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Note on the divisoriality of domains of the form $k[[X^p, X^q]]$, $k[X^p, X^q]$, $k[[X^p, X^q, X^r]]$, and $k[X^p, X^q, X^r]$

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Abstract: Let k be a field and X an indeterminate over k. In this note we prove that the domain $k[[X^p, X^q]]$ (resp. $k[X^p, X^q]$) where p, q are relatively prime positive integers is always divisorial but $k[[X^p, X^q, X^r]]$ (resp. $k[X^p, X^q, X^r]$) where p, q, r are positive integers is not. We also prove that $k[[X^q, X^{q+1}, X^{q+2}]]$ (resp. $k[X^q, X^{q+1}, X^{q+2}]$) is divisorial if and only if q is even. These are very special cases of well-known results on semigroup rings, but our proofs are mainly concerned with the computation of the dual (equivalently the inverse) of the maximal ideal of the ring.

Key words: Divisorial ideal, divisorial domain, Noetherian domain

1. Introduction

Let R be an integral domain and L its quotient field. For a nonzero (fractional) ideal I of R, the inverse (also called the dual) of I is the R-submodule of L given by $I^{-1} = \{x \in L | xI \subseteq R\}$. The v-closure of I is the (fractional) ideal I_v of R defined by $I_v = (I^{-1})^{-1}$. Clearly $I \subseteq I_v$ and I is said to be divisorial (or a v-ideal) if $I = I_v$, and the domain R is called divisorial provided that every nonzero ideal of R is divisorial. The class of domains in which each nonzero ideal is divisorial was studied, independently and with different methods, by Bass [10], Matlis [27], and Heinzer [19] in the 1960s. Following Bazzoni and Salce [12, 11], these domains are now called divisorial domains. Among other results, Heinzer proved that an integrally closed domain is divisorial if and only if it is a Prüfer domain with certain finiteness properties [19, Theorem 5.1]. According to [5], the domain $R = k[X^2, X^3]$ is probably the simplest example of an atomic domain that is not a half-factorial domain (HFD for short) since X^2 and X^3 are each irreducible elements of R and $X^6 = X^3X^3 = X^2X^2X^2$. (Clearly R is atomic since R is a (one-dimensional) Noetherian domain. This may also be shown by an easy degree argument.) The domain R is also of interest and has been studied extensively in several other contexts. For example, R is also the simplest example of a non-seminormal domain, and hence $Pic(R) \neq Pic(R[T])$ (see [28]). Domains of the form $k[[X^p, X^q]]$, $k[X^p, X^q]$, $k[[X^p, X^q, X^r]]$, and $k[X^p, X^q, X^r]$ where p, q, and r are positive integers are extensively used as sources of examples and counter-examples in studying different properties of integral domains (see, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17, 21, 20, 22, 25, 26, 29]). The objective of this note is to study the divisoriality of those domains and present it as a unified reference for interested authors. First we prove that $k[[X^p, X^q]]$ (resp. $k[X^p, X^q]$) where p and q are relatively prime

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positive integers is always a divisorial domain; however, $k[[X^p, X^q, X^r]]$ (resp. $k[X^p, X^q, X^r]$) is not. We also prove that $k[[X^q, X^{q+1}, X^{q+2}]]$ (resp. $k[X^q, X^{q+1}, X^{q+2}]$) is divisorial if and only is q is even. It is worth mentioning that the results in this paper are very special cases of well-known results on numerical semigroup rings. For instance, it is well known that if the semigroup has only two generators, then the ring is a hypersurface and therefore Gorenstein, so every ideal is reflexive (equivalently divisorial). Also, a numerical semigroup ring (power series or polynomials) is Gorenstein if and only if the semigroup is symmetric [24]. A one-dimensional domain is Gorenstein if and only if the inverse of the maximal ideal is generated by two elements [10]. Thus, the three-generator semigroups are symmetric if and only if the first generator is even. However, our proofs are mainly concerned with the computation of the dual (equivalently the inverse) of the maximal ideal of the ring. Unreferenced material is standard as in [18] and [23].

