minimum by the maximum principle. Further, if ε is small enough, $J(x,\tau) > 0$ since $u_x(x,t) > 0$ in $(0,1]\times(0,T)$. Furthermore, if ε is small enough,

$$\begin{aligned} J(1-\eta,t) &= u_x(1-\eta,t) > 0, \\ J(1,t) &= (1-v(1,t))^{-q_1} - \varepsilon\eta > 1-\varepsilon\eta > 0, \end{aligned}$$

for $t \in (\tau,T)$. By the maximum principle, we obtain that $J(x,t) > 0$, i.e. $u_x > \varepsilon\,(x-(1-\eta))$ for $(x,t) \in [1-\eta,1]\times[\tau,T)$. Integrating with respect to x from $1-\eta$ to 1, we have

$$u(1-\eta,t) < u(1,t) - \frac{\varepsilon\eta^2}{2} < 1 - \frac{\varepsilon\eta^2}{2}.$$

Thus, u does not quench in $[0,1)$. Similarly, we show that v does not quench in $[0,1)$. The theorem is proved. □

Theorem 4 (u_t, v_t) *blows up at the quenching time.*

Proof We will prove that (u_t,v_t) blows up at quenching, as in [3]. Suppose that u_t and v_t are bounded on $[0,1]\times[0,T)$. Then there exist positive constants M_1 and M_2 such that $u_t < M_1$ and $v_t < M_2$. We have

$$\begin{aligned} u_{xx} + (1-v)^{-p_1} &< M_1 \Rightarrow u_{xx} < M_1, \\ v_{xx} + (1-u)^{-p_2} &< M_2 \Rightarrow v_{xx} < M_2. \end{aligned}$$

Integrating this twice with respect to x from x to 1, and then from 0 to 1, we have

$$\begin{aligned} \frac{1}{(1-v(1,t))^{q_1}} &< \frac{M_1}{2} + u(1,t) - u(0,t), \\ \frac{1}{(1-u(1,t))^{q_2}} &< \frac{M_2}{2} + v(1,t) - v(0,t). \end{aligned}$$

As $t \to T^-$, the left-hand side tends to infinity, while the right-hand side is finite. This contradiction shows that (u_t,v_t) blows up somewhere. □

Lemma 3 ***(a)*** *If* $p_1 \ge p_2$, $q_1 \ge q_2$ *and* $u_0(x) \ge v_0(x)$ *for* $x \in [0,1]$, *then* $u(x,t) \ge v(x,t)$ *in* $[0,1]\times(0,T)$,

(b) *If* $p_2 \ge p_1$, $q_2 \ge q_1$ *and* $v_0(x) \ge u_0(x)$ *for* $x \in [0,1]$, *then* $v(x,t) \ge u(x,t)$ *in* $[0,1]\times(0,T)$.

Proof **(a)** Define $M(x,t) = u - v$ in $[0,1]\times[0,T)$. Then $M(x,t)$ satisfies

$$\begin{aligned} M_t - M_{xx} &= (1-v)^{-p_1} - (1-u)^{-p_2} \\ &= (1-v)^{-p_1} - (1-u)^{-p_1} + (1-u)^{-p_1} - (1-u)^{-p_2} \\ &\ge -p_1(1-\beta_1)^{-p_1-1}M,\ (0,1)\times[0,T), \end{aligned}$$

where $\beta_1(x,t)$ lies between $u(x,t)$ and $v(x,t)$ and $p_1 \geq p_2$. Thus, $M(x,t)$ cannot attain a negative interior minimum by the maximum principle. Further, $M(x,0) \geq 0$ since $u_0 \geq v_0$ for $x \in (0,1)$. Furthermore,