2. Main result

Theorem 2.1 *([10, Theorem 6.2, 6.3], [27, Theorem 3.8]) Let R be a local Noetherian domain with maximal ideal M. The following are equivalent:*

(1) *R is divisorial.*

(2) *R has Krull dimension one and M^{-1}/R is simple.*

Our first theorem shows that the domain $k[[X^p, X^q]]$ (resp. $k[X^p, X^q]$) where p and q are relatively prime is always divisorial.

Theorem 2.2 *Let $1 < p < q$ be positive integers such that p and q are relatively prime, $R = k[[X^p, X^q]]$ (resp. $R = k[X^p, X^q]$) and $M = (X^p, X^q)$. Then $M^{-1} = k[[X^p, X^q, X^{p(q-1)-q}]]$ (resp. $M^{-1} = k[X^p, X^q, X^{p(q-1)-q}]$) and R is divisorial.*

Proof For simplicity we put $R = k[[X^p, X^q]]$ and $M = (X^p, X^q)$. Since p and q are relatively prime, $(p-1)(q-1) - 1$ is the largest positive integer that is not expressible as $p\alpha + q\beta$ with α, β positive integers. Thus, for every $n \geq (p-1)(q-1)$, $n = p\alpha + q\beta$ with α, β positive integers and so $X^n \in R$. Now let $f \in M^{-1} \subseteq X^{-p}R$ and set $f = X^{-p}g$ for some $g \in R$. Write $g = \sum\limits_{\alpha,\beta\geq 0} a_{(\alpha,\beta)}X^{p\alpha+q\beta}$. Since $X^q \in M$, $X^{q-p}g = fX^q \in R$. Thus, $\sum\limits_{\alpha,\beta\geq 0} a_{(\alpha,\beta)}X^{p(\alpha-1)+q(\beta+1)} \in R$. If $\alpha \geq 1$, then $X^{p(\alpha-1)+q(\beta+1)} \in R$. If $\alpha = 0$ and $\beta \geq p-1$, then $\beta + 1 \geq p$ and so $\beta + 1 = sp + r$ for some positive integers $s \geq 1$ and $0 \leq r < p$. Thus, $-p + (\beta+1)q = -p + (sp+r)q = (sq-1)p + rq$ and since $sq - 1$ and r are positive integers, $X^{p(\alpha-1)+q(\beta+1)} = X^{-p+(\beta+1)q} = X^{(sq-1)p+rq} \in R$. Hence, if $\alpha = 0$ and $\beta < p-1$, $X^{p(\alpha-1)+q(\beta+1)} \notin R$ and so $a_{(\alpha,\beta)} = 0$. Therefore, $g = \sum\limits_{\beta\geq p-1} a_{(0,\beta)}X^{q\beta} + \sum\limits_{\alpha\geq 1,\beta\geq 0} a_{(\alpha,\beta)}X^{p\alpha+q\beta}$. Thus, $f = X^{-p}g = \sum\limits_{\beta\geq p-1} a_{(0,\beta)}X^{-p+q\beta} + \sum\limits_{\alpha\geq 1,\beta\geq 0} a_{(\alpha,\beta)}X^{p(\alpha-1)+q\beta}$. Set $h = \sum\limits_{\alpha\geq 1,\beta\geq 0} a_{(\alpha,\beta)}X^{p(\alpha-1)+q\beta}$. For every $\alpha \geq 1$, $X^{p(\alpha-1)+q\beta} \in R$ and so $h \in R$. Also, for every $\beta \geq p$, if $\beta = sp + r$ for some positive integers $s \geq 1$ and $0 \leq r < p$, then $-p + q\beta = -p + (sp+r)q = (sq-1)p + rq$ and so $X^{-p+q\beta} = X^{(sq-1)p+rq} \in R$. Thus, $\sum\limits_{\beta\geq p-1} a_{(0,\beta)}X^{-p+q\beta} = a_{(0,p-1)}X^{-p+q(p-1)} + \sum\limits_{\beta\geq p} a_{(0,\beta)}X^{-p+q\beta} = a_{(0,p-1)}X^{(p-1)q-q} + U$ where $U =$

$\sum_{\beta\geq p} a_{(0,\beta)}X^{-p+q\beta} \in R$. Hence, $f = a_{(0,p-1)}X^{(p-1)q-q} + U + h \in R + kX^{p(q-1)-q} \subseteq k[[X^p, X^q, X^{p(q-1)-q}]]$ and therefore $M^{-1} \subseteq k[[X^p, X^q, X^{p(q-1)-q}]]$.

Conversely, since $X^pX^{p(q-1)-q} = X^{(p-1)q} = (X^q)^{p-1} \in R$ and $X^qX^{p(q-1)-q} = X^{p(q-1)} = (X^p)^{q-1} \in R$, $k[[X^p, X^q, X^{p(q-1)-q}]] \subseteq M^{-1}$. Hence, $M^{-1} =$
$k[[X^p, X^q, X^{p(q-1)-q}]] = R + kX^{p(q-1)-q}$ and so $R \subseteq M^{-1}$ is a minimal extension. By Theorem 2.1, R is divisorial.
A similar argument shows that if $R = k[X^p, X^q]$ and $M = (X^p, X^q)$, then $M^{-1} = k[X^p, X^q, X^{p(q-1)-q}] = R + kX^{p(q-1)-q}$. Now let Q be any maximal ideal of R. If $Q \neq M$, then $R_Q = k[X]_N$ for some maximal ideal N of $k[X] = R'$ and so R_Q is divisorial. If $Q = M$, then $R_M \subseteq (MR_M)^{-1} = M^{-1}R_M$ is a minimal extension and by Theorem 2.1, R_M is divisorial. It follows that R is divisorial. □

While the domain $k[[X^p, X^q]]$ (resp. $k[X^p, X^q]$) where p and q are relatively prime is always divisorial, this is not the case for $k[[X^p, X^q, X^r]]$ (resp. $k[X^p, X^q, X^r]$) even if $p < q < r$ are pairwise relatively prime positive integers as is shown by the next proposition. Since the domain $R = k[[X^p, X^q, X^r]]$ (resp. $R = k[X^p, X^q, X^r]$) is a Noetherian domain with integral closure $R' = k[[X]]$ (resp. $R' = k[X]$) and $M = (X^p, X^q, X^q)$ is a noninvertible maximal ideal of R of height one, it is a t-ideal (and so a v-ideal or divisorial), $R \subsetneq M^{-1} = (M : M) \subseteq R'$.

Proposition 2.3 *Let k be a field, q a positive integer, $R_q = k[[X^q, X^{q+1}, X^{q+2}]]$ (resp. $R_q = k[X^q, X^{q+1}, X^{q+2}]$), and $M_q = (X^q, X^{q+1}, X^{q+2})$. Then $M_q^{-1} = k[[X]]$ (resp. $M_q^{-1} = k[X]$) if and only if $q = 3$. In this case R_q is not divisorial.*

Proof Set $R = R_q$ and $M = M_q$ and suppose that $M^{-1} = k[[X]]$ (resp. $M^{-1} = k[X]$). Then $X^{q+3} = X.X^{q+2} \in R$ and so $X^{q+3} = (X^q)^r$ for some positive integer r. Then $q + 3 = rq$ and so $(r-1)q = 3$. Thus, $r = 2$ and $q = 3$.
Conversely, assume that $q = 3$. Then $R = k[[X^3, X^4, X^5]]$ (resp. $R = k[X^3, X^4, X^5]$) and clearly $X^n \in R$ for every $n \geq 3$. Let $f \in M^{-1}$ and set $f = X^{-3}g$ for some $g \in R$. Write $g = a_0 + a_3X^3 + a_4X^4 + a_5X^5 + X^6h$ where $h \in k[[X]]$ (resp. $h \in k[X]$). Since $X^4, X^5 \in M$, $Xg = X^4f, X^2g = X^5f \in R$. Thus, $a_0 = 0$ and so $f = X^{-3}g = a_3 + a_4X + a_5X^2 + X^3h \in k[[X]]$. Thus $M^{-1} \subseteq k[[X]]$ and so $M^{-1} = k[[X]]$ (resp. $M^{-1} = k[X]$). Finally, since $R \subsetneq k[[X^2, X^3]] \subsetneq k[[X]] = M^{-1}$, R is not divisorial. □