$$\begin{aligned} M_x(0,t) &= 0,\ t \in (0,T), \\ M_x(1,t) &= (1-v(1,t))^{-q_1} - (1-u(1,t))^{-q_2} \\ &= (1-v(1,t))^{-q_1} - (1-u(1,t))^{-q_1} + (1-u(1,t))^{-q_1} - (1-u(1,t))^{-q_2} \\ &\geq -q_1(1-\beta_2(1,t))^{-q_1-1}M,\ t \in (0,T), \end{aligned}$$

where $\beta_2(1,t)$ lies between $u(1,t)$ and $v(1,t)$ and $q_1 \geq q_2$. By the maximum principle and Hopf's lemma for parabolic equation, we obtain that $M(x,t) \geq 0$ in $[0,1] \times [0,T)$, i.e. $u(x,t) \geq v(x,t)$ in $[0,1] \times [0,T)$.

(b) Similarly, we get $v(x,t) \geq u(x,t)$ in $[0,1] \times (0,T)$ since $v_0(x) \geq u_0(x), p_2 \geq p_1$, and $q_2 \geq q_1$. □

Corollary 1 *From the statement of problem* (1.1), *we show that*

$$\begin{aligned} \textit{if } \lim_{t\to T^-} v(1,t) &= 1,\ \textit{then } \lim_{t\to T^-} u_t(1,t) = \infty, \\ \textit{if } \lim_{t\to T^-} u(1,t) &= 1,\ \textit{then } \lim_{t\to T^-} v_t(1,t) = \infty. \end{aligned}$$

Assume that (u_0, v_0) *satisfies* (3.1)−(3.2). *We can obtain the following results from Theorems 2–4 and Lemma 5:*

(a) *If* $q_2 \geq q_1$, $p_2 \geq p_1$ *and* $v_0(x) \geq u_0(x)$ *for* $x \in [0,1]$ *then the only quenching point is* $x = 1$, v *quenches* $\left(\lim_{t\to T^-} v(1,t) = 1\right)$, *and* u_t *blows up* $\left(\lim_{t\to T^-} u_t(1,t) = \infty\right)$ *at the quenching time* T.

(b) *If* $q_1 \geq q_2$, $p_1 \geq p_2$ *and* $u_0(x) \geq v_0(x)$ *for* $x \in [0,1]$ *then the only quenching point is* $x = 1$, u *quenches* $\left(\lim_{t\to T^-} u(1,t) = 1\right)$, *and* v_t *blows up* $\left(\lim_{t\to T^-} v_t(1,t) = \infty\right)$ *at the quenching time* T.

Remark 1 If we select $p_2 \geq p_1$, $q_1 = 1$, $q_2 = 2$ and $u_0(x) = \frac{1}{2}x^4, v_0(x) = \frac{1}{2}x^8$ as in Remark 1, then we get $\lim_{t\to T} v(1,t) \to 1$ and $\lim_{t\to T^-} u_t(1,t) = \infty$ at the quenching time T from Corollary 1(a).

4. Bounds for the quenching time

In this section, we assume that

$$u_0'(x) \geq x(1-v_0(x))^{-q_1}, 0 \leq x \leq 1, \tag{4.1}$$

$$v_0'(x) \geq x(1-u_0(x))^{-q_2}, 0 \leq x \leq 1. \tag{4.2}$$

Theorem 5 *If* (u_0, v_0) *satisfies* (3.1) − (3.2) *and* (4.1) − (4.2), *then there exist positive constants* C_1 *and* C_2 *such that*

$$\begin{aligned} u(1,t) &\geq 1 - C_1(T-t)^{1/(2q_1+2)}, \\ v(1,t) &\geq 1 - C_2(T-t)^{1/(2q_2+2)}, \end{aligned}$$

for t *sufficiently close to* T.