Theorem 2.4 *Let $q \geq 2$ be a positive integer, $R_q = k[[X^q, X^{q+1}, X^{q+2}]]$ (resp. $R_q = k[X^q, X^{q+1}, X^{q+2}]$), and $M = (X^q, X^{q+1}, X^{q+2})$.*

(1) *If q is odd, then $M^{-1} = R + kX^{\frac{q(q-1)}{2}-1} + kX^{\frac{q(q-1)}{2}-2}$ and so R_q is not divisorial.*

(2) *If q is even, then $M^{-1} = R + kX^{\frac{q^2}{2}-1}$ and so R_q is divisorial.*

Proof (1) Assume that $q = 2r + 1$. Then $\frac{q(q-1)}{2} - 1 = rq - 1$ and $\frac{q(q-1)}{2} - 2 = rq - 2$. Now since $X^{rq-1}X^q = X^{rq+q-1} = X^{rq+2r} = X^{r(q+2)} = (X^{q+2})^r \in R$, $X^{rq-1}X^{q+1} = X^{rq+q} = X^{q(r+1)} = (X^q)^{r+1} \in R$ and $X^{rq-1}X^{q+2} = X^{rq+q+1} = X^{rq}X^{q+1} \in R$, $kX^{\frac{q(q-1)}{2}-1} = kX^{rq-1} \subseteq M^{-1}$. Similarly, since $X^{rq-2}X^q =$

$X^{(r-1)(q+2)+q+1} = (X^{q+2})^{r-1}X^{q+1} \in R$, $X^{rq-2}X^{q+1} = X^{rq+q-1} \in R$ and $X^{rq-2}X^{q+2} = X^{rq+q} = X^{q(r+1)} = (X^q)^{r+1} \in R$, $kX^{\frac{q(q-1)}{2}-2} = kX^{rq-2} \subseteq M^{-1}$. Thus, $R \subsetneq R + kX^{\frac{q(q-1)}{2}-1} \subsetneq R + kX^{\frac{q(q-1)}{2}-1} + kX^{\frac{q(q-1)}{2}-2} \subseteq M^{-1}$ and therefore R is not divisorial.

(2) Assume that $q = 2r$. Then $\frac{q^2}{2} - 1 = rq - 1$. Since $X^{rq-1}X^q = X^{rq+q-1} = (X^{q+2})^{r-1}X^{q+1} \in R$, $X^{rq-1}X^{q+1} = (X^q)^{r+1} \in R$ and $X^{rq-1}X^{q+2} = X^{rq+q+1} = (X^q)^rX^{q+1} \in R$, $kX^{rq-1} \subseteq M^{-1}$ and so $R + kX^{rq-1} \subseteq M^{-1}$.