Proof Define $J_1(x,t) = u_x - x(1-v)^{-q_1}$ and $J_2(x,t) = v_x - x(1-u)^{-q_2}$ in $[0,1]\times[0,T)$. Then $J(x,t)$ satisfies

$$(J_1)_t - (J_1)_{xx} - q_1(1-v)^{-q_1-1}J_2 = p_1(1-v)^{-p_1-1}v_x + q_1(1-v)^{-q_1-1}v_x + q_1(q_1+1)x(1-v)^{-q_1-2}v_x^2,$$

since $v_x > 0$, $J_1(x,t)$ cannot attain a negative interior minimum. On the other hand, $J_1(x,0) \geq 0$ by (4.1) and

$$J_1(0,t) = 0, \;\; J_1(1,t) = 0,$$

for $t \in (0,T)$. By the maximum principle, we obtain that $J_1(x,t) \geq 0$ for $(x,t) \in [0,1]\times[0,T)$. Therefore,

$$(J_1)_x(1,t) = \lim_{h\to 0^+}\frac{J(1,t) - J(1-h,t)}{h} = \lim_{h\to 0^+}\frac{-J(1-h,t)}{h} \leq 0.$$

Thus, we get

$$\begin{aligned}(J_1)_x(1,t) &= u_{xx}(1,t) - (1-v(1,t))^{-q_1} - q_1(1-v(1,t))^{-q_1-1}(1-u(1,t))^{-q_2}\\ &= u_t(1,t) - (1-v(1,t))^{-p_1} - (1-v(1,t))^{-q_1} - q_1(1-v(1,t))^{-q_1-1}(1-u(1,t))^{-q_2} \leq 0\end{aligned}$$

and

$$u_t(1,t) \leq (2+q_1)(1-v(1,t))^{-q_1-1}(1-u(1,t))^{-q_2},$$

where $q_1 \geq p_1$, and using Lemma 5(a),

$$\begin{aligned}(1-v(1,t))^{-q_1-1} &\leq (1-u(1,t))^{-q_1-1},\\ (1-u(1,t))^{-q_2} &\leq (1-u(1,t))^{-q_1}\end{aligned}$$

since $p_1 \geq p_2$, $q_1 \geq q_2$ and $u_0(x) \geq v_0(x)$. Thus, we get

$$u_t(1,t) \leq (2+q_1)(1-u(1,t))^{-2q_1-1}.$$

Integrating for t from t to T we get

$$u(1,t) \geq 1 - C_1(T-t)^{1/(2q_1+2)} \tag{4.3}$$

where $C_1 = [2(q_1+1)(q_1+2)]^{1/(2q_1+2)}$.

From $q_2 \geq p_2$, Lemma 5(b), and (4.2), if we follow the above process, then we get

$$v(1,t) \geq 1 - C_2(T-t)^{1/(2q_2+2)} \tag{4.4}$$

where $C_2 = [2(q_2+1)(q_2+2)]^{1/(2q_2+2)}$. The theorem is proved. □

Corollary 2 *If we put $t = 0$ in (4.3) and (4.4), then we get following results:*

(a) *if $q_1 \geq p_1 \geq p_2$, $q_1 \geq q_2$, and $u_0(x) \geq v_0(x)$, then a lower bound for the quenching time is $(1-u_0(1))^{2q_1+2}/2(q_1+1)(q_1+2)$,*

(b) *if $q_2 \geq p_2 \geq p_1$, $q_2 \geq q_1$, and $v_0(x) \geq u_0(x)$, then a lower bound for the quenching time is $(1-v_0(1))^{2q_2+2}/2(q_2+1)(q_2+2)$.*

Remark 2 From Theorem 5 and Corollary 2, if we choose $p_1 \leq p_2 \leq 2$, $q_1 = 1, q_2 = 2$ and $u_0(x) = \frac{1}{2}x^4, v_0(x) = \frac{1}{2}x^8$ as in Remark 1, then we get $\lim_{t\to T} v(1,t) \to 1$ and $\lim_{t\to T} u_t \to \infty$ at the quenching time $T = \frac{1}{3\times 2^9}$ as in Remark 3.

Theorem 6 *If* $\min_{x\in[0,1]} u_0(x) = c_1$, $\min_{x\in[0,1]} v_0(x) = c_2$, *and* (u_0, v_0) *satisfies* $(3.1)-(3.2)$, *then we get the following results:*

(a) if $p_2 \geq p_1$ *and* $c_2 \geq c_1$, *then an upper bound for the quenching time is* $(1-c_1)^{p_1+1}/(p_1+1)$.

(b) if $p_1 \geq p_2$ *and* $c_1 \geq c_2$, *then an upper bound for the quenching time is* $(1-c_2)^{p_2+1}/(p_2+1)$.

Proof We consider the following problem to construct lower solutions:

$$\begin{gathered}\mu_t = \mu_{xx} + (1-\lambda)^{-p_1}, \lambda_t = \lambda_{xx} + (1-\mu)^{-p_2}, (x,t) \in (0,1)\times(0,T),\\ \mu_x(0,t) = \lambda_x(0,t) = \mu_x(1,t) = \lambda_x(1,t) = 0, t \in (0,T),\\ \mu(x,0) = u_0(x), \lambda(x,0) = v_0(x), x \in [0,1].\end{gathered}$$

By Definition 2, (μ, λ) is a lower solution of problem (1.1). A solution of the initial-value problem is

$$\begin{cases} \frac{d}{dt}\mu(t) = (1-\lambda(t))^{-p_1} \text{ for } t > 0 \text{ and } \mu(0) = c_1,\\ \frac{d}{dt}\lambda(t) = (1-\mu(t))^{-p_2} \text{ for } t > 0 \text{ and } \lambda(0) = c_2, \end{cases} \tag{4.5}$$

where $c_1 = \min_{x\in[0,1]} u_0(x)$ and $c_2 = \min_{x\in[0,1]} v_0(x)$. Here, $\mu_x(t) = \mu_{xx}(t) = 0$, $\lambda_x(t) = \lambda_{xx}(t) = 0$ and $\mu(0) \leq u_0(x), \lambda(0) \leq v_0(x)$ on $[0,1]$.

Similar to Lemma 5, if we select $c_2 = \lambda(0) \geq \mu(0) = c_1$ and $p_2 \geq p_1$, then we easily get $\lambda(t) \geq \mu(t)$ in $[0,T)$, and if we select $c_1 = \mu(0) \geq \lambda(0) = c_2$ and $p_1 \geq p_2$, then we get $\mu(t) \geq \lambda(t)$ in $[0,T)$, respectively.

Quenching of (4.5) occurs since $\lim_{\mu\to 1-}(1-\mu)^{-p_2} = \infty$ (λ_t blow-up) and $\lim_{\lambda\to 1-}(1-\lambda)^{-p_1} = \infty$ (μ_t blow-up). Since function (μ, λ) does not depend on x the quenching set is $[0,1]$.

Thus, if we select $c_2 = \lambda(0) \geq \mu(0) = c_1$ and $p_2 \geq p_1$, then we get

$$\mu_t = (1-\lambda)^{-p_1} \geq (1-\mu)^{-p_1}.$$

Therefore, integrating for t from t to T we get an upper bound for the quenching time of

$$T \leq (1-c_1)^{p_1+1}/(p_1+1),$$

and if we select $c_1 = \mu(0) \geq \lambda(0) = c_2$ and $p_1 \geq p_2$, then we get

$$\lambda_t = (1-\mu)^{-p_2} \geq (1-\lambda)^{-p_2}.$$

Thus, integrating for t from t to T we get an upper bound for the quenching time of

$$T \leq (1-c_2)^{p_2+1}/(p_2+1).$$

The theorem is proved. □

Remark 3 From Theorem 6, if we choose $p_1 \leq p_2$ and $u_0(x) = \frac{1}{2}x^4, v_0(x) = \frac{1}{2}x^8$ as in Remark 1, then we get $\lim_{t\to T} v(1,t) \to 1$ and $\lim_{t\to T} u_t \to \infty$ at the quenching time $T \leq 1/(p_1+1)$ as in Remark 3.