Conversely, it is easy to check that $X^n \in R$ for every $n \geq rq$. Now let $f \in M^{-1} \subseteq X^{-q}R$ and write $f = X^{-q}g$ for some $g \in R$. Set $g = a_0 + a_qX^q + a_{q+1}X^{q+1} + a_{q+2}X^{q+2} + a_{2q}X^{2q} + a_{2q+1}X^{2q+1} + a_{2q+2}X^{2q+2} + a_{2q+3}X^{2q+3} + a_{2q+4}X^{2q+4} + a_{3q}X^{3q} + a_{3q+1}X^{3q+1} + a_{3q+2}X^{3q+2} + a_{3q+3}X^{3q+3} + a_{3q+4}X^{3q+4} + a_{3q+5}X^{3q+5} + a_{3q+6}X^{3q+6} + a_{4q}X^{4q} + \cdots + a_{4q+6}X^{4q+6} + a_{4q+7}X^{4q+7} + a_{4q+8}X^{4q+8} + \cdots + a_{(r-1)q}X^{(r-1)q} + a_{(r-1)q+1}X^{(r-1)q+1} + \cdots + a_{(r-1)q+2r-4}X^{(r-1)q+2r-4} + a_{(r-1)q+2r-3}X^{(r-1)q+2r-3} + a_{(r-1)q+2(r-1)}X^{(r-1)q+2(r-1)} + a_{rq}X^{rq} + a_{rq+1}X^{rq+1} + a_{rq+2}X^{rq+2} + a_{rq+3}X^{rq+3} + a_{rq+4}X^{rq+4} + \cdots + a_{(r+1)q-2}X^{(r+1)q-2} + a_{(r+1)q-1}X^{(r+1)q-1} + a_{(r+1)q}X^{(r+1)q} + \ldots$.

Since $X^{q+1}, X^{q+2} \in M$, $Xg = X^{q+1}f$ and $X^2g = X^{q+2}f$ are in R and so $a_0 = a_{q+2} = a_{2q+4} = a_{3q+6} = \cdots = a_{(r-1)q+2(r-1)} = 0$ and $a_0 = a_{q+1} = a_{2q+3} = a_{3q+5} = \cdots = a_{(r-1)q+2r-3} = 0$. Hence,

$g = a_qX^q + a_{2q}X^{2q} + a_{2q+1}X^{2q+1} + a_{2q+2}X^{2q+2} + a_{3q}X^{3q} + a_{3q+1}X^{3q+1} + a_{3q+2}X^{3q+2} + a_{3q+3}X^{3q+3} + a_{3q+4}X^{3q+4} + a_{4q}X^{4q} + \cdots + a_{4q+6}X^{4q+6} + \cdots + a_{(r-1)q}X^{(r-1)q} + a_{(r-1)q+1}X^{(r-1)q+1} + \cdots + a_{(r-1)q+2r-4}X^{(r-1)q+2r-4} + a_{rq}X^{rq} + a_{rq+1}X^{rq+1} + a_{rq+2}X^{rq+2} + a_{rq+3}X^{rq+3} + a_{rq+4}X^{rq+4} + \ldots$.

Thus, $f = X^{-q}g = a_q + a_{2q}X^q + a_{2q+1}X^{q+1} + a_{2q+2}X^{q+2} + a_{3q}X^{2q} + a_{3q+1}X^{2q+1} + a_{3q+2}X^{2q+2} + a_{3q+3}X^{2q+3} + a_{3q+4}X^{2q+4} + a_{4q}X^{3q} + \cdots + a_{4q+6}X^{3q+6} + \cdots + a_{(r-1)q}X^{(r-2)q} + a_{(r-1)q+1}X^{(r-2)q+1} + \cdots + a_{(r-1)q+2r-4}X^{(r-2)q+2r-4} + a_{rq}X^{(r-1)q} + a_{rq+1}X^{(r-1)q+1} + a_{rq+2}X^{(r-1)q+2} + a_{rq+3}X^{(r-1)q+3} + a_{rq+4}X^{(r-1)q+4} + \cdots + a_{(r+1)q-2}X^{rq-2} + a_{(r+1)q-1}X^{rq-1} + a_{(r+1)q}X^{rq} + \cdots \in R + kX^{rq-1}$ and therefore $M^{-1} = R + kX^{rq-1}$. Since the extension $R \subsetneq M^{-1}$ is minimal, by Theorem 2.1, R is divisorial. □

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