Theorem 7 *If* $q_1, q_2 \leq 1\,(\neq 1)$, $\min_{x\in[0,1]} u_0(x) = c_1$, $\min_{x\in[0,1]} v_0(x) = c_2$, $\alpha = (1-q_1)/(1-q_1q_2)$, $\beta = (1-q_2)/(1-q_1q_2)$, $\delta \leq \min(\alpha/2, \beta/2)$ *and* (u_0, v_0) *satisfies* $(3.1)-(3.2)$, *then an upper bound for the quenching time is*

$$T \leq \max\left\{(1-c_1)^{1/\alpha}/\delta, (1-c_2)^{1/\beta}/\delta\right\}.$$

Proof Let $\min_{x\in[0,1]} u_0(x) = c_1$ and $\min_{x\in[0,1]} v_0(x) = c_2$. Define

$$\begin{aligned}
\mu(x,t) &= 1 - \left(\delta\left(1 - x^2 + \tau - t\right)\right)^{\alpha} \text{ in } [0,1]\times[0,\tau], \\
\lambda(x,t) &= 1 - \left(\delta\left(1 - x^2 + \tau - t\right)\right)^{\beta} \text{ in } [0,1]\times[0,\tau],
\end{aligned}$$

where $\alpha = (1-q_1)/(1-q_1q_2)$, $\beta = (1-q_2)/(1-q_1q_2)$, $\delta \leq \min(\alpha/2, \beta/2)$, $\tau = \max\left\{(1-c_1)^{1/\alpha}/\delta, (1-c_2)^{1/\beta}/\delta\right\}$.

We have

$$\begin{aligned}
\mu_t - \mu_{xx} &= -\delta\alpha\left(\delta\left(1-x^2+\tau-t\right)\right)^{\alpha-1} + 4\delta^2x^2\alpha(\alpha-1)\left(\delta\left(1-x^2+\tau-t\right)\right)^{\alpha-2} \leq 0, \\
\lambda_t - \lambda_{xx} &= -\delta\beta\left(\delta\left(1-x^2+\tau-t\right)\right)^{\beta-1} + 4\delta^2x^2\beta(\beta-1)\left(\delta\left(1-x^2+\tau-t\right)\right)^{\beta-2} \leq 0,
\end{aligned}$$

for $x \in (0,1), t \in (0,\tau]$. Further,

$$\begin{aligned}
\mu_x(0,t) &= 0, \mu_x(1,t) \leq (1-\lambda(1,t))^{-q_1}, \\
\lambda_x(0,t) &= 0, \lambda_x(1,t) \leq (1-\mu(1,t))^{-q_2},
\end{aligned}$$

for $t \in (0,\tau]$. Furthermore,

$$\begin{aligned}
\mu(x,0) &= 1 - \left(\delta\left(1-x^2+\tau\right)\right)^{\alpha} \leq 1 - (\delta\tau)^{\alpha} = c_1, \\
\lambda(x,0) &= 1 - \left(\delta\left(1-x^2+\tau\right)\right)^{\beta} \leq 1 - (\delta\tau)^{\beta} = c_2,
\end{aligned}$$

for $x \in [0,1]$. Thus, $(\mu(x,t), \lambda(x,t))$ is a lower solution of problem (1.1). In addition, at $t = \tau$ and $x = 1$, we get

$$\mu(1,\tau) = 1 \text{ or } \lambda(1,\tau) = 1.$$

Hence, we have

$$u(1,\tau) \geq \mu(1,\tau) = 1 \text{ or } v(1,\tau) \geq \lambda(1,\tau) = 1$$

by Lemma 5(a). Thus, $x = 1$ is a quenching point. Also, we get

$$T \leq \tau = \max\left\{(1-c_1)^{1/\alpha}/\delta, (1-c_2)^{1/\beta}/\delta\right\}.$$

The theorem is proved. □

